

# **PHYSICS**

## **FOR ENGINEERS**

### **Part-1**

**DR. GIAS UDDIN AHMAD**

**HAFIZ BOOK CENTRE**

# PHYSICS FOR ENGINEERS

## VOLUME ONE

(Waves & Oscillations, Properties of Matter,  
Heat & Thermodynamics and Optics)

*Jannat  
Hossain*

**Dr. Gias uddin Ahmed.**

B.Sc. (Hons.), M.Sc. (Dhaka), Ph.D. (Glasgow)

Ex-Professor of Physics,

Bangladesh University of Engineering and Technology, Dhak

and

Vice-Chancellor,

Primeasia University, Dhaka

Revised and enlarged edition of  
Outlines of physics (vol. 1)

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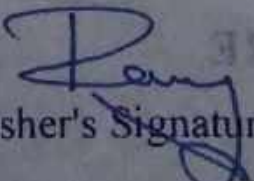
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## PREFACE

**Dedicated to the memory of  
Late Kazi Hafizur Rahman,  
Founder of Hafiz Book Centre.**



## PREFACE

Physics for Engineers, vol. 1, is the revised and enlarged edition of Outlines of Physics, vol. 1. Many new topics have been added to the contents of the present edition, primarily to meet the needs of the students of various disciplines of engineering. It is for this reason that the title of the book has been changed from Outlines of Physics to Physics for Engineers.

The book is based on the syllabi as followed by the different universities of Bangladesh – both public and private. It covers Waves and Oscillations, General Properties of Matter, Heat and Thermodynamics and Geometrical and Physical Optics. Although written primarily for engineering students, degree students in Physics should find the book equally useful.

The author would like to take this opportunity to express his deep gratitude to Mr. Kazi Mahfuzur Rahman and Mr. Kazi Masudur Rahman, who like their father Late Kazi Hafizur Rahman have boldly stuck to the principle of their late father of publishing standard text books written in english without bothering about the financial consequences of such a venture. The author would also like to take this opportunity to thank their brother Mr. Kazi Mashiur Rahman for doing an excellent job in computer composing this book.

Many friends and colleagues have assisted and encouraged the author in preparing the manuscript of the present edition. Particular mention must be made of Prof. Dr. A. A. Z. Ahmad, Ex-Chairman, Space Research And Remote Sensing Organization (SPAARSO) and presently Professor of physics, Brac University, Prof. Dr. Tafazzal Hossain, Pro-Vice Chancellor, American International University of Bangladesh (AIUB), Dr. Engr. Ayub Nabi Khan, Head of the Department of Textile Engineering, Primeasia University and Mr. Sultan Mahmood, Head of the Department of Physics, Asia Pacific University for their many useful suggestions and encouragement. As always, the author is deeply grateful to his erstwhile colleagues of the Department of Physics, Bangladesh University of Engineering and Technology, for their continued support and encouragement.

The book has been hurried through the press and as such there may be errors, typographical or worse. The author would be grateful to learn of any such errors or of any suggestions for the improvement of the book.

Gias uddin Ahmed



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## CHAPTER I

## SIMPLE HARMONIC MOTION

*Periodic motion - Simple Harmonic Motion - Phase and epoch of a particle executing simple harmonic motion - Time period and frequency of a body executing SHM - Energy of a body executing SHM - Average value of kinetic and potential energies of a harmonic oscillator - Some examples of simple harmonic motion - Relation between simple harmonic motion and uniform circular motion - Solved problems - Exercises.*

**1.1 Periodic motion**

A motion which repeats itself over and over again after a regular interval of time is referred to as a *periodic motion*. The time required for each repetition is called *time-period*. The motion of moon about the earth, the oscillation of a pendulum, the motion of a mass suspended from a coil spring are examples of periodic motion.

When the particle, undergoing periodic motion, covers the same path back and forth about a mean position, it is said to be executing an *oscillatory or vibratory motion*. The oscillatory motion is, therefore, a *to and fro* (forward and backward) motion. One complete to and fro motion is called an *oscillation or vibration or a cycle*. Further, the oscillatory motion is not only periodic but also *bounded*, i.e., the displacement of the particle on either side of its mean position remains confined within a well-defined limit. The number of complete oscillations or cycles in unit time is called the *frequency* of vibration.

Of all the trigonometrical ratios, the sines and cosines alone are periodic as well as bounded. Thus the displacement of a particle executing an oscillatory motion is usually expressed in terms of sines or cosines or combination of both. This, coupled with the fact that this type of motion is generally associated with musical instruments, oscillatory motion is also referred to as *harmonic motion*.

**Simple Harmonic Motion**

Let a particle oscillate along a straight line within some fixed limits.

The magnitude and direction of its displacement  $y$  will be changed periodically and along with displacement the velocity and acceleration of the particle will also be changed periodically.

Fig. 1.1 shows a particle oscillating between  $y_1$  and  $y_2$  with the point  $O$  as its equilibrium or mean position. When the particle is at the equilibrium position its potential energy is minimum but its kinetic energy is maximum. When the particle is at its maximum displacement, its potential energy is maximum but its kinetic energy is minimum.

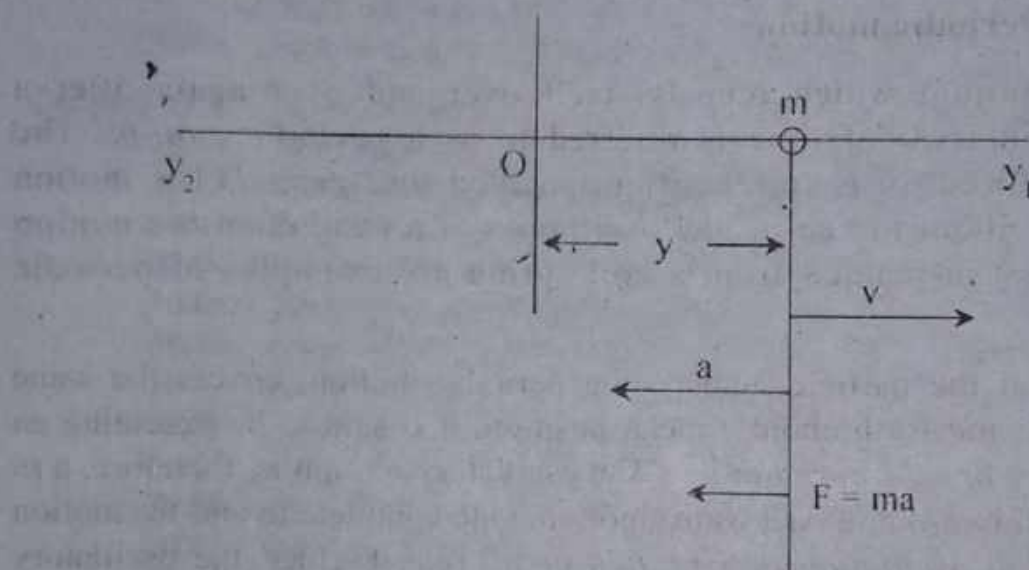


Fig. 1.1

Fig. 1.2 shows the relationship of the potential energy of the oscillating particle with its displacement. For a displacement  $y$  of the particle, the potential energy is represented by  $U(y)$ . It can be observed that  $U(y) = 0$  at the equilibrium or mean position. A restoring force  $F = -\frac{dU(y)}{dy}$  acts upon the particle at every point of its displacement, except at the mean position (where it is zero), tending to bring it back to its mean or equilibrium position. Or, in other words, if the displacement is on the right side of the mean position the force acts towards the left side and *vice versa*.



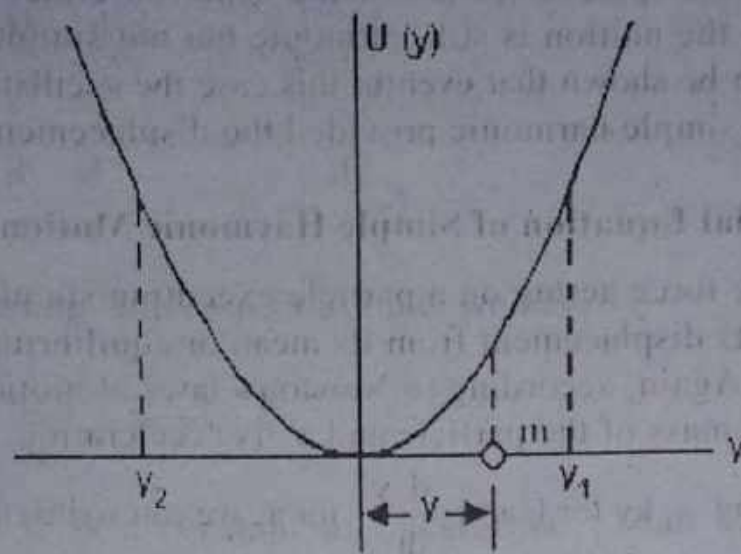


Fig. 1.2

The potential energy of an oscillating particle is  $U(y) = \frac{1}{2}ky^2$  ;  
then the force acting on the particle is

$$F(y) = - \frac{dU(y)}{dy} = -ky \quad (1.1)$$

where  $k$  is a *positive constant* called the *force constant*

Eqn. (1.1) indicates that a restoring force acts on the particle tending to bring it back to its mean or equilibrium position.

The motion executed by such an oscillating particle is called harmonic oscillation and the particle itself a harmonic oscillator. In case the limits ( $y_1$  and  $y_2$ ) are equally spaced about the equilibrium position, it is called simple harmonic motion (S. H. M) and the particle a simple harmonic oscillator.

Thus a clear-cut definition to simple harmonic motion can be given as follows :

*Whenever a force acting on a particle, and hence the acceleration of the particle, is proportional to its displacement from its equilibrium position or any other fixed point in its path, but is always directed in a direction opposite to the direction of the displacement (i.e., directed towards the mean position) and if the maximum displacement of the particle is the same on either side of the mean position, the particle is said to execute a simple harmonic motion.*

In case the displacement is not the same on either side of the mean position, the motion is still harmonic but not simple harmonic. However, it can be shown that even in this case the oscillation can still be regarded as simple harmonic provided the displacement is small.

## 1.2 Differential Equation of Simple Harmonic Motion

If  $F$  be the force acting on a particle executing simple harmonic motion and  $y$  its displacement from its mean or equilibrium position, then  $F = -ky$ . Again, according to Newton's laws of motion,  $F = ma'$  where  $m$  is the mass of the particle and  $a'$  its acceleration.

Substituting  $-ky$  for  $F$  and  $\frac{d^2y}{dt^2}$  for  $a'$ , we can write (for  $F = ma'$ )

$$-ky = m \frac{d^2y}{dt^2}$$

$$\text{or, } \frac{d^2y}{dt^2} + \frac{k}{m}y = 0 \quad (1.2)$$

Equation (1.2) is called the differential equation of motion of a body executing simple harmonic motion, because the solution of eqn. (1.2) gives us the nature of variation of displacement with time. Thus the correct nature of the motion of the particle can be known.

Rearranging eqn. (1.2), we can write,

$$\frac{d^2y}{dt^2} = -\frac{k}{m}y = -w^2y \quad (1.3)$$

$$= -\mu y \quad (1.4)$$

where  $w = \sqrt{\frac{k}{m}}$  is the angular velocity of the particle and  $\mu$  is a constant equal to  $w^2$ . Since  $\frac{d^2y}{dt^2} = -\mu$  when  $y = 1$ ,  $\mu$  may be defined as the *acceleration per unit displacement of the particle*.

To obtain a general solution of the differential equation of simple harmonic motion, let us multiply both sides of eqn. (1.3) by  $2 \frac{dy}{dt}$  when we get



$$2. \frac{dy}{dt} \cdot \frac{d^2y}{dt^2} = -w^2 y \cdot 2. \frac{dy}{dt}$$

$$\text{or, } 2 \frac{dy}{dt} \cdot \frac{d^2y}{dt^2} = -w^2 \cdot 2. \frac{dy}{dt} \cdot y$$

Integrating with respect to time, we have

$$\left( \frac{dy}{dt} \right)^2 = w^2 y^2 + C \quad (1.5)$$

where  $C$  is a constant of integration.  $C$  can be evaluated by applying boundary conditions. To evaluate  $C$  we recall that velocity and acceleration of a simple harmonic motion are oppositely directed. We further recall that the velocity is zero (or K.E. is zero) at maximum displacement (or amplitude).

$$\text{or, } \frac{dy}{dt} = 0 \text{ when } y = a \text{ (amplitude).}$$

Substituting these values in eqn. (1.5)

$$0 = -w^2 a^2 + C ; \text{ or, } C = w^2 a^2$$

Substituting this value of  $C$  in eqn. (1.5), we have

$$\left( \frac{dy}{dt} \right)^2 = w^2 y^2 + w^2 a^2 = w^2 (a^2 + y^2) \quad (1.6)$$

$$\text{or, } \frac{dy}{dt} = \pm w \sqrt{a^2 - y^2}$$

$$= \pm \sqrt{\frac{k}{m}} \sqrt{a^2 - y^2} \quad (1.7)$$

Eqn. (1.7) can be rearranged as

$$\frac{dy}{\sqrt{a^2 - y^2}} = w \cdot dt$$

Integrating again with respect to time, we have

$$\sin^{-1} \frac{y}{a} = wt + \phi$$

$$\text{or, } y = a \sin (\omega t + \phi) \quad (1.8)$$

Equation (1.8) gives the displacement of the particle at an instant  $t$  in terms of its amplitude  $a$  and its total phase  $(\omega t + \phi)$  and is the general solution of the differential equation of simple harmonic motion.

Expanding eqn. (1.8), we have

$$\begin{aligned} y &= a \sin \omega t \cos \phi + a \cos \omega t \sin \phi \\ &= A \sin \omega t + B \cos \omega t \end{aligned}$$

where  $A = a \cos \phi$  and  $B = a \sin \phi$

In special cases where either  $A$  or  $B$  may be zero, the displacement may be written as either

$$y = A \sin \omega t$$

$$\text{or, } y = B \cos \omega t$$

Hence the most general form of the differential equation is

$$y = A \sin \omega t + B \cos \omega t \quad (1.9)$$

which is a combination of both the sine and the cosine terms.

### Velocity and acceleration of a body executing SHM

The displacement of a particle executing simple harmonic motion is given by

$$y = a \sin (\omega t + \phi)$$

(i) Hence, the velocity of the particle at any instant of time  $t$  is

$$\begin{aligned} \frac{dy}{dt} &= \omega a \cos (\omega t + \phi) \\ &= \pm \omega a \sqrt{1 - \sin^2 (\omega t + \phi)} \end{aligned}$$

$$\text{Now } \sin (\omega t + \phi) = \frac{y}{a}$$

$$\therefore \frac{dy}{dt} = \pm \omega a \sqrt{1 - \frac{y^2}{a^2}}$$



$$\begin{aligned}
 &= \pm wa \sqrt{\frac{a^2 - y^2}{a^2}} \\
 &= \pm wa \sqrt{a^2 - y^2} \\
 &= \pm \sqrt{\frac{k}{m}} \sqrt{a^2 - y^2} \quad (1.10)
 \end{aligned}$$

(ii) The acceleration of the particle at any time  $t$  is given by

$$\begin{aligned}
 \frac{d^2 y}{dt^2} &= -w^2 a \sin (wt + \phi) \\
 &= w^2 a \cdot \frac{y}{a} = -w^2 y \\
 &= -\frac{k}{m} y. \quad (1.11)
 \end{aligned}$$

From eqns. (1.10) and (1.11) it can be seen that *maximum value of velocity*, i.e.,  $v_{\max} = \pm wa$  or  $\pm a \sqrt{\frac{k}{m}}$  and occurs when  $y = 0$  i.e., when the particle is passing through its mean position.

Similarly, the *maximum value of acceleration* occurs when  $y$  is maximum i.e., the particle is at the position of one of its extreme displacements.

### 1.3 Phase and epoch of a particle executing simple harmonic motion

Going back to eqn. (1.8), we see that the total phase of a particle executing simple harmonic motion is made up of the *phase angle*  $wt$  and  $\phi$  which is called the *initial phase* or *phase constant* or the *epoch* of the particle, usually denoted by the letter  $e$ . This initial phase or epoch is a direct consequence of the fact that we start to count time, not from the instant when the particle is in some standard position, like its mean position or one of its extreme position, but from the instant when it is anywhere else in between. The following cases will then arise :

(i) if we start counting time when the particle is in its mean position, i.e., when  $y = 0$  at  $t = 0$ , we have  $\phi = 0$ . Eqn. (1.8), therefore, reduces to

$$y = a \sin \omega t$$

(ii) If the counting of time starts *when the particle is in one of its extreme positions, i.e.,* when  $y = a$  at  $t = 0$ , we have from eqn (1.8),

$$a = a \sin (0 + \phi)$$

$$\text{or, } \sin \phi = \frac{a}{a} = 1; \quad \text{or, } \phi = \frac{\pi}{2}.$$

Eqn. (1.9), therefore, becomes

$$y = a \sin \left( \omega t + \frac{\pi}{2} \right) = a \cos \omega t.$$

As can be seen, both  $y = a \sin \omega t$  and  $y = a \cos \omega t$  represent the equations of simple harmonic motion with the difference lying only in the initial position of the vibrating particle. Either of these equations can, therefore, be taken as the general equation of simple harmonic motion. In fact, the general solution of the differential equation of simple harmonic motion is of the form

$$y = A \sin \omega t + B \cos \omega t$$

which is a combination of both the sine and cosine terms.

(iii) If, on the other hand, we start counting time from an instant  $t'$ , before the particle has passed through its mean position, we have  $y = 0$  at  $t = t'$ .

Then we have

$$0 = a \sin (\omega t' + \phi)$$

or,  $(\omega t' + \phi) = 0$  whence  $\phi = -\omega t' = -e$ , say. Therefore, the expression for simple harmonic motion becomes

$$y = a \sin (\omega t - e)$$

(iv) Similarly, if the counting of time is started from an instant  $t'$ , after the particle has passed through its mean position, we have

$$y = a \sin (\omega t + e)$$

#### 1.4 Time period, frequency and angular frequency of a body executing SHM



If the time  $t$  in the relation  $y = a \sin (wt + \phi)$  is increased by  $2\pi/w$ , then we have

$$y = a \sin \left[ w \left( t + \frac{2\pi}{w} \right) + \phi \right]$$

$$\begin{aligned} \text{or, } y &= a \sin (wt + 2\pi + \phi) \\ &= a \sin (wt + \phi) \end{aligned}$$

the same as before. This indicates that the particle executing simple harmonic motion *repeats its motion after every  $2\pi/w$  seconds*. In other words, the *time period of the particle* is given by

$$T = \frac{2\pi}{w} = 2\pi \sqrt{\frac{1}{w^2}} = 2\pi \sqrt{\frac{1}{\mu}}$$

$$\begin{aligned} \text{or, } T &= 2\pi \sqrt{\frac{1}{\text{acceleration per unit displacement}}} \\ &= 2\pi \sqrt{\frac{\text{displacement}}{\text{acceleration}}} \end{aligned}$$

Since  $w^2 = \frac{k}{m}$ , the time period may also be expressed as

$$T = 2\pi \sqrt{\frac{m}{k}} \quad (1.12)$$

The number of oscillations (or vibrations) made by the particle per second (unit time) is called its *frequency of oscillation* or, simply, its *frequency*, usually denoted by the letter  $n$ .

Thus, frequency is the reciprocal of the time period.

$$\text{or, } n = \frac{1}{T} = \frac{w}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{k}{m}}$$

From the above relation we also have  $w = 2\pi n = \frac{2\pi}{T}$ .  $w$  is also referred to as the angular frequency of the particle. It is the angle described per second and has the unit radians per second, the same as angular velocity, also designated  $w$ . Angular frequency is very

closely related to the angular velocity of a circular motion which is associated with simple harmonic motion. The use of  $\omega$  instead of  $n$  to describe the frequency of simple harmonic motion simplifies these expressions by avoiding factor of  $2\pi$ , which usually accompany  $n$  in such expressions.

It may be noted that the angular frequency and thus the time period  $T = \frac{2\pi}{\omega}$  are determined only by the force constant and the mass.

They do not depend either on the amplitude  $a$  or the initial phase  $\phi$ . This means that the oscillations of a simple harmonic oscillator are isochronous, i.e., they take the same time irrespective of the values of  $a$  and  $\phi$ . This is an important property of simple harmonic motion where the frequency and time period are independent of the amplitude and epoch.

**Example 1.1.** A spring, hung vertically, is found to be stretched by 0.02 m from its equilibrium position when a force of 4N acts on it. Then a 2 kg body is attached to the end of the spring and is pulled 0.04 m from its equilibrium position along the vertical line. The body is then released and it executes simple harmonic motion.

(i) what is the force constant of the spring?

A force of 4N on the spring produces a displacement of 0.02 m. Hence,

$$\text{from } F = ky \text{ we have } k = \frac{F}{y} = \frac{4\text{N}}{0.02\text{m}}$$

$$= 200 \text{ N per m.}$$

(ii) what is the force executed by the spring on the 2 kg body just before it is released?

The spring is stretched 0.04 m. Hence the force executed by the spring is

$$F = -ky = -(200 \text{ N.m}^{-1})(0.04 \text{ m})$$

$$= -8 \text{ N.}$$

The minus sign indicates that the force is directed opposite to the displacement.



(iii) what is the period and frequency of oscillation after release ?

$$T = 2\pi \sqrt{\frac{m}{k}} = 2\pi \sqrt{\frac{2 \text{ kg}}{200 \text{ N.m}^{-1}}}$$

$$= \frac{\pi}{5} \text{ sec} = 0.628 \text{ sec.}$$

$$n = \frac{1}{T} = \frac{1}{0.628} = 1.59 \text{ Hz.}$$

$$\omega = 2\pi n = (2)(3.14)(1.59)$$

$$= 10 \text{ sec}^{-1}.$$

Also,

$$\omega = 2\pi n = 2\pi \cdot \frac{1}{T} = \frac{1}{2\pi} = \sqrt{\frac{k}{m}}$$

$$= \sqrt{\frac{k}{m}} = \sqrt{\frac{200 \text{ N.m}^{-1}}{2 \text{ kg}}} = 10 \text{ sec}^{-1}$$

(iv) what is the amplitude of motion ?

Amplitude is the initial displacement to the body, i.e. **0.04 m.**

(v) what is the maximum velocity of the oscillating body ?

The maximum velocity occurs when the body passes through the position of equilibrium, i.e.,  $y = 0$ .

Hence, from

$$v = \pm \omega \sqrt{a^2 - y^2}, \text{ we get}$$

$$v_{\max} = \pm \omega a$$

$$\therefore v_{\max} = (10 \text{ sec}^{-1})(0.04 \text{ m})$$

$$= \pm 0.4 \text{ m sec}^{-1}.$$

(vii) what is the mechanical (total) energy of the oscillating system ?

Total energy = P.E + K.E

$$= \frac{1}{2} ka^2 = \frac{1}{2} (200)(0.04)^2$$

$$= 0.16 \text{ joules.}$$

Also, total energy

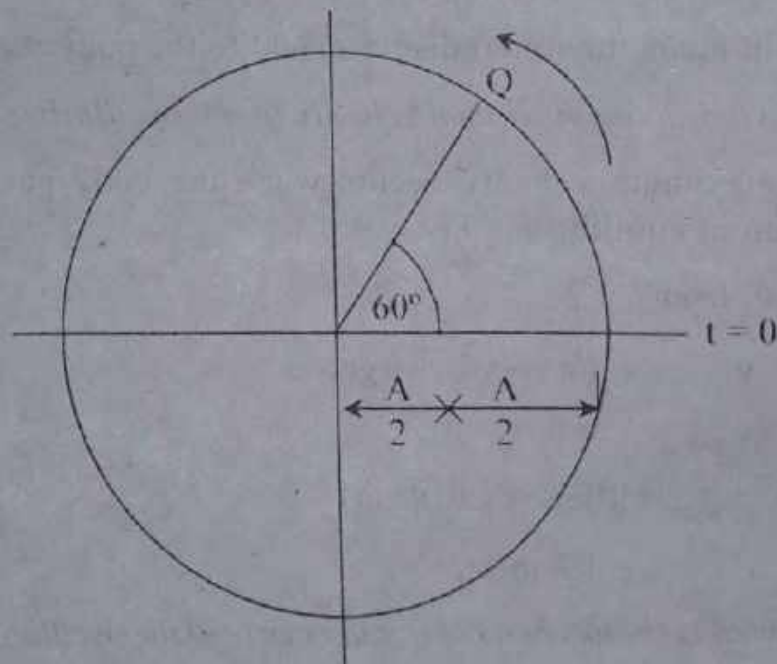
$$\frac{1}{2} k a^2 = 2\pi^2 m a^2 n^2 \quad (\text{Art. 1.5})$$

$$= 2 \cdot (3.14)^2 \cdot 2 \cdot (0.04)^2 \cdot (1.59)^2$$

$$= 0.16 \text{ joules.}$$

(ix) how much time is required for the body to move half-way in to the centre from its initial position?

The motion is neither one of constant velocity nor one of constant acceleration. The simplest method is to make use of the reference circle. While the body moves half-way in, the reference point revolves through an angle of  $60^\circ$ . Since the reference point moves with constant angular speed and in this example makes one complete revolution in  $\frac{\pi}{5}$  sec, the time to rotate through  $60^\circ$  is



$$\frac{1}{6} \cdot \frac{\pi}{5} \text{ sec} = \frac{\pi}{30} \text{ sec} = 0.105 \text{ sec.}$$

The time can also be computed directly from the relation,

$$y = a \sin \omega t$$



$$\frac{a}{2} = a \sin (10 \text{ sec}^{-1}) \cdot t$$

$$\sin (10 \text{ sec}^{-1}) \cdot t = \frac{1}{2}$$

$$10 \text{ sec}^{-1} \cdot t = \sin^{-1} \frac{1}{2} = \frac{\pi}{3}$$

$$\therefore t = \frac{\pi}{3 \times 10} \text{ sec}$$

(x) what is the displacement of the body as a function of time ?

The general equation for displacement of a body executing simple harmonic motion is

$$y = a \sin (\omega t + \phi)$$

The value of  $\omega$ , as already obtained, is

$$\omega = \frac{2\pi}{T} = 10 \text{ radian/sec.}$$

$$\therefore y = a \sin (10t + \phi)$$

At  $t = 0$ ,  $y = a = 0.04 \text{ m}$ , so that at that instant

$$y = 0.04 \sin \phi = 0.04$$

$$\therefore \sin \phi = 1$$

$$\text{or, } \phi = \sin^{-1} 1 = \frac{\pi}{2} \text{ radian.}$$

Therefore, with  $a = 0.04 \text{ m}$ ,  $\omega = 10 \text{ rad./sec}$  and  $\phi = \frac{\pi}{2}$  radian, we get

$$\begin{aligned} y &= 0.04 \sin \left( 10t + \frac{\pi}{2} \right) \\ &= 0.04 \sin 10t. \end{aligned}$$

**Example 1.2.** A body is vibrating with simple harmonic motion of amplitude 15 cm and frequency 4 Hz. Compute (a) the maximum values of the acceleration and velocity and (b) the acceleration and velocity when the displacement is 9 cm.

**Soln.** (a)  $v_{\max} = \omega a$

$$a = 15 \text{ cm}$$

$$n = 4 \text{ Hz}$$

$$\therefore \omega = 2\pi n = 2 \times 3.14 \times 4 \\ = 25.12 \text{ rad/sec.}$$

$$\therefore v_{\max} = 25.12 \times 15 = 376.8 \text{ cm/sec.}$$

$$(a_{\text{accln}})_{\max} = -\omega^2 a$$

$$= -(25.12)^2 \times 15$$

$$= -9470 \text{ cm/sec}^2.$$

$$(b) \text{ when } y = 9 \text{ cm}$$

$$v = \omega \sqrt{a^2 - y^2} = 25.12 \sqrt{(15)^2 - 9^2}$$

$$= 300 \text{ cm/sec}$$

$$a_{\text{accln.}} = -\omega^2 y = -(25.12)^2 \times 9$$

$$= -5680 \text{ cm/sec}^2.$$

**Example 1.3.** A particle executes linear harmonic motion about the point  $x = 0$ . At  $t = 0$ , it has displacement  $y = 0.37 \text{ cm}$  and zero velocity. If the frequency of the motion is  $0.25/\text{sec}$ , determine (a) the period, (b) the amplitude, (c) the maximum speed and (d) the maximum acceleration.

**Soln.**

$$(a) \quad T = \frac{1}{n} = \frac{1}{0.25} = 4 \text{ sec.}$$

$$(b) \quad y = a \sin (\omega t + \delta)$$

$$\text{at } t = 0, y = 0.37 \text{ cm.}$$

$$\therefore a \sin \delta = 0.37 \quad (i)$$

$$\text{Again } v = \frac{dy}{dt} = \omega a \cos (\omega t + \delta)$$

$$\text{at } t = 0, v = 0$$

$$\therefore \omega a \cos \delta = 0$$



$$\text{or, } a \cos \delta = 0$$

(ii)

from (i) and (ii),

$$a^2 = (0.37)^2$$

$$\text{or, } a = \pm 0.37 \text{ cm.}$$

$$(c) \quad v = w \sqrt{a^2 - y^2}$$

$v$  is maximum when  $y = 0$ .

$$\begin{aligned} \therefore v_{\max} &= w \cdot a = a \times 2\pi n \\ &= 0.37 \times 2 \times 3.14 \times 0.25 \\ &= 0.5809 \text{ cm/sec.} \end{aligned}$$

$$\begin{aligned} (d) \quad (acceln)_{\max} &= -w^2 \cdot a \\ &= -(2\pi n)^2 \cdot a \\ &= -(2 \times 3.14 \times 0.25)^2 \times 0.37 \\ &= -0.912013 \text{ cm/sec}^2. \end{aligned}$$

**Example 1.4.** The displacement of an oscillating particle at an instant  $t$  is given by

$$y = a \cos wt + b \sin wt.$$

Show that it is executing a simple harmonic motion.

If  $a = 5 \text{ cm}$ ,  $b = 12 \text{ cm}$  and  $w = 4 \text{ radian/sec}$ , calculate (i) the amplitude, (ii) the time period, (iii) the maximum velocity and (iv) the maximum acceleration of the particle.

**Soln.**

$$y = a \cos wt + b \sin wt$$

$$\text{or, } \frac{dy}{dt} = -aw \sin wt + bw \cos wt$$

$$\text{or, } \frac{d^2y}{dt^2} = -w^2 \cdot a \cos wt - w^2 b \sin wt$$

$$= -w^2 (a \cos wt + b \sin wt)$$

$$= -w^2 y$$

$$\text{or, } \frac{d^2 y}{dt^2} + w^2 y = 0$$

Hence the motion is simple harmonic.

$$(i) \text{ Let } a = A \sin \alpha \text{ and } b = A \cos \alpha.$$

Then

$$\begin{aligned} y &= A \sin \alpha \cos wt + A \cos \alpha \sin wt \\ &= A \sin (wt + \alpha) \end{aligned}$$

This represents a simple harmonic motion with amplitude  $A$ .

$$\therefore A^2 \sin^2 \alpha + A^2 \cos^2 \alpha = a^2 + b^2$$

$$\text{or, } A = \sqrt{a^2 + b^2}$$

$$a = 5 \text{ cm}$$

$$b = 12 \text{ cm}$$

$$\begin{aligned} \therefore A &= \sqrt{5^2 + 12^2} \\ &= 13 \text{ cm} \end{aligned}$$

$$(ii) T = \frac{2\pi}{w} = \frac{2\pi}{4} = \frac{\pi}{2} \text{ sec.}$$

$$\begin{aligned} (iii) v_{\max} &= w.A = 13 \times 4 \\ &= 52 \text{ cm/sec} \end{aligned}$$

$$\begin{aligned} (iv) (a_{\max})_{\max} &= -w^2.A = -(4)^2 \times 13 \\ &= -208 \text{ cm/sec}^2. \end{aligned}$$

**Example 1.5.** The positions of a particle executing simple harmonic motion along the  $x$ -axis are  $x = A$  and  $x = B$  at time  $t$  and  $2t$  respectively. Show that its period of oscillation is given by

$$T = (2\pi) / \cos^{-1} (B/2A)$$

**Soln.**



$$A = a \sin \omega t$$

$$B = a \sin (\omega \cdot 2t)$$

$$= a \sin 2\omega t$$

$$= a 2 \sin \omega t \cos \omega t$$

$$\therefore \frac{A}{B} = \frac{a \sin \omega t}{a 2 \sin \omega t \cos \omega t}$$

$$= \frac{1}{2 \cos \omega t}$$

$$\text{or, } \cos \omega t = \frac{B}{2A}$$

$$\text{or, } \omega t = \cos^{-1} \left( \frac{B}{2A} \right)$$

$$\text{or, } \omega = \frac{1}{t} \cos^{-1} \left( \frac{B}{2A} \right)$$

$$\text{or, } \frac{2\pi}{T} = \frac{\cos^{-1} \left( \frac{B}{2A} \right)}{t}$$

$$\therefore T = \frac{2\pi t}{\cos^{-1} \left( \frac{B}{2A} \right)}$$

**Example 1.6.** For a particle executing simple harmonic motion the displacement is 8 cm at the instant the velocity is 6 cm/sec and the displacement is 6 cm at the instant the velocity is 8 cm/sec. Calculate (i) amplitude, (ii) frequency and (iii) time period.

**Soln.**

Velocity of a particle executing simple harmonic motion,

$$v = \frac{dy}{dt} = \omega \sqrt{a^2 - y^2}$$

Now  $v = 6$  cm/sec when  $y = 8$  cm.

$$\therefore 6 = w \sqrt{a^2 - 64} \quad (i)$$

Again  $v = 8$  cm/sec when  $y = 6$  cm

$$\therefore 8 = w \sqrt{a^2 - 64} \quad (ii)$$

Dividing (ii) by (i) and squaring

$$\frac{64}{36} = \frac{a^2 - 36}{a^2 - 64}$$

or,  $a = 10$  cm.

Substituting  $a = 10$  cm in eqn. (i)

$$\therefore 6 = w \sqrt{100 - 64}$$

or,  $w = 1$  rad/sec.

Hence frequency,

$$n = \frac{w}{2\pi} = \frac{1}{2\pi} \text{ Hz}$$

time period

$$T = \frac{1}{n} = 2\pi \text{ seconds.}$$

**Example 1.7.** A simple harmonic motion is represented by

$$y = 10 \sin \left( 10t - \frac{\pi}{6} \right)$$

where  $y$  is measured in metres,  $t$  in seconds and the phase angle in radians. Calculate (i) the frequency, (ii) the time period, (iii) the maximum displacement, (iv) the maximum velocity and (v) the maximum acceleration and (vi) displacement, velocity and acceleration at time  $t = 0$  and  $t = 1$  second.

**Soln.**

$$\text{Here } y = 10 \sin \left( 10t - \frac{\pi}{6} \right) \quad (1)$$

Comparing with the displacement equation

$$y = a \sin (wt + \delta) \quad (2)$$



for Engineers

we get, (i)  $w = 2\pi n = 10$

or,  $n = \frac{10}{2\pi} = 1.6 \text{ Hz.}$

(ii) time period,  $T = \frac{1}{n} = \frac{2\pi}{10} = 0.63 \text{ sec.}$

(iii) maximum displacement (amplitude)  
 $a = 10 \text{ m.}$

(iv) maximum velocity,

$$v_{\max} = w \cdot a = 10 \times 10 = 100 \text{ m/sec.}$$

(v)  $(\text{accln.})_{\max} = -w^2 a = -(10)^2 \times 10$   
 $= -1000 \text{ m/sec}^2.$

minus sign shows that the acceleration is directed towards the mean position.

(vi) From eqn. (1)

(a) at  $t = 0$

$$y = 10 \sin \left( -\frac{\pi}{6} \right) = -5 \text{ m.}$$

velocity,  $\frac{dy}{dt} = a w \cos \delta$

$$= 10 \times 10 \cos \left( -\frac{\pi}{6} \right)$$

$$= 100 \times 0.866 = 86.6 \text{ m/sec.}$$

Acceleration,  $\frac{d^2y}{dt^2} = -aw^2 \sin \delta$

$$= -10 \times 10^2 \times \sin \left( -\frac{\pi}{6} \right)$$

$$= -10 \times 100 \times 0.5$$

$$= -500 \text{ m/sec}^2.$$

(b) From eqn. (1), at  $t = 1$ ,

displacement

$$y = 10 \sin \left( 10 - \frac{\pi}{6} \right)$$

$$= 10 \sin \left( \frac{60 - 3.142}{6} \right)$$

$$= 10 \sin \left( \frac{56.858}{6} \right)$$

$$= 10 \sin (3\pi) \text{ approximately}$$

$$= 10 \sin \pi$$

$$= 0$$

$$\text{velocity, } \frac{dy}{dt} = a\omega \cos \left( 10 - \frac{\pi}{6} \right)$$

$$= a\omega \cos (\pi) \text{ approximately}$$

$$= 10 \times 10 \times (-1)$$

$$= -100 \text{ m/sec.}$$

$$\text{Accn., } \frac{d^2y}{dt^2} = -a\omega^2 \sin \left( 10 - \frac{\pi}{6} \right)$$

$$= -a\omega^2 \sin (\pi) \text{ approximately.}$$

$$= 0.$$

**Example 1.8.** A particle performs simple harmonic motion given by the equation

$$y = 20 \sin (\omega t + \alpha)$$

If the time period is 30 seconds and the particle has a displacement of 10 cm at  $t = 0$ , find (i) epoch, (ii) the phase angle at  $t = 5$  seconds and (iii) the phase difference between two positions of the particle 15 seconds apart.

**Soln.**

Here

$$y = 20 \sin (\omega t + \alpha)$$

$$T = 30 \text{ secs.}$$

$$\therefore \omega = \frac{2\pi}{T} = \frac{2\pi}{30} = \frac{\pi}{15} \text{ rad/sec.}$$



(i) at  $t = 0$ ,  $y = 10$  cm.

$$\therefore 10 = 20 \sin \left( \frac{\pi}{15} \times 0 + \alpha \right)$$

$$\text{or, } \sin \alpha = \frac{10}{20} = 0.5$$

$$\text{or, } \alpha = \frac{\pi}{6} \text{ radian.}$$

(ii) at  $t = 5$  sec,

the phase angle  $= (\omega t + \alpha)$

$$= \left( \frac{\pi}{15} \times 5 + \frac{\pi}{6} \right)$$

$$= \frac{\pi}{2}$$

(iii) at  $t = 0$  the phase angle

$$\theta_1 = \left( \frac{\pi}{15} \times 0 + \frac{\pi}{6} \right) = \frac{\pi}{6} \text{ radian.}$$

at  $t = 15$  sec, the phase angle

$$\theta_2 = \left( \frac{\pi}{15} \times 15 + \frac{\pi}{6} \right)$$

$$= \frac{7\pi}{6} \text{ radian.}$$

$\therefore$  the phase difference,

$$\theta_2 - \theta_1 = \frac{7\pi}{6} - \frac{\pi}{6} = \pi \text{ radian.}$$

**Example 1.9.** A body describing SHM has a maximum acceleration of  $8\pi \text{ m/s}^2$  and a maximum speed of  $1.6 \text{ m/s}$ . Find the period  $T$  and the amplitude  $a$ .

**Soln :**

$$a_{\max} = \omega^2 a \quad (\text{ignoring the minus sign})$$

$$= (2\pi n)^2 a = \left( \frac{2\pi}{T} \right)^2 a = \frac{4\pi^2}{T^2} a = 8\pi \text{ m/s}^2.$$

again  $v_{\max} = \omega a = \left(\frac{2\pi}{T}\right) a = 1.6 \text{ m/s.}$

$$\frac{a_{\max}}{v_{\max}} = \frac{2\pi}{T} = \frac{8\pi}{1.6}; \quad \text{or, } T = 0.4 \text{ s.}$$

Then  $\frac{2\pi a}{T} = \frac{2\pi a}{0.4} = 1.6; \quad \text{or, } a = \frac{0.4 \times 1.6}{2\pi}$

$$= 0.102 \text{ m}$$

**Example 1.10.** A ball moves in a circular path of 0.15 m diameter with a constant angular speed of 20 rev/min. Its shadow performs simple harmonic motion on the wall behind it. Find the acceleration and speed of the shadow (i) at a turning point of the motion (ii) at the equilibrium position, and (iii) at a point 6 cm from the equilibrium position.

**Soln :**

Amplitude is the radius of the circle.

$$\therefore a = \frac{0.15}{2} \text{ m} = 0.075 \text{ m}$$

The period and frequency of the shadow are the same as that of the circular motion.

So

$$20 \text{ rev/min} \times \frac{20}{60} \text{ rev./sec.}$$

$$\omega = (2\pi) \times \frac{20}{60} \text{ rad/s} = \frac{2\pi}{3} \text{ rad/sec.}$$

From the general formulas for SHM, we have

(i) at the turning point  $v = 0$  and  $a = a_{\max}$ .

$$\therefore a_{\max} = \omega^2 a = \left(\frac{4\pi^2}{9}\right) (0.075) = 0.329 \text{ m/sec}^2.$$

(ii) at the equilibrium position,



$$a = 0 \text{ and } v = v_{\max}$$

$$\therefore v_{\max} = \omega a = \left( \frac{2\pi}{3} \right) (0.075) = \mathbf{0.157 \text{ m/sec.}}$$

(iii) at  $y = 0.06 \text{ m (6 cm)}$

$$\begin{aligned} v &= \omega \sqrt{a^2 - y^2} = \frac{2\pi}{3} \sqrt{(0.075)^2 - (0.06)^2} \\ &= \frac{2\pi}{3} (0.045) = \mathbf{0.0942 \text{ m/sec.}} \end{aligned}$$

$$a = \omega^2 y = \left( \frac{4\pi^2}{9} \right) (0.06) = \mathbf{0.263 \text{ m/sec}^2}.$$

### 1.5 Energy of a body executing S.H.M.

The acceleration of a particle and hence the force acting on the particle executing simple harmonic motion is, as we know, directed towards its mean or equilibrium position, *i.e.*, opposite to the direction in which the displacement  $y$  increases. Work is, therefore, done during the displacement of the particle. Hence the particle possesses *potential energy* ( $U$ ). As the particle also possesses velocity, it possesses *kinetic energy* ( $K$ ) too. Thus the mechanical energy  $E$  of a particle executing simple harmonic motion is partly kinetic and partly potential. *If no non-conservative forces, such as the force of friction act on the particle, the sum of its kinetic energy and potential energy remains constant.* or,

$$E = K + U = \text{constant.}$$

As the displacement increase, the potential energy increases and the kinetic energy decreases and *vice versa*. But the total energy  $E (= K + U)$  is conserved.

Let the displacement of a particle executing simple harmonic motion at any instant be  $y$ . If the mass of the particle be  $m$  and its velocity at that instant be  $v$ , then its kinetic energy is  $\frac{1}{2} mv^2$ . The potential energy of the particle at the same instant is the amount of work that

must be done in overcoming the force through a displacement  $y$  and is given by the relation  $\int_0^y F \cdot dy$  where  $F$  is the force required to maintain the displacement and  $dy$  is a small displacement.

Now the displacement is given by the relation  $y = a \sin(\omega t + \phi)$ .

Hence the acceleration  $\frac{d^2 y}{dt^2}$  is given by

$$\frac{d^2 y}{dt^2} = -a\omega^2 \sin(\omega t + \phi) = -\omega^2 \cdot y.$$

Then, force  $F = \text{mass} \times \text{acceleration}$

$$= m \cdot (-\omega^2 y) = -m\omega^2 \cdot y.$$

Then the potential energy of the particle is

$$\begin{aligned} \text{P.E.} &= \int_0^y F \cdot dy = \int_0^y m\omega^2 \cdot y \cdot dy \\ &= \frac{1}{2} m \cdot \omega^2 \cdot y \\ &= \frac{1}{2} m \cdot \omega^2 \cdot a^2 \sin^2(\omega t + \phi) \\ &= \frac{1}{2} k \cdot a^2 \sin^2(\omega t + \phi) \quad (\because \omega^2 = \frac{k}{m}) \end{aligned} \quad (1.14)$$

ignoring the minus sign in the expression for  $F$ , which simply shows that the direction of the force and displacement are opposite to each other.

Now the kinetic energy of the particle is given by

$$\begin{aligned} \text{K.E.} &= \frac{1}{2} mv^2 \\ &= \frac{1}{2} m \left( \frac{dy}{dt} \right)^2 \\ &= \frac{1}{2} m \cdot [a\omega \cos(\omega t + \phi)]^2 \\ &= \frac{1}{2} m \cdot a^2 \cdot \omega^2 \cos^2(\omega t + \phi) \end{aligned}$$



$$= \frac{1}{2} k \cdot a^2 \cos^2 (wt + \phi) \quad (1.15)$$

It can be seen from eqns. (1.14) and (1.15) that both the potential and kinetic energies have a maximum value of  $\frac{1}{2} ka^2$  or  $\frac{1}{2} m (wa)^2$ . During the motion the potential energy as well as the kinetic energy vary between zero to this maximum value.

The total energy which is just the sum of the kinetic and potential energy is

$$\begin{aligned} E = K + U &= \frac{1}{2} ka^2 \sin^2 (wt + \phi) + \frac{1}{2} ka^2 \cos^2 (wt + \phi) \\ &= \frac{1}{2} ka^2. \end{aligned} \quad (1.16)$$

We see that the total energy, as expected, is constant and has the value  $\frac{1}{2} ka^2$ . Thus, *the total energy of the system is the same as the maximum value of any one of the two forms of energy.* At the maximum displacement the kinetic energy is zero, but the potential energy has the value  $\frac{1}{2} ka^2$ . At the position of equilibrium the potential energy is zero but the kinetic energy has the value  $\frac{1}{2} ka^2$ . At any other position the kinetic and potential energies each contribute energy whose sum is always  $\frac{1}{2} ka^2$ .

Fig. 1.3 shows the kinetic, potential and total energies as a function of time while in Fig. 1.4, the same energies are plotted as a function of displacement from the equilibrium position. *The total energy of a particle executing simple harmonic motion is proportional to the square of the amplitude of the motion.*

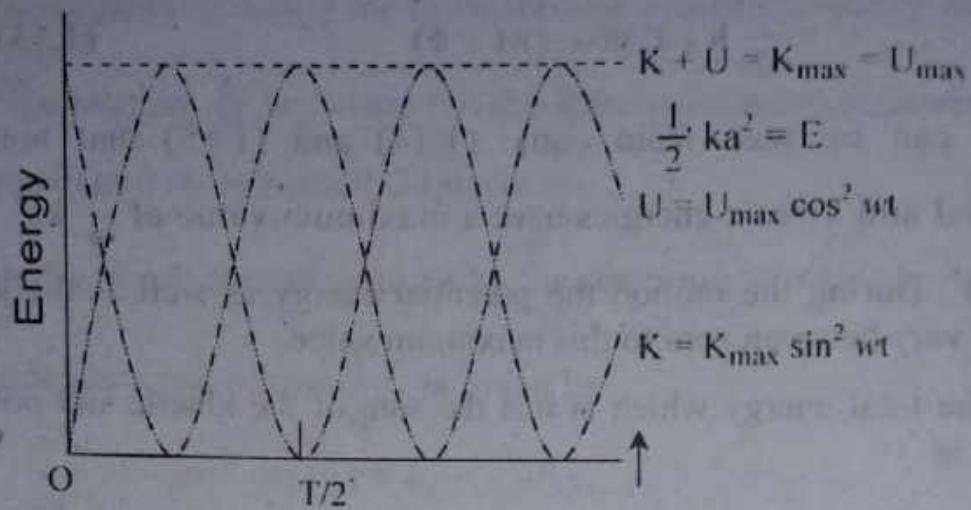


Fig. 1.3

Now

$$\begin{aligned} \frac{1}{2} ka^2 &= \frac{1}{2} m \omega^2 a^2 \\ &= \frac{1}{2} m \cdot \left( \frac{2\pi}{T} \right)^2 a^2 = \frac{2\pi^2 ma^2}{T^2} \end{aligned}$$

But, since  $\frac{1}{T} = n$ , the frequency of oscillation, the total energy of the system can also be expressed as

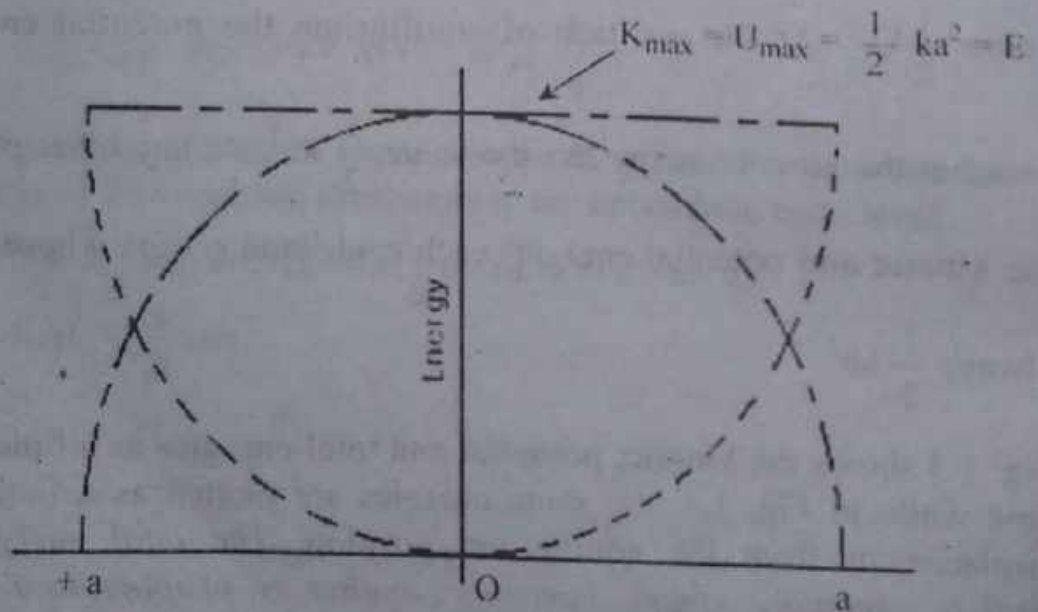


Fig. 1.4



$$E = \frac{1}{2} k a^2 = 2\pi^2 m \cdot a^2 \cdot n^2 \quad (1.17)$$

Thus, *total energy*

= maximum value of potential energy

= maximum value of kinetic energy

$$= \frac{1}{2} k a^2 = \frac{1}{2} m \omega^2 a^2 = 2\pi^2 n^2 a^2 m.$$

The total energy in Fig. 1.3 is represented by the upper horizontal line parallel to the time axis and touching the two curves at points representing the maximum values of kinetic and potential energies respectively. The upper horizontal line in Fig. 1.4 which passes through the two points of maximum potential energy corresponding to the two points of maximum displacement  $+a$  and  $-a$  on either side of the mean position represents the total energy. This line is also parallel to the displacement axis. Since the line representing total energy in either figure is a horizontal line, being parallel to either the time axis or the displacement axis, it follows that *total energy of the particle executing simple harmonic motion remains constant throughout and is independent of both time and displacement.*

### 1.6 Average value of kinetic and potential energies of a harmonic oscillator

The potential energy (P.E.) of the particle at a displacement  $y$  is given by

$$\begin{aligned} &= \frac{1}{2} m \omega^2 y^2 \\ &= \frac{1}{2} m \cdot \omega^2 a^2 \sin^2 (\omega t + \phi) \end{aligned}$$

So the average P.E. of the particle over a complete cycle or a whole time period  $T$

$$= \frac{1}{T} \int_0^T \frac{1}{2} m \omega^2 a^2 \sin^2 (\omega t + \phi) dt$$

$$\begin{aligned}
 &= \frac{1}{T} \cdot \frac{m \omega^2 a^2}{4} \int_0^T 2 \sin^2 (\omega t + \phi) dt \\
 &= \frac{m \omega^2 a^2}{4T} \int_0^T [1 - \cos 2 (\omega t + \phi)] dt \\
 &= \frac{m \omega^2 a^2}{4T} \left[ \int_0^T dt - \int_0^T \cos 2 (\omega t + \phi) dt \right]
 \end{aligned}$$

The average value of both a sine and a cosine function for a complete cycle or a whole time period  $T$  is zero. We, therefore, have

average P.E. of the particle

$$\begin{aligned}
 &= \frac{1}{4T} m \omega^2 a^2 \left[ t \right]_0^T - 0 \\
 &= \frac{1}{4T} m \omega^2 a^2 T \\
 &= \frac{1}{4} m \omega^2 a^2 \\
 &= \frac{1}{4} k a^2 \quad [ \because m \omega^2 = K ] \quad (1.18)
 \end{aligned}$$

The kinetic energy (K.E.) of the particle at displacement  $y$  is given by

$$\begin{aligned}
 &= \frac{1}{2} m \left( \frac{dy}{dt} \right)^2 \\
 &= \frac{1}{2} m \left[ \frac{d}{dt} a \sin (\omega t + \phi) \right]^2 \\
 &= \frac{1}{2} m \omega^2 a^2 \cos^2 (\omega t + \phi)
 \end{aligned}$$

The average K.E. of the particle over a complete cycle or a whole time period  $T$ , as in the case of P.E., is given by

$$\frac{1}{T} \int_0^T \frac{1}{2} m \omega^2 a^2 \cos^2 (\omega t + \phi) dt$$



$$\begin{aligned}
 &= \frac{m \omega^2 a^2}{4T} \int_0^T 2 \cos^2 (wt + \phi) dt \\
 &= \frac{m \omega^2 a^2}{4T} \int_0^T [1 + \cos 2 (wt + \phi)] dt \\
 &= \frac{m \omega^2 a^2}{4T} \left[ \int_0^T dt + \int_0^T \cos 2 (wt + \phi) dt \right]
 \end{aligned}$$

Again, the average value of a sine or cosine function over a complete cycle or a whole time period is zero. Hence

average K.E. of the particle

$$\begin{aligned}
 &= \frac{m \omega^2 a^2}{4T} [t]_0^T \\
 &= \frac{m \omega^2 a^2}{4T} \cdot T \\
 &= \frac{1}{4} m \omega^2 a^2 = \frac{1}{4} k a^2. \quad (1.19)
 \end{aligned}$$

Thus,

average value of P.E. of the particle

= average value of K.E. of the particle

$$= \frac{1}{4} m \omega^2 a^2 = \frac{1}{4} k a^2$$

= half the total energy.

**Example 1.11.** (i) What is the mechanical (total) energy of the oscillating system of Example 1.1?

**Soln.**

$$\text{Total energy} = \text{P.E.} + \text{K.E.} = \frac{1}{4} k a^2$$

Now  $k = 200 \text{ N/m}$  and  $a = 0.04 \text{ m}$  [Ex. 1.1 (i) and (iv)]

$$\therefore \text{Total energy} = \frac{1}{2} (200) (0.04)^2 = \mathbf{0.16 \text{ joules}}$$

Also total energy,

$$\frac{1}{4} k a^2 = 2\pi^2 m a^2 n^2$$

$$= 2 (3.14)^2 (2) (0.04)^2 (159)$$

$$= \mathbf{0.16 \text{ joules}}$$

From Ex. 1.1

$$a = 0.04 \text{ m}$$

$$m = 2 \text{ Kg}$$

$$n = 159 \text{ Hz}$$

(ii) Compute the velocity, the acceleration and the kinetic and potential energies of the body when it has moved in half-way from its initial position toward the centre of motion.

$$v = \pm w \sqrt{a^2 - y^2}$$

At half-way,

$$= \pm (10 \text{ sec}^{-1}) \sqrt{(0.04\text{m})^2 - (0.02\text{m})^2} \quad y = \frac{a}{2} = \frac{0.04}{2} \text{ m} = 0.02\text{m}$$

$$= \pm \frac{2\sqrt{3}}{10} \text{ m/sec}^{-1}$$

$$w = 10 \text{ sec}^{-1}$$

$$= \pm \mathbf{0.346 \text{ m sec}^{-1}}$$

$$\text{acceleration} = -w^2 y = - (10 \text{ sec}^{-1})^2 (0.02 \text{ m})$$

$$= \mathbf{-2.0 \text{ m sec}^{-2}}$$

$$\text{K. E.} = \frac{1}{2} m v^2 = \frac{1}{2} (2) (0.346)^2 \simeq \mathbf{0.12 \text{ joules}}$$

$$\text{P.E.} = \frac{1}{2} k y^2 = \frac{1}{2} (200) (0.02)^2 \simeq \mathbf{0.04 \text{ joules}}$$

$$\therefore \text{Total energy} = \text{P.E.} + \text{K.E.} = (0.04 + 0.12) \text{ joules}$$

$$= \mathbf{0.16 \text{ joules.}}$$

Note that total energy is constant.

*Example 1.12.* A block whose mass is 680 gm is at rest on a table top and is fastened to an anchored horizontal spring. The



spring constant of the spring is 65 N/m. There is negligible friction between the block and the table top. The block is pulled a distance  $x = 11$  cm from its equilibrium position at  $x = 0$  and released from rest at  $t = 0$ .

(i) What force does the spring exert on the block just before the block is released?

from Hooke's law

$$F = -ky = -(65 \text{ N/m})(0.11 \text{ m}) \\ = 7.2 \text{ N}$$

minus sign simply indicates that force and displacement are oppositely directed.

(ii) What are the angular frequency, the frequency, and the period of the resulting oscillation?

$$\text{angular frequency, } \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{65 \text{ N/m}}{0.68 \text{ Kg}}} = 9.78 \text{ rad/s.}$$

$$\text{frequency, } n = \frac{\omega}{2\pi} = \frac{9.78 \text{ N/m}}{2 \times 3.14} \approx 1.56 \text{ Hz}$$

$$\text{and the period, } T = \frac{1}{n} = \frac{1}{1.56 \text{ Hz}} = 0.64 \text{ s}$$

(iii) What is the amplitude of the oscillation?

since the block is released from rest 11 cm from its equilibrium position, its kinetic energy will be zero whenever it is again 11 cm from that position.

Hence its maximum displacement, i.e., amplitude is zero.

or,  $a = 11 \text{ cm} = 0.11 \text{ m}$ .

(iv) What are the maximum velocity and acceleration of the oscillating block?

$$v_{\max} = \omega a = (9.78 \text{ rad/s})(0.11 \text{ m}) = 1.1 \text{ m/s.}$$

$$a_{\max} = -\omega^2 a = (9.78 \text{ rad/s})^2 (0.11 \text{ m}) = -11 \text{ m/s}^2.$$

(v) What is the phase constant  $\phi$  for the motion?

At  $t = 0$  i.e., the moment of release, the displacement of the block has its maximum value. Hence the equation for the displacement is

$$y = a \cos (wt + \phi) \dots\dots\dots (1)$$

and the expression for velocity

$$\frac{dy}{dt} = -wa \sin (wt + \phi) \dots\dots\dots (2)$$

At  $t = 0$   $y = a$ . So from eqn. (1) we have

$$1 = \cos \phi \dots\dots\dots (3)$$

Again at  $t = 0$ ,  $\frac{dy}{dt} = 0$ . So from eqn (2) we get

$$0 = \sin \phi \dots\dots\dots (4)$$

The smallest angle that satisfies both these requirements is  $\phi = 0$ .

(Any angle that is an integral multiple of  $2\pi$  also satisfies the requirement)

(vi) What is the mechanical energy of the oscillating system?

$$\begin{aligned} E &= \frac{1}{2} ka^2 = \left(\frac{1}{2}\right) (65 \text{ N/m}) (0.11 \text{ m})^2 \\ &= 0.393 \text{ J.} \end{aligned}$$

(vii) What are the potential and kinetic energies of the oscillator when  $y = \frac{dy}{dt}$  ?

$$\begin{aligned} \text{P.E.} &= \frac{1}{2} ky^2 = \frac{1}{2} k \left(\frac{a}{2}\right)^2 = \frac{1}{4} \left(\frac{1}{2} ka^2\right) \\ &= \left(\frac{1}{4}\right) (0.393 \text{ J}) = 0.098 \text{ J.} \end{aligned}$$

Since the total energy remains constant,

$$\begin{aligned} \text{K. E.} &= E - \text{P. E.} = 0.393 \text{ J} - 0.098 \text{ J} \\ &\simeq 0.30 \text{ J.} \end{aligned}$$



Thus when the oscillator is half-way to its end point, 25% of the energy is potential and 75% is Kinetic.

**Example 1.13.** A 0.2 kg mass suspended from a spring describes a simple harmonic motion with a period  $T$  of 3 s and amplitude  $R$  of 10 cm. At  $t = 0$  the mass passed upward through the equilibrium position. (i) Find the force constant  $k$  of the spring (ii) Find the displacement, velocity and acceleration of the mass when  $t = 1$  s. (iii) Also show that at  $t = 1$ , the sum of the potential and kinetic energy is equal to  $\frac{1}{2} ka$ ?

**Soln.**

$$(i) \quad \omega = \sqrt{\frac{k}{m}} \quad \text{or, } \omega^2 = \frac{k}{m}$$

$$\therefore (2\pi n)^2 = \frac{k}{m} \quad ; \quad \left(\frac{2\pi}{T}\right)^2 = \frac{k}{m}$$

$$\frac{4\pi^2}{T^2} = \frac{k}{m} \quad ; \quad \text{or, } k = \frac{4\pi^2}{T^2} \times m = \frac{4\pi^2}{3^2} \times (0.2)$$

$$= 0.88 \text{ N/m.}$$

(ii) Let us consider the upward direction as the positive direction. Then from

$$y = a \sin \omega t, \quad \text{we have}$$

$$\text{at } t = 0, y = 0 \text{ and } v > 0.$$

At  $t = 1$  s,

$$y = 0.10 \sin \left[ \left( \frac{2\pi}{3} \right) (1) \right] \quad \omega = \frac{2\pi}{T}$$

$$= 0.10 \sin 120^\circ$$

$$= 0.0866 \text{ m.}$$

$$v = \frac{dy}{dt} = \omega a \cos \omega t$$

$$= \left( \frac{2\pi}{T} \right) (a) \cos \left[ \left( \frac{2\pi}{T} \right) (1) \right]$$

$$= \left( \frac{2\pi}{3} \right) (0.10) \cos 120^\circ$$

$$= -0.105 \text{ m/s.}$$

$$\text{acc in.} = \frac{d^2 y}{dt^2} = -\omega^2 a \sin \omega t$$

$$= -\left( \frac{2\pi}{T} \right)^2 (a) \sin \left[ \left( \frac{2\pi}{T} \right) (t) \right]$$

$$= -\left( \frac{2\pi}{3} \right)^2 (0.10) \sin 120^\circ$$

$$= -0.38 \text{ m/s}^2.$$

$$(iii) E = P. E. + K. E.$$

$$\frac{1}{2} k a^2 = \frac{1}{2} k y^2 + \frac{1}{2} m v^2$$

$$\left( \frac{1}{2} \right) (0.88) (0.1)^2 = \left( \frac{1}{2} \right) (0.88) (0.0866)^2 + \left( \frac{1}{2} \right) (0.2) (0.105)^2$$

$$4.4 \times 10^{-3} \text{ J} = (3.3 \times 10^{-3} + 1.1 \times 10^{-3}) \text{ J}$$

$$= 4.4 \times 10^{-3} \text{ J.}$$

## 1.7 Some examples of simple harmonic motion

Some important examples of simple harmonic motion will be examined below.

### (i) Motion of a body suspended from a coil spring

Fig. 1.5 shows a coil spring whose upper end is fixed to a rigid support. A mass  $m$  is attached to its free end. Let  $l$  be the no-load length of the spring as shown in (a). When the load  $m$  is attached to the spring, it hangs in equilibrium with the spring extended by an amount  $\Delta l$  as in (b). Under this condition the upward force  $F$  exerted by the spring is equal to the weight of the body,  $mg$ . If the spring obeys Hooke's law (Art. 8.6), then the force on the body is given by

$$F = -k \Delta l$$

where  $k$  is the force constant of the spring and is referred to as the *spring constant*. Again, the minus sign simply indicates that the force and displacement are oppositely directed. Since  $F = mg$ , ignoring the minus sign, we have

$$k\Delta l = mg ; \quad \text{or, } k = \frac{mg}{\Delta l} \quad (1.20)$$

Thus spring constant may be defined as the tension ( $mg$ ) per unit displacement ( $\Delta l$ ).

If the body is now displaced from its equilibrium position and released, it will oscillate along the vertical direction. Suppose the body is at a distance  $y$  above its equilibrium position as in (c). Then the extension of the spring is  $(\Delta l - y)$ ; the upward force it exerts on

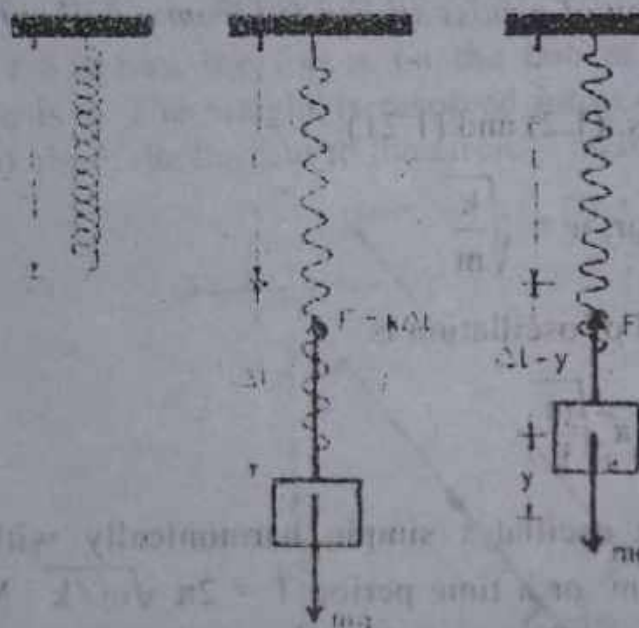


Fig. 1.5

the body is  $k(\Delta l - y)$  and the downward force acting on the body is  $mg$ . Hence the resultant force  $F$  on the body is

$$F = k(\Delta l - y) - mg = -ky$$

The resultant force is, therefore, proportional to the displacement of the body from its equilibrium position and is oppositely directed.

According to Newton's second law of motion,



$$F = m \frac{d^2 y}{dt^2} = -ky = 0$$

$$\text{or, } m \frac{d^2 y}{dt^2} + ky = 0$$

$$\text{or, } m \frac{d^2 y}{dt^2} + \left( \frac{k}{m} \right) \cdot y = 0 \quad (1.21)$$

The equation is similar to the differential equation of simple harmonic motion

$$\frac{d^2 y}{dt^2} + w^2 y = 0$$

Hence the motion of a mass suspended from a coil spring is simple harmonic.

Comparing eqns. (1.2) and (1.21)

$$w^2 = \frac{k}{m} ; \text{ or, } w = \sqrt{\frac{k}{m}}$$

The time period of oscillation is

$$T = \frac{2\pi}{w} = 2\pi \sqrt{\frac{m}{k}}$$

Thus the body oscillates simple harmonically with angular frequency  $w = \sqrt{k/m}$  or a time period  $T = 2\pi \sqrt{m/k}$ . Measuring the time period, the spring constant  $k$  can be determined. The constant  $k$  depends on the shear modulus ( $n$ ) of the wire, its radius ( $r$ ), the radius of the coil ( $R$ ) and the number of turns ( $N$ ) in the coil and

$$\text{is given by } n = \frac{4NR^3k}{r^4}$$

Alternately,  $k$  can also be determined from the relation  $k = mg/\Delta l$  where  $\Delta l$  is the increase in length of the spring when a small mass  $m$  is attached to its free end (eqn. 1.20). It is to be noted that  $mg/\Delta l$  is constant for a given spring provided it obeys Hooke's law.

### (ii) The simple pendulum

An important example of periodic motion is that of a simple pendulum. A simple pendulum consists of a point mass suspended by an inextensible weightless string in a uniform gravitational field. When pulled to one side of its equilibrium position and released, the bob of the pendulum vibrates about the position of equilibrium. We would like to investigate whether the motion is simple harmonic.

Let the mass of the bob be  $m$  and the length of the string be  $l$ . The path of the bob is not a straight line, but the arc of a circle of radius  $l$ ,  $l$  being the length of the supporting string. The co-ordinate  $s$  refers to distances measured along this arc. The necessary condition for the motion to be simple harmonic is that the restoring force  $F$  shall be directly proportional to the co-ordinate  $s$  and oppositely directed i.e.,

$$F = -ks \quad \text{where } k \text{ is the force constant.}$$

Fig. 1.6 shows the forces on the bob at an instant when its coordinate is  $s$ . The weight is resolved into components along the radius and along the tangent to the circle. The restoring force  $F$  is

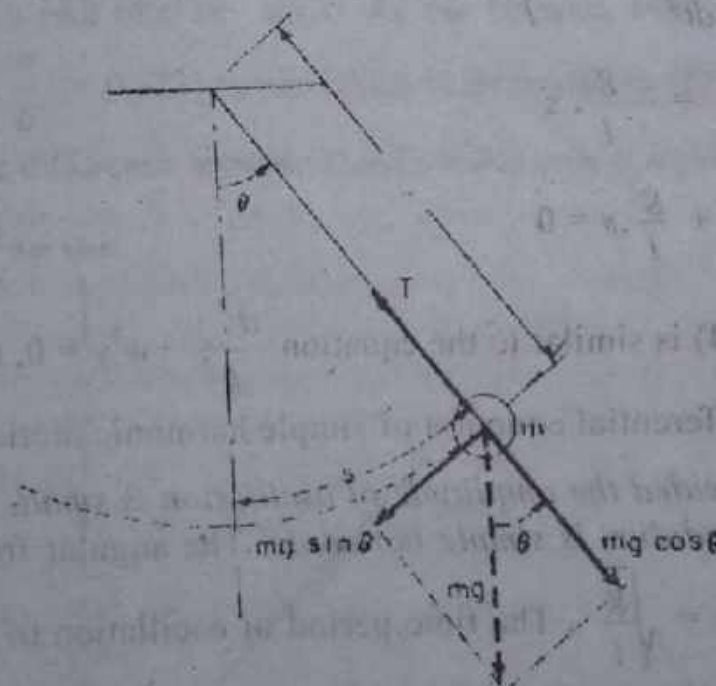


Fig. 1.6

$$F = -mg \sin \theta$$

(1.22)



The restoring force is, therefore, proportional not to  $\theta$  but to  $\sin \theta$ . Hence the motion is not simple harmonic. However, if the angle  $\theta$  is small,  $\sin \theta$  is very nearly equal to  $\theta$ . For example, when  $\theta = 0.1$  radian (about  $6^\circ$ ),  $\sin \theta = 0.0998$ , a difference of only 0.2 per cent. With this approximation eqn. (1.22) becomes

$$F = -mg \theta$$

$$\text{But } \theta = \frac{s}{l}$$

$$\therefore F = -mg \frac{s}{l} = -\frac{mg}{l} \cdot s \quad (1.23)$$

Thus, for small displacements, the restoring force is proportional to  $s$ , and the constant  $mg/l$  represents the force constant  $k$ .

Now  $F = m \cdot a$  where  $a$  is the tangential acceleration  $\frac{d^2s}{dt^2}$ . Eqn. (1.23) then becomes

$$m \cdot \frac{d^2s}{dt^2} = -\frac{mg}{l} \cdot s$$

$$\text{or, } \frac{d^2s}{dt^2} = -\frac{g}{l} \cdot s$$

$$\text{or, } \frac{d^2s}{dt^2} + \frac{g}{l} \cdot s = 0 \quad (1.24)$$

Eqn. (1.24) is similar to the equation  $\frac{d^2y}{dt^2} + w^2y = 0$ , the familiar form of the differential equation of simple harmonic motion.

Thus, provided the amplitude of oscillation is small, the motion of a simple pendulum is simple harmonic. The angular frequency of oscillation is  $w = \sqrt{\frac{g}{l}}$ . The time period of oscillation of the simple pendulum when its amplitude is small is, therefore,

$$T = \frac{2\pi}{w} = 2\pi \sqrt{\frac{l}{g}} \quad (1.25)$$

It may be noted that the above expressions do not contain the mass



of the particle; this is because *the restoring force which is a component of the particle's weight is proportional to  $m$* . Hence mass appears on both sides of  $F = ma$  and is cancelled out. For small oscillations the period of a pendulum for a given value of  $g$  is determined entirely by its length.

It should, however, be emphasized that the motion of pendulum is only *approximately* simple harmonic, and when the amplitude is not small, the departures from simple harmonic motion can be substantial. *What then is a small amplitude?*

It can be shown that the general equation for the time of swing, when the maximum angular displacement is  $\theta$ , is

$$T = 2\pi \sqrt{\frac{l}{g}} \left[ 1 + \frac{2}{2^2} \sin^2 \frac{\theta}{2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \sin^4 \frac{\theta}{2} + \dots \right]$$

Thus the period of oscillation may be computed to any desired degree of precision by taking enough terms in the infinite series. When  $\theta = 15^\circ$  on either side of the central position, the true period differs from the approximate period as given by eqn. (1.25), by less than 0.5 per cent. Fig. 1.7 shows a plot of  $\theta$  and  $\sin \theta$ . As can be seen, even for a rather large angle  $\theta = \frac{\pi}{6}$  ( $= 0.523$ ) radian which corresponds to  $30^\circ$ ,  $\sin \theta = \sin 30^\circ = 0.500$ ; the difference between  $\theta$  and  $\sin \theta$  is only 4.6 per cent.

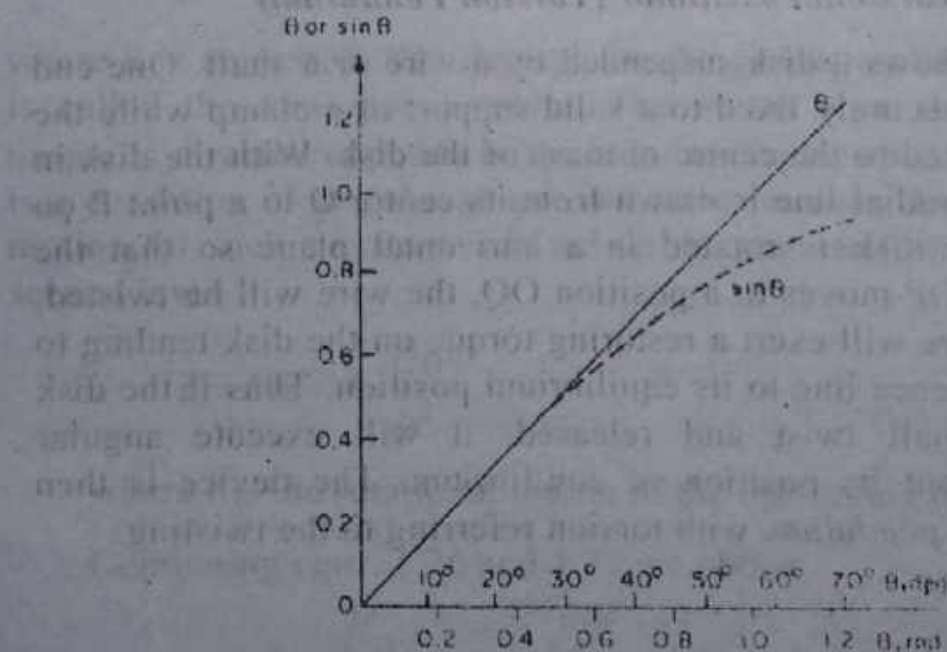


Fig. 1.7

The utility of the pendulum as a time keeper is based on the fact that the period is practically independent of the amplitude. Thus, as a clock runs down and the amplitude of the swings becomes slightly smaller, the clock will keep very nearly correct time. However, for accurate time keeping, the amplitude of the swing must be kept constant despite the frictional losses that affect all mechanical systems. Even so small a change in amplitude as from  $5^\circ$  to  $4^\circ$  would cause a pendulum clock to run fast by 0.25 minute per day, an unacceptable amount even for household time-keeping. To keep the amplitude constant in a pendulum clock, energy is automatically supplied in small increments from a weight or a spring by an escapement mechanism to compensate for friction losses. The pendulum clock with escapement was invented by Christiaan Huygens.

The simple pendulum was used for early determinations of the value of acceleration due to gravity because both the period and the length are easily measured. Direct measurement by observation of free fall, in contrast, is difficult because the time of fall over reasonable distance is too short for easy measurement. From eqn.

(1.25), we have, in terms of  $l$  and  $T$ ,  $g = \frac{4\pi^2 l}{T^2}$ .

### (iii) The Torsional oscillator (Torsion Pendulum)

Fig. 1.8 shows a disk suspended by a wire or a shaft. One end of the wire is securely fixed to a solid support or a clamp while the other end is fixed to the centre of mass of the disk. With the disk in equilibrium, a radial line is drawn from its centre  $O$  to a point  $P$  on its rim. If the disk is rotated in a horizontal plane so that the reference line  $OP$  moves to a position  $OQ$ , the wire will be twisted. The twisted wire will exert a restoring torque on the disk tending to return the reference line to its equilibrium position. Thus if the disk is given a small twist and released, it will execute angular oscillations about its position of equilibrium. The device is then called a *torsion pendulum*, with torsion referring to the twisting.



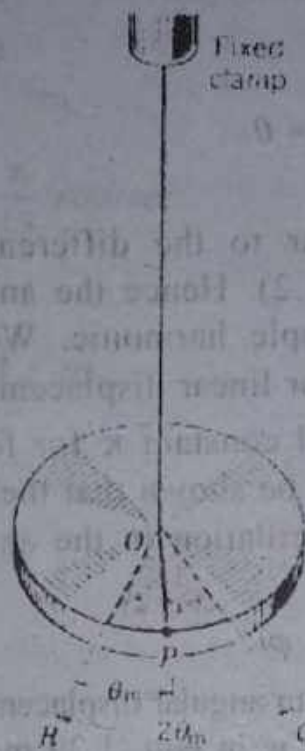


Fig. 1.8

For small twists the restoring torque is found to be proportional to the angular displacement (Hooke's law), so that

$$\tau = -k\theta \quad (1.26)$$

where  $k$  is a constant that depends on the properties of the wire and is called the *torsional constant*. The minus sign shows that the torque is directed opposite to the angular displacement  $\theta$ . Eqn. 1.26 is a condition of *angular simple harmonic motion*. The equation of motion for such a system is based on the angular form of Newton's second law,

$$\tau = I\alpha = I \frac{d^2\theta}{dt^2}$$

where  $I$  is the rotational inertia of the oscillating disk.

Combining eqns. 1.26 and 1.27 we obtain

$$K\theta = I \frac{d^2\theta}{dt^2}$$



**Soln.**

Here

$$y = 20 \sin (wt + \alpha)$$

$$T = 30 \text{ secs.}$$

$$\therefore w = \frac{2\pi}{T} = \frac{2\pi}{30} = \frac{\pi}{15} \text{ rad/sec.}$$

$$(i) \text{ at } t = 0, y = 10 \text{ cm.}$$

$$\therefore 10 = 20 \sin \left( \frac{\pi}{15} \times 0 + \alpha \right)$$

$$\text{or, } \sin \alpha = \frac{10}{20} = 0.5$$

$$\text{or, } \alpha = \frac{\pi}{6} \text{ radian.}$$

$$(ii) \text{ at } t = 5 \text{ sec,}$$

$$\text{the phase angle} = (wt + \alpha)$$

$$= \left( \frac{\pi}{15} \times 5 + \frac{\pi}{6} \right)$$

$$= \frac{\pi}{2}$$

$$(iii) \text{ at } t = 0 \text{ the phase angle}$$

$$\theta_1 = \left( \frac{\pi}{15} \times 0 + \frac{\pi}{6} \right) = \frac{\pi}{6} \text{ radian.}$$

$$\text{at } t = 15 \text{ sec, the phase angle}$$

$$\theta_2 = \left( \frac{\pi}{15} \times 15 + \frac{\pi}{6} \right)$$

$$= \frac{7\pi}{6} \text{ radian.}$$

$$\therefore \text{ the phase difference,}$$

$$\theta_2 - \theta_1 = \frac{7\pi}{6} - \frac{\pi}{6} = \pi \text{ radian.}$$

$$\text{or, } \frac{d^2\theta}{dt^2} = - \left( \frac{\kappa}{I} \right) \theta$$

$$\text{or, } \frac{d^2\theta}{dt^2} + \left( \frac{\kappa}{I} \right) \theta = 0 \quad (1.28)$$

Eqn. 1.28 is similar to the differential equation of simple harmonic motion (eqn. 1.2). Hence the angular oscillations of the torsion pendulum is simple harmonic. We can simply substitute angular displacement  $\theta$  for linear displacement  $y$ , rotational inertia  $I$  for mass  $m$ , and torsional constant  $\kappa$  for force constant  $k$ . Making these substitutions, it can be shown that the solution of eqn. 1.28 to be a simple harmonic oscillation in the angle coordinate  $\theta$  and is given by

$$\theta = \theta_m \sin (wt + \phi) \quad (1.29)$$

Here  $\theta_m$  is the maximum angular displacement, that is, the amplitude of the angular oscillation.  $w$  in eqn. 1.29 means angular frequency, not angular velocity. In eqn. 1.29  $w \neq d\theta/dt$ .

In Fig. 1.8 the disk oscillates about the equilibrium position  $\theta = 0$ , the total angular range being  $2\theta_m$  (from OQ to OR). By analogy with eqn. 1.12, the period of oscillation is given by

$$T = 2\pi \sqrt{\frac{I}{\kappa}}$$

If  $\kappa$  is known and  $T$  is measured, the rotational inertia  $I$  about the axis of rotation of any oscillating rigid body can be determined. If  $I$  is known and  $T$  is measured, the torsional constant  $\kappa$  of any sample of wire can be determined.

**Example 1.14.** A particle performs simple harmonic motion given by the equation.

$$y = 20 \sin (wt + \alpha)$$

If the time period is 30 seconds and the particle has a displacement of 10 cm at  $t = 0$ , find (i) epoch, (ii) the phase angle at  $t = 5$  seconds and (iii) the phase difference between two positions of the particle 15 seconds apart.

**Example. 1.15.** A vertically suspended spring of negligible mass and force constant  $R$  is stretched by an amount  $l$  when a body of mass  $m$  is hung on it. The body is pulled by hand an additional distance  $y$  (positive direction downward) and then released.

(a) Show that the motion of the body is governed by  $a' = -\frac{k}{m}y$ , so

that the body executes harmonic motion about its equilibrium position, and

(b) show that the period of this motion is the same as that of a simple pendulum of length  $l$ .

**Soln.**

When the mass  $m$  is attached to the spring it hangs in equilibrium with the spring extended by an amount  $l$ . From Hooke's law, the upward force exerted by the spring on the body is

$$F = kl \quad \text{where } k \text{ is the force constant of the spring.}$$

Under the condition of equilibrium we therefore have  $mg = kl$  where  $mg$  is the downward force on the spring.

When the spring is stretched by an additional length  $y$  and released, the net force acting on the spring is

$$mg - k(l + y)$$

According to Newton's second law of motion,

$$\begin{aligned} F = ma' &= mg - k(l + y) \\ &= mg - kl - ky = -ky \quad (mg = kl) \end{aligned}$$

$$\therefore a' = -\frac{k}{m}y.$$

(b) The period of oscillation of the spring is given by

$$\begin{aligned} T &= 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{mg/l}} \quad (\text{from } mg = kl) \\ &= 2\pi\sqrt{\frac{l}{g}} \end{aligned}$$



The above expression is the period of oscillation of a simple pendulum of length  $l$  placed in uniform gravitational field  $g$ .

**Example 1.16.** A thin rod of length  $L = 12.4$  cm and mass  $m = 135$  g is suspended at its midpoint from a long wire. Its period  $T_a$  of angular simple harmonic motion is measured to be 2.53 s. An irregular object  $X$  is then hung from the same wire and its period  $T_b$  of the angular SHM is found to be 4.76 s.

(i) What is the rotational inertia of object  $X$  about its suspension axis?

The rotational inertia of the thin rod about a perpendicular axis through its midpoint is given by  $\frac{1}{12} mL^2$ . Thus we have

$$I_a = \frac{1}{12} mL^2 = \left(\frac{1}{12}\right)(0.135 \text{ kg})(0.124 \text{ m})^2 \\ = 1.73 \times 10^{-4} \text{ kg.m}^2$$

Now

$$T_a = 2\pi\sqrt{\frac{I_a}{\kappa}} \quad \text{and} \quad T_b = 2\pi\sqrt{\frac{I_b}{\kappa}}$$

The torsion constant  $\kappa$ , which is a property of the wire, is the same for both the objects. Only the periods and rotational inertias are different.

$$\frac{T_b}{T_a} = \sqrt{\frac{I_b}{I_a}} \quad \text{or,} \quad \frac{T_b^2}{T_a^2} = \frac{I_b}{I_a}$$

$$\text{or, } I_b = \frac{T_b^2}{T_a^2} I_a = \frac{(4.76 \text{ s})^2}{(2.53 \text{ s})^2} (1.73 \times 10^{-4} \text{ kg.m}^2)$$

$$= 6.12 \times 10^{-4} \text{ kg.m}^2$$

(ii) If both the objects are fastened together and hung from the same wire, what would be the period of oscillation?

$$T_a = 2\pi\sqrt{\frac{I_a}{\kappa}} \quad \text{and} \quad T_c = 2\pi\sqrt{\frac{I_c}{\kappa}}$$

$$\text{or, } \frac{T_c}{T_a} = \sqrt{\frac{I_c}{I_a}}$$

$$\therefore T_c = T_a \sqrt{\frac{I_c}{I_a}} = T_a \sqrt{\frac{I_a + I_b}{I_a}}$$

$$\text{or, } T_c = T_a \sqrt{\frac{I_c}{I_a}}$$

$$= T_a \sqrt{1 + \frac{I_b}{I_a}} = (2.53) \sqrt{1 + \frac{6.12 \times 10^{-4} \text{ Kg.m}^2}{1.73 \times 10^{-4} \text{ Kg.m}^2}}$$

$$= 5.37 \text{ s.}$$

$$\text{Now } I_c = I_a + I_b$$

#### (iv) LC circuit

Just as harmonic oscillation is in mechanical system, so also we come across harmonic oscillations in electrical systems. Under suitable conditions, charge, current or voltage can execute simple harmonic motion. As an example let us consider the case of an LC circuit.

In an LC circuit, a capacitor of capacitance  $C$  and an inductance coil of inductance  $L$  (of negligible ohmic resistance) are connected with a battery through a Morse key  $K$  in the manner as shown in Fig. 1.9. When the key is pressed, the capacitor gets directly connected to the battery and thus gets charged. On being released, the key gets disconnected to the inductance coil. The capacitance thus discharges itself through the inductance coil.

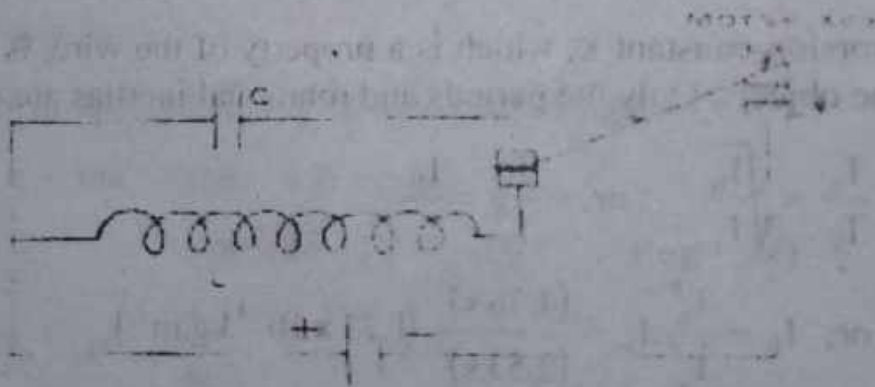


Fig. 1.9

An inductive circuit, i.e., a circuit containing an inductance coil, opposes both growth and decay of current in the circuit. Thus as the capacitor discharges itself through the inductance coil and the current in the latter grows, the magnetic flux due to the current increases. This increasing magnetic flux induces an electro-motive



force (*emf*) in the circuit which opposes the growth of current in it. Thus, if  $i$  be the instantaneous value of the current in the coil (or the circuit) at any given instant, then the opposing *emf* set up across the coil is  $-L \frac{di}{dt}$  where  $\frac{di}{dt}$  is the rate of change of current through the coil or the circuit.

If  $Q$  be the charge on the capacitor at the instant considered, then the voltage across it tending to drive the current through the circuit (or the coil) is  $\frac{Q}{C}$ . Since there is no external *emf* in the circuit (the battery being cut off), the net *emf* in the circuit is

$$\frac{Q}{C} + L \frac{di}{dt} = 0 \quad (1.27)$$

But current is the rate of flow of charge ;

hence  $i = \frac{dQ}{dt}$

Eqn. (1.27) thus takes the form

$$\begin{aligned} \frac{Q}{C} + L \frac{d^2 Q}{dt^2} &= 0 \\ \text{or, } \frac{d^2 Q}{dt^2} + \frac{Q}{LC} &= 0 \\ \text{or, } \frac{d^2 Q}{dt^2} + w^2 Q &= 0 \end{aligned} \quad (1.28)$$

where  $w^2 = \frac{1}{LC}$

or,  $\frac{1}{\sqrt{LC}}$  is a constant  $t$ .

Eqn. (1.28) is similar to eqn. (1.2), the differential equation of simple harmonic motion with  $y$  replaced by  $Q$ , mass ( $m$ ) by inductance  $L$  and the force constant by the reciprocal of the capacitance ( $\frac{1}{C}$ ).

Thus the charge on the capacitor oscillates simple harmonically with time, *i.e.*, the discharge of the capacitor is oscillatory in nature, its time period  $T$  is given by



$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{LC}$$

and, therefore, its frequency

$$n = \frac{1}{T} = \frac{1}{2\pi\sqrt{LC}} \quad (1.29)$$

The solution of eqn. (1.28) is given by

$$Q = Q_0 \sin(\omega t + \phi) \quad (1.30)$$

where  $Q_0$  is the maximum value or amplitude of the charge and  $\omega = \frac{1}{\sqrt{LC}}$ , the angular frequency (of the variation of charge) and  $\phi$ , the phase constant which depends, as usual, on the initial conditions.

The charge in the circuit, therefore, oscillates between  $+Q_0$  and

$$-Q_0 \text{ with a frequency } n = \frac{1}{2\pi\sqrt{LC}}$$

Differentiating eqn. (1.29) with respect to time, we have

$$i = \frac{dQ}{dt} = Q_0 \omega \cos(\omega t + \phi)$$

where  $i = \frac{dQ}{dt}$  is the instantaneous value of the current,  $Q_0\omega$  is the maximum value or the amplitude of current when

$$\cos(\omega t + \phi) = 1.$$

Denoting this maximum value by  $i_0$ , we get

$$i = i_0 \cos(\omega t + \phi) \quad (1.30)$$

Thus current in the circuit is also oscillatory in character and has the same frequency as the charge, viz.  $n = \frac{1}{2\pi\sqrt{LC}}$

### 1.8 Simple harmonic motion and uniform circular motion

Let us consider a particle  $P$  moving along the circumference of a circle with a constant angular speed  $\omega$  radians per seconds.  $Q$  is the perpendicular projection of  $P$  on the horizontal diameter i.e.,

along the x-axis while  $R$  is the perpendicular projection of  $P$  on the vertical diameter *i.e.*, along the y-axis. Let us consider the moment when  $P$  crosses the position  $X$  at time  $t = 0$  (Fig. 1.10). At this instant the displacement of  $P$  from the centre  $O$  measured along the horizontal direction is equal to radius  $a$  of the circle while that measured along the vertical direction is zero. When it arrives at  $P$  after, say, an interval of time  $t$ , its respective displacements along the x-and y-axes are  $a \cos wt$  and  $a \sin wt$ . On arrival at  $Y$ , the horizontal displacement vanishes but the vertical displacement attains its maximum value namely  $a$ . In the second quadrant of the

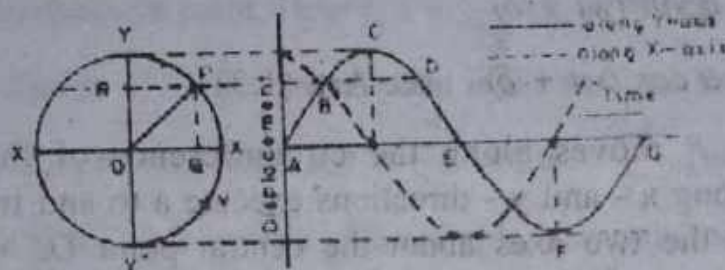


Fig. 1.10

circle, the vertical displacement gradually decreases, becoming zero as the particle arrives at  $X'$ . But, at the same time, the horizontal displacement increases attaining the maximum value  $a$  at  $X'$ . However, the maximum value  $a$  attained this time is in the negative x-direction. Thus as the particle completes one half-cycle from  $X$  to  $X'$ , the displacement measured along the vertical diameter (y-axis), starts from zero, rises to a maximum ( $= a$ ) and then falls back to zero. In the next half - cycle (from  $X'$  to  $X$ ) the vertical displacement again increases (though in the negative direction), becomes maximum ( $= -a$ ) when the particle reaches  $Y'$  and starts decreasing till it becomes zero when the particle arrives back at  $X$ . At each instant, the displacement in the y - direction as measured from the central point  $O$  is given by

$$y = a \sin wt \quad (1.31)$$

where  $t$  is measured from the instant when the particle crosses the point  $X$ . If  $y$  is plotted against  $t$ , the sine curve given in Fig. 1.10 is obtained.

Similarly, the displacement along the x-axis at each instant is given by

$$x = a \cos wt \quad (1.32)$$



The dotted line in Fig. (1.10) represents a plot of  $x$  vs.  $t$ . As can be seen, the cosine curve can be obtained by displacing the sine curve to the left by an interval  $T/4$  where  $T$  is the time required by the particle to complete one revolution along the circumference of the circle.

If the particle is already ahead of  $X$  when we start counting time, *i.e.*, if the radius  $OP$  has already made an angle  $\delta$  with the  $x$ -axis at  $t = 0$ , then the angle between  $OP$  and the  $x$ -axis at any subsequent time  $t$  is  $(\omega t + \delta)$ .

Eqns. (1.31) and (1.32) will then assume the forms

$$y = a \sin (\omega t + \delta) \quad (1.33)$$

$$\text{and } x = a \cos (\omega t + \delta) \quad \text{see Art. (1.3)} \quad (1.34)$$

Thus as  $P$  moves along the circumference of the circle, its projections along  $x$  - and  $y$  - directions execute a to and fro (vibratory) motion along the two axes about the central point  $O$ . Moreover, by differentiating eqn. (1.31) twice with respect to time we find that.

$$\frac{dy}{dt} = a\omega \cos (\omega t + \delta)$$

$$\text{and } \frac{d^2 y}{dt^2} = -a\omega^2 \sin (\omega t + \delta) = -\omega^2 y$$

$$\text{or, acceleration} = -\omega^2 y.$$

similarly the acceleration along the  $x$ -axis is given by  $-\omega^2 x$ .

Thus the acceleration of the projection whether along the  $x$  - or along the  $y$  - axis is directly proportional to the instantaneous displacement of the projection and is oppositely directed to this displacement. Thus motion of the projection either along the  $x$ - or along the  $y$ - axis satisfies the necessary condition of simple harmonic motion.

Thus the motion of the projection of uniform circular motion on a diameter of the circle in which the circular motion occurs is simple harmonic.

Conversely, the uniform motion of a particle along the circumference of a circle can be described as a combination of two simple harmonic motions. It is the combination of those simple harmonic motions which occur along perpendicular lines and have the same amplitude and



frequency but differ in phase by  $90^\circ$ . When one component is at the maximum displacement, the other component is at the equilibrium point. Combining eqns. (1.33) and (1.34) we at once obtain the relation

$$r = \sqrt{x^2 + y^2} = a$$

The point  $P$  is called the reference point and the circle on which it moves is called the reference circle. The angular frequency  $\omega$  of simple harmonic motion is the same as the angular speed of the reference point. The frequency of simple harmonic motion is the number of revolutions per unit time of the reference point. Hence,  $n = \frac{\omega}{2\pi}$  or,  $\omega = 2\pi n$ . The time for

a complete revolution of the reference point is the same as the period  $T$  of the simple harmonic motion. Hence,  $T = \frac{2\pi}{\omega}$  or,  $\omega = \frac{2\pi}{T}$ .  $(\omega t + \delta)$  in eqn.

(1.33) or (1.34) is called the phase of the simple harmonic motion while  $\delta$  is called the epoch or initial phase of the motion. The amplitude of the simple harmonic motion is the same as the radius of the reference circle.

### Two-body oscillations

Many objects on the microscopic level, such as molecules, atoms, nuclei, execute oscillations that are approximately simple harmonic. One example is a diatomic molecule, in which the two atoms are bound together with a force. If displaced a small distance from its equilibrium position, the molecule will oscillate about the equilibrium position. Let us suppose that the molecule can be represented by two particles of masses  $m_1$  and  $m_2$  connected by a spring of force constant  $k$ , as shown in Fig. 1.11.

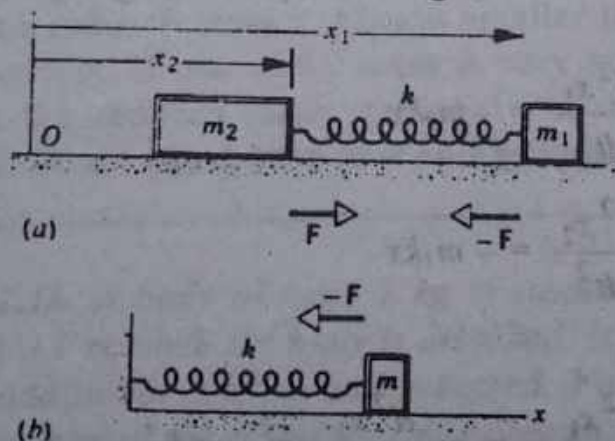


Fig. 1.11

The motion of the system can be described in terms of the separate motions of the two particles, which are located relative to the origin  $O$  by the two coordinates  $x_1$  and  $x_2$ , as shown in Fig. 1.11 a. As we shall see, this leads to a different and often more useful description, which is given in terms of the *relative separation* and velocity of the two particles. In effect, the two coordinates  $x_1$  and  $x_2$  are replaced with two other coordinates: the relative separation  $x_1 - x_2$  and the location  $x_{cm}$  of the centre of mass. But, in the absence of external forces, the centre of mass moves at constant velocity and as such, its motion is of no real interest in studying the oscillation of the system. So the system can be analyzed in terms of the relative coordinate alone.

The relative separation  $x_1 - x_2$  gives the length of the spring at any time. If the unstretched length of the spring is  $L$ , then the change in length of the spring is given by  $x = (x_1 - x_2) - L$ . The magnitude of the force that the spring exerts on *each particle* is  $F = Kx$ . As can be seen in Fig. 1.11 a, if the spring exerts a force  $-F$  on  $m_1$ , then it exerts a force  $+F$  on  $m_2$ .

Taking the force component along the  $x$  axis, let us apply Newton's second law separately to the two particles,

$$m_1 \frac{d^2 x_1}{dt^2} = -kx$$

$$m_2 \frac{d^2 x_2}{dt^2} = +kx$$

Multiplying the first of these equations by  $m_2$  and the second by  $m_1$ , we get

$$m_1 m_2 \frac{d^2 x_1}{dt^2} = -m_2 kx$$

$$m_1 m_2 \frac{d^2 x_2}{dt^2} = +m_1 kx$$

Subtracting,

$$m_1 m_2 \frac{d^2 x_1}{dt^2} - m_1 m_2 \frac{d^2 x_2}{dt^2} = -m_2 kx - m_1 kx$$



$$= -kx (m_1 - m_2)$$

$$\text{or, } \frac{m_1 m_2}{m_1 + m_2} \frac{d^2}{dt^2} (x_1 - x_2) = -kx \quad (1.35)$$

The quantity  $m_1 m_2 / (m_1 + m_2)$  has the dimension of mass and is known as the *reduced mass*  $m$  of the system.

The unstretched length  $L$  of the spring is a constant. Hence the derivatives of  $(x_1 - x_2)$  are the same as the derivatives of  $x$ .

$$\frac{d}{dt} (x_1 - x_2) = \frac{d}{dt} (x + L) = \frac{dx}{dt}$$

Eqn. 1.35 therefore becomes

$$\frac{d^2 x}{dt^2} = -\frac{k}{m} x$$

$$\text{or, } \frac{d^2 x}{dt^2} + \frac{k}{m} x = 0 \quad (1.36)$$

Eqn. 1.36 is identical to the differential equation of simple harmonic motion of a single oscillating mass. Hence, from the standpoint of oscillation, the system of Fig. 1.11 a can be replaced by a single particle, as represented by Fig. 1.11, whose mass is given by the reduced mass of the system. In particular, the frequency of oscillation of the system is given by

$$n = \frac{1}{T} = \frac{1}{2\pi} \sqrt{\frac{k}{m}}$$

where  $m$  in the expression is the reduced mass of the system. It may be noted that the reduced mass is always smaller than either of the masses of the system. If one of the mass is very much smaller than the other, then the reduced mass is roughly equal to the smaller mass. If the masses are equal, then  $m$  is half as large as either mass.

---

**Example 1.16.** A body of mass 5 kg is suspended by a spring which stretches 0.1 m when the body is attached. It is then displaced downward an additional 0.05 m and released. Find the amplitude, period and frequency of the resulting simple harmonic motion.



**Soln.**

Since the initial position is 0.05 m from equilibrium and there is no initial velocity, the amplitude.

$$a = 0.05 \text{ m.}$$

A force of (5 kg) (9.8 m/sec<sup>2</sup>) produces a displacement of 0.1 m. Hence from  $F = ky$ ,

$$\text{the force constant } k = \frac{F}{y} = \frac{mg}{y}$$

$$= \frac{(5 \text{ kg})(9.8 \text{ m/sec}^2)}{0.1 \text{ m}}$$

$$= 490 \text{ N/m.}$$

the time period,

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{5 \text{ kg}}{490 \text{ N/m}}}$$

$$= 0.635 \text{ sec.}$$

the frequency

$$n = \frac{1}{T} = \frac{1}{0.635} = 1.57 \text{ Hz}$$

**Example 1.17.** The scale of a spring balance reading from 0 to 10 kg is 0.25 m long. A body suspended from the balance is found to oscillate vertically with a frequency of  $10/\pi$  hertz. Calculate the mass of the body attached to the spring.

**Soln.**

Clearly, a mass of 10 kg suspended from the spring balance extends the spring by 0.25 m. Hence from  $F = ky$ , the force constant

$$k = \frac{F}{y} = \frac{10 \times 9.8}{0.25} \text{ N/m.}$$

From

$$T = 2\pi\sqrt{\frac{m}{k}} \text{ we have}$$

$$T = \frac{1}{n} = 2\pi \sqrt{\frac{m}{k}}$$

$$\text{or, } \frac{1}{n^2} = 4\pi^2 \cdot \frac{m}{k}$$

$$\text{or, } m = \frac{k}{4\pi^2 n^2}$$

$$k = \frac{10 \times 9.8}{0.25} \text{ N/m.}$$

$$n = \frac{10}{\pi} \text{ Hz.}$$

$$m = ?$$

$$\begin{aligned} \therefore m &= \frac{10 \times 9.8}{0.25 \times 4 \times (3.14)^2 \times \frac{100}{(3.14)^2}} \\ &= \frac{98}{100} \\ &= 0.98 \text{ kg.} \end{aligned}$$

**Example 1.18.** A body of mass 0.5 kg is suspended from a spring of negligible mass and it stretches the spring by 0.07 m. For a displacement of 0.03 m it has a downward velocity of 0.4 m/sec. Calculate (i) the time period, (ii) the frequency and (iii) the amplitude of vibration of the spring.

**Soln.**

From  $F = ky$ , the force constant of the spring

$$k = \frac{F}{y} = \frac{0.5 \times 9.8}{0.07}$$

(i) the time period,

$$T = 2\pi \sqrt{\frac{m}{k}}$$

$$m = 0.5 \text{ kg}$$

$$k = \frac{0.5 \times 9.8}{0.07}$$

$$\therefore T = 2 \times 3.14 \times \sqrt{\frac{0.5 \times 0.07}{0.5 \times 9.8}} = 0.5307 \text{ sec.}$$

(ii) the frequency,

$$n = \frac{1}{T} = \frac{1}{0.5309} = 1.8843 \text{ Hz.}$$

the angular frequency,  $w = 2\pi n$

$$= 2 \times 3.14 \times 1.8843$$

$$= 11.833 \text{ rad/sec.}$$

(iii) For the amplitude, we have

$$v = w \sqrt{a^2 - y^2}$$

$$\text{or, } 0.4 = 11.383 \sqrt{a^2 - (0.03)^2}$$

$$\text{or, } (0.4)^2 = (11.383)^2 [a^2 - (0.03)^2]$$

$$\text{or, } a = 0.04526 \text{ m.}$$

**Example 1.19.** A simple pendulum of length 100 cm has an energy equal to  $2 \times 10^6$  ergs when its amplitude is 4 cm. Calculate its energy when (i) its length is doubled and (ii) its amplitude is doubled.

**Soln.**

The energy of a simple pendulum

= its maximum P.E.

= its maximum K.E.

$$= \frac{1}{2} m w^2 a^2$$

Now  $w = \frac{2\pi}{T}$  where  $T$  is the time period of the pendulum  $= 2\pi \sqrt{\frac{l}{g}}$

$$\text{or, } w = 2\pi \cdot \frac{1}{2\pi} \sqrt{\frac{g}{l}} = \sqrt{\frac{g}{l}}$$

Initially  $l = 100 \text{ cm.} \therefore w = \sqrt{g/100}.$

So, initially with  $a = 4 \text{ cm}$ , the energy of the pendulum

$$E = \frac{1}{2} m \left( \sqrt{g/100} \right)^2 \cdot (4)^2 = 8mg/100$$

$$= 2 \times 10^6 \text{ ergs.}$$

(i) If the length of the pendulum is doubled,  $l = 200 \text{ cm.}$  and  $w = \sqrt{g/200}.$  Hence energy of the pendulum



$$E' = \frac{1}{2} \cdot m \cdot (\sqrt{g/200})^2 \cdot (4)^2$$

$$= 8mg/200$$

$$\therefore \frac{E'}{E} = \frac{8mg}{200} / \frac{8mg}{100} = \frac{1}{2}$$

$$\text{or, } E' = \frac{1}{2} \cdot E = \frac{1}{2} \times 2 \times 10^6 \text{ ergs}$$

$$= 10^6 \text{ ergs.}$$

(ii) If the amplitude is doubled,  $a = 8$  cm, its length remaining 100 cm. Hence energy of the pendulum

$$E'' = \frac{1}{2} \cdot m \cdot (\sqrt{g/100})^2 \cdot (8)^2$$

$$= 32mg / 100$$

$$\therefore \frac{E''}{E} = \frac{32mg}{100} / \frac{8mg}{100} = 4$$

$$\text{or, } E'' = 4.E = 4 \times 2 \times 10^6 \text{ ergs}$$

$$= 8 \times 10^6 \text{ ergs.}$$

**Example 1.20.** A small body of mass 0.10 kg is undergoing simple harmonic motion of amplitude 1.0 metre and period 0.20 sec. (i) what is the maximum value of the force acting on it? (ii) if the oscillations are produced by a spring, what is the force constant of the spring?

**Soln.**

(ii) From  $T = 2\pi \sqrt{\frac{m}{k}}$ , we have

$$T^2 = 4\pi^2 \cdot \frac{m}{k}$$

$$T = 0.20 \text{ sec}$$

$$m = 0.10 \text{ kg.}$$

$$\therefore (0.2)^2 = 4 \times (3.14)^2 \times \frac{0.10}{k}$$

$$\text{or, } k = \frac{4 \times 9.86 \times 0.10}{0.04}$$

$$= \frac{3.944}{0.04} \simeq 99 \text{ nt/m.}$$

$\therefore$  force constant of the spring

$$\simeq 99 \text{ nt/m.}$$

(i) force acting on the body is given by

$$F = ky.$$

Obviously  $F$  is maximum when  $y$  is maximum. The maximum value of  $y$  is its amplitude *i.e.*, 1.0 metre.

$$\therefore \text{maximum force} = k.y = 99 \times 1$$

$$= 99 \text{ nt.}$$

**Example 1.21.** An oscillating mass-spring system has a mechanical (total) energy of 1.0 joule, amplitude of 0.10 metre and maximum speed of 1.0 m/sec. Find (i) the force constant of the spring, (ii) the mass, and (iii) the frequency of oscillation.

$$\text{(iii) } v_{\max} = w.a$$

$$\text{or, } 1.0 = w \times 0.1$$

$$\text{or, } w = \frac{1.0}{0.1} = 10$$

Now  $w = 2\pi n$  where  $n$  is the frequency of oscillation

$$\therefore 10 = 2 \times 3.14 \times n$$

$$\text{or, } n = \frac{10}{6.28}$$

$$= 1.6 \text{ cycle/sec.}$$

(ii) Mechanical (or total energy),

$$2\pi^2 n^2 a^2 m = 1.0 \text{ joule.}$$

$$2 \times 9.8 \times (1.6)^2 \times (0.1)^2 \times m \\ = 1.0 \text{ joule.}$$

$$\text{or, } m = \frac{1.0}{0.5} \text{ kg} = 2 \text{ kg.}$$

$$(i) \quad T = \frac{1}{n} = \frac{1}{1.6} = 0.625 \text{ sec.}$$

$$\text{But } T = 2\pi \sqrt{\frac{m}{k}}$$

$$\text{or, } T^2 = 4\pi^2 \frac{m}{k}$$

$$\text{or, } (0.625)^2 = \frac{4 \times 9.8 \times 2.0}{k}$$

$$\text{or, } k = \frac{78.4}{0.30} \approx 200 \text{ nt/m.}$$

### EXERCISES

- [1] Define simple harmonic motion and discuss its characteristics.
- [2] Establish the differential equation of simple harmonic motion and solve it to obtain an expression for the displacement of a particle executing simple harmonic motion.
- [3] Show that for a body vibrating simple harmonically the time period is given by

$$T = 2\pi \sqrt{\frac{\text{displacement}}{\text{acceleration}}}$$

- [4] Establish the differential equation of simple harmonic motion and show that the time period is equal to  $2\pi$  times the square root of displacement per unit acceleration.
- [5] Define simple harmonic motion. Prove that the motion of a simple pendulum is simple harmonic. Hence obtain an expression for the time period of a simple pendulum.
- [6] Solve the differential equation  $\frac{d^2y}{dt^2} + w^2y = 0$  to obtain the expression  $y = a \sin (wt + \phi)$  for the displacement of a particle executing simple harmonic motion.



- [7] Explain simple harmonic motion. Obtain expressions for the frequency, amplitude, velocity and acceleration of a body executing simple harmonic motion.
- [8] Show that the motion of a body suspended from a coil spring is simple harmonic.
- [9] Show that a simple harmonic motion may be expressed as either a sine or a cosine function, there being only a difference of initial phase in the two cases.
- [10] Show that for a body executing simple harmonic motion, mechanical energy remains conserved and that its energy is, on an average, half kinetic and half potential in form. At what particular displacement is this exactly so? What is the ratio between its kinetic and potential energies at a displacement equal to half its amplitude.
- [11] Calculate the average kinetic energy and the total energy of a body executing simple harmonic motion. Show that the principle of conservation of energy is obeyed by a harmonic oscillator.
- [12] Show that the motion of a simple pendulum is simple harmonic. Obtain an expression for the frequency of oscillations of the simple pendulum. Is the motion of a simple pendulum really simple harmonic?
- [13] Suppose we have a block of unknown mass and a spring of unknown force constant. Show how we can predict the period of oscillation of this block-spring system simply by measuring the extension of the spring produced by attaching the block to it.
- [14] What is a torsion pendulum? Show that for small angular displacement the oscillation of a torsion pendulum is simple harmonic. Obtain an expression for the period of oscillation.
- [15] Show that the motion of the projection of uniform circular motion on a diameter of the circle in which the circular motion occurs is simple harmonic.
- [16] Show that the uniform motion of a particle along the circumference of a circle can be described as a combination of two simple harmonic motions which occur along perpendicular lines and have the same amplitude and frequency but differ in phase by  $90^\circ$ .
- [17] Show that the charge on the capacitor in an LC circuit oscillates simple harmonically and hence obtain an expression for the frequency of oscillations.

- [18] A mass suspended on a vertical spring oscillates with a period of 0.5 sec. When the mass is attached to the spring and allowed to hang at rest, by how much is the spring stretched? [6.205 cm].
- [19] A body is vibrating with simple harmonic motion of amplitude 15 cm and frequency 4.0 cycles per second. Calculate (i) the maximum values of its velocity and acceleration, (ii) the acceleration and speed when the body is 9.0 cm from the equilibrium position, and (iii) the time required for the body to move from the equilibrium position to a point 12 cm away. [(i) 3.8 m/sec; 94 m/sec<sup>2</sup>. (ii) 57 m/sec<sup>2</sup>; 3.0 m/sec. (iii)  $3.7 \times 10^{-2}$  sec.]
- [20] The general equation of simple harmonic motion,  $y = A \sin (\omega t + \theta_0)$  can be written in the equivalent form  $y = B \sin \omega t + C \cos \omega t$ . Find the expressions for the amplitude B and C in terms of the amplitude A and the initial phase angle  $\theta_0$ . [ $B = A \cos \theta_0$  and  $C = A \sin \theta_0$ ]
- [21] A particle executes simple harmonic motion given by the equation
- $$y = 12 \sin \left( \frac{2\pi t}{10} + \frac{\pi}{4} \right)$$
- Calculate (i) amplitude (ii) frequency (iii) epoch (iv) displacement at  $t = 1.25$  sec (v) velocity at  $t = 2.5$  sec and (vi) acceleration at  $t = 5$  sec. [(i) 12 units (ii) 0.1 Hz (iii)  $\frac{\pi}{4}$  (iv) 12 units (v) -5.552 units (vi) 3.35 units]
- [22] (a) What is the frequency of a simple pendulum 2.0 metre long? (b) Assuming small amplitudes, what would its frequency be in an elevator accelerating upward at a rate of 2.0 m/sec<sup>2</sup>? (c) What would its frequency be in free fall? [(a) 0.35 cycles/sec (b) 0.39 cycles per sec (c) zero]
- [23] (a) When the displacement is one-half the amplitude A, what fraction of the total energy is kinetic and what fraction is potential in simple harmonic motion? (b) At what displacement is the energy half kinetic and half potential?

$$[(a) \frac{3}{4}, \frac{3}{4} (b) A\sqrt{2}]$$



## CHAPTER II

### COMPOSITION OF SIMPLE HARMONIC MOTIONS

*Introduction - Composition of two simple harmonic vibrations of same frequency but different phase and amplitude - Composition of two simple harmonic vibrations at right angles to each other having equal frequencies but differing in phases and amplitudes - Lissajous' figures - Composition of two simple harmonic motion at right angles to each other and having time periods in the ratio 1:2 - Experimental determination of Lissajous' figures - Uses of Lissajous' figures - Solved problems - Exercises.*

#### 2.1 Introduction

Very often problems are encountered in physics where a particle is simultaneously acted upon by more than one simple harmonic vibrations acting either along the same straight line or at right angles to each other. The resultant displacement, velocity, acceleration, etc., of the particle is given by the vector (algebraic) sum of the corresponding quantities due to the individual waves. There are two general methods for solving such problems based on (i) a graphical treatment and (ii) analytical treatment of the dynamics of the particle. The analytical treatment is based on finding the vector sum of the individual motions either with the help of trigonometric functions or writing them as complex quantities. This is usually the easiest to handle and is at the same time most informative. This method of treatment will be followed in the ensuing articles.

#### 2.2 Composition of two simple harmonic vibrations of same frequency but different phase and amplitude

Let a particle in a medium be simultaneously acted upon by two simple harmonic vibrations of same frequency but different phase and amplitude given by the following equations.

$$y_1 = a_1 \sin (wt + \alpha_1) \quad (2.1)$$

$$y_2 = a_2 \sin (wt + \alpha_2) \quad (2.1)$$

where  $y_1$  and  $y_2$  are the displacements of the particles due to the individual vibrations of amplitudes  $a_1$  and  $a_2$  and angles of epochs  $\alpha_1$  and  $\alpha_2$  respectively. The two vibrations have the same angular frequency  $\omega$ . The resultant displacement  $y$  of the particle will be given by the vector sum of the individual displacements so that

$$\begin{aligned}
 y &= y_1 + y_2 \\
 &= a_1 \sin(\omega t + \alpha_1) + a_2 \sin(\omega t + \alpha_2) \\
 &= a_1 (\sin \omega t \cos \alpha_1 + \cos \omega t \sin \alpha_1) \\
 &\quad + a_2 (\sin \omega t \cos \alpha_2 + \cos \omega t \sin \alpha_2) \\
 &= (a_1 \cos \alpha_1 + a_2 \cos \alpha_2) \sin \omega t \\
 &\quad + (a_1 \sin \alpha_1 + a_2 \sin \alpha_2) \cos \omega t \quad (2.3)
 \end{aligned}$$

The amplitudes  $a_1$  and  $a_2$  and the angles of epoch  $\alpha_1$  and  $\alpha_2$  of the two vibrations are constant.

Hence putting

$$a_1 \cos \alpha_1 + a_2 \cos \alpha_2 = A \cos \phi$$

$$\text{and } a_1 \sin \alpha_1 + a_2 \sin \alpha_2 = A \sin \phi$$

the resultant amplitude can be written as

$$\begin{aligned}
 y &= A \cos \phi \sin \omega t + A \sin \phi \cos \omega t \\
 &= A \sin(\omega t + \phi) \quad (2.4)
 \end{aligned}$$

Thus the equation of the resultant vibration as given by eqn. (2.4) is simple harmonic and is very much similar to either eqn. (2.1) or (2.2). The amplitude of the resultant vibration is  $A$  while the epoch angle is  $\phi$ , the time period of the resultant vibration remaining same as the original vibrations. The values of  $A$  and  $\phi$  in eqn. (2.4) can be determined as follows :

$$\begin{aligned}
 A^2 \sin^2 \phi + A^2 \cos^2 \phi &= a_1^2 \sin^2 \alpha_1 + a_2^2 \sin^2 \alpha_2 + 2a_1 a_2 \sin \alpha_1 \sin \alpha_2 \\
 &\quad + a_1^2 \cos^2 \alpha_1 + a_2^2 \cos^2 \alpha_2 + 2a_1 a_2 \cos \alpha_1 \cos \alpha_2 \\
 &= a_1^2 (\sin^2 \alpha_1 + \cos^2 \alpha_1) + a_2^2 (\sin^2 \alpha_2 + \cos^2 \alpha_2) + 2a_1 a_2 (\sin \alpha_1 \sin \alpha_2 + \cos \alpha_1 \cos \alpha_2) \\
 \text{or, } A^2 &= a_1^2 + a_2^2 + 2a_1 a_2 \cos(\alpha_1 - \alpha_2) \quad (2.5)
 \end{aligned}$$

and



$$\begin{aligned}\tan \phi &= \frac{A \sin \phi}{A \cos \phi} \\ &= \frac{a_1 \sin \alpha_1 + a_2 \sin \alpha_2}{a_1 \cos \alpha_1 + a_2 \cos \alpha_2}\end{aligned}$$

### special cases :

(i) *same phase* : if the two simple harmonic vibrations are in the same phase, then  $\alpha_1 = \alpha_2 = \alpha$  (say). Thus if the two vibrations acting on the particle are in the same phase or if the phase difference

$(\alpha_1 - \alpha_2) = 0, 2\pi, 4\pi, \dots = 2n\pi$  where  $n = 0, 1, 2, \dots$ , then we get from eqn. (2.5)

$$\cos(\alpha_1 - \alpha_2) = 1 \text{ and}$$

$$A^2 = a_1^2 + a_2^2 + 2a_1a_2$$

$$= (a_1 + a_2)^2$$

$$\text{or, } A = a_1 + a_2$$

$$\text{and } \tan \phi = \frac{(a_1 + a_2) \sin \alpha}{(a_1 + a_2) \cos \alpha} = \tan \alpha.$$

In that case eqn. (2.4) can be rewritten as

$$y = (a_1 + a_2) \sin(\omega t + \alpha).$$

(ii) *opposite phase* : If the two vibrations acting on the particle are in opposite phase i.e., if the phase difference  $(\alpha_1 - \alpha_2) = \pi, 3\pi, 5\pi, \dots = (2n + 1)\pi$  where  $n = 0, 1, 2, \dots$ , we have  $\cos(\alpha_1 - \alpha_2) = -1$  and

$$A^2 = a_1^2 + a_2^2 - 2a_1a_2$$

$$= (a_1 - a_2)^2$$

$$\text{or, } A = a_1 - a_2.$$

If, in addition, the amplitudes of the individual vibrations are equal, i.e.,

$$a_1 = a_2 = a \text{ (say), then}$$

for the *same phase* condition

$$A = 2a, \text{ and } A^2 = 4a^2$$

while in case of *opposite phase*

$$A = 0 \text{ (i.e., the resultant vibration is zero).}$$

**Example 2.1.** Two simple harmonic motions acting simultaneously on a particle are given by the equations

$$y_1 = 2 \sin (wt + \pi/6)$$

$$y_2 = 3 \sin (wt + \pi/3)$$

Calculate (i) amplitude, (ii) phase constant and (iii) time period of the resultant vibration.

What is the equation of the resultant vibration ?

**Soln :**

$$y_1 = 2 \sin (wt + \pi/6) \quad \text{(i)}$$

$$y_2 = 3 \sin (wt + \pi/3) \quad \text{(ii)}$$

The equations are similar to the equations

$$y_1 = a_1 \sin (wt + \alpha_1) \quad \text{(iii)}$$

$$y_2 = a_2 \sin (wt + \alpha_2) \quad \text{(iv)}$$

The equation of the resultant vibration is given by

$$y = A \sin (wt + \phi) \quad \text{(v)}$$

$$\text{where } A^2 = a_1^2 + a_2^2 + 2a_1a_2 \cos (\alpha_1 - \alpha_2) \quad \text{(vi)}$$

$$\text{and } \phi = \tan^{-1} \left( \frac{a_1 \sin \alpha_1 + a_2 \sin \alpha_2}{a_1 \cos \alpha_1 + a_2 \cos \alpha_2} \right) \quad \text{(vii)}$$

$$\text{Here } a_1 = 2, \quad a_2 = 3, \quad \alpha_1 = \pi/6, \quad \alpha_2 = \pi/3$$

Hence,

(i) the resultant amplitude

$$A = \sqrt{a_1^2 + a_2^2 + 2a_1a_2 \cos (\alpha_2 - \alpha_1)}$$



$$= \sqrt{4+9+(2)(2)(3) \cos \left( -\frac{\pi}{6} \right)}$$

$$= 4.939$$

$$(ii) \quad \phi = \tan^{-1} \left[ \frac{(2)(0.5) + (3)(0.866)}{(2)(0.866) + (3)(0.5)} \right]$$

$$= \tan^{-1} [1.114]$$

$$= 48.1^\circ \simeq \frac{4\pi}{15}$$

$$\therefore \text{phase constant} = (wt + \phi)$$

$$= \left( wt + \frac{4\pi}{15} \right)$$

(iii) the resultant time period is the same as the time period of the individual vibrations. The equation of the resultant vibration is

$$y = 4.939 \sin \left( wt + \frac{4\pi}{15} \right).$$

### 2.3 Composition of two simple harmonic vibrations at right angles to each other having equal frequencies but differing in phases and amplitudes

Let us consider two simple harmonic motions of the same frequency (*i.e.*, time period) but of amplitude  $a$  and  $b$  and having their vibrations mutually perpendicular to one another (*i.e.*, if one vibrates along the  $X$ -axis, the other vibrates along the  $Y$ -axis). If  $\phi$  is the phase difference between the two vibrations, then their equations can be written as

$$x = a \sin (wt + \phi) \quad (2.7)$$

$$\text{and } y = b \sin wt \quad (2.8)$$

The vibrations given by eqn. (2.7) leads the vibration given by eqn. (2.8).

From eqn. (2.7) we get

$$\begin{aligned}
 \frac{x}{a} &= \sin(\omega t + \phi) \\
 &= \sin \omega t \cos \phi + \cos \omega t \sin \phi \\
 &= \sin \omega t \cos \phi + \sqrt{1 - \sin^2 \omega t} \sin \phi \quad (2.9)
 \end{aligned}$$

Again from eqn. (2.8) we have

$$\sin \omega t = \frac{y}{b} \quad (2.10)$$

Substituting this value of  $\sin \omega t$  in eqn. (2.9), we get

$$\begin{aligned}
 \frac{x}{a} &= \frac{y}{b} \cos \phi + \sqrt{1 - \frac{y^2}{b^2}} \sin \phi \\
 \text{or, } \left( \frac{x}{a} - \frac{y}{b} \cos \phi \right) &= \sqrt{1 - \frac{y^2}{b^2}} \sin \phi
 \end{aligned}$$

Squaring both sides, we get

$$\begin{aligned}
 \frac{x^2}{a^2} + \frac{y^2}{b^2} \cos^2 \phi - 2 \frac{x}{a} \frac{y}{b} \cos \phi &= \left( 1 - \frac{y^2}{b^2} \right) \sin^2 \phi \\
 \text{or, } \frac{x^2}{a^2} + \frac{y^2}{b^2} \cos^2 \phi + \frac{y^2}{b^2} \sin^2 \phi - 2 \frac{xy}{ab} \cos \phi &= \sin^2 \phi \\
 \text{or, } \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} \cos \phi &= \sin^2 \phi \quad (2.11)
 \end{aligned}$$

Eqn. (2.11) gives the general equation of the resultant vibration of the two vibrations given by eqns. (2.7) and (2.8). This is the general equation of a conic whose shape will depend upon the value of the phase difference  $\phi$  between the two vibrations.

case I :

$$\phi = 0, 2\pi, 4\pi, \dots = 2n\pi$$

where  $n = 0, 1, 2, \dots$



Since there is no phase difference between the two vibrations,  $\phi = 0$  and hence  $\sin \phi = 0$  and  $\cos \phi = 1$ . Putting these values in eqn. (2.11), we get

$$\begin{aligned}\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} &= 0 \\ \text{or, } \left(\frac{x}{a} - \frac{y}{b}\right)^2 &= 0 \\ \text{or, } \pm \left(\frac{x}{a} - \frac{y}{b}\right) &= 0 \\ \text{or, } y &= \frac{b}{a} x \quad (2.12)\end{aligned}$$

Eqn. (2.12) is the equation of a straight line passing through the origin and inclined to the direction of first motion i.e., the X-axis, at an angle  $\tan^{-1} \frac{b}{a}$  (Fig. 2.1 (i)). The resultant amplitude is  $\sqrt{a^2 + b^2}$ . If in addition  $a = b$ , then the line will be inclined at an angle of  $45^\circ$ .

case II :

$$\begin{aligned}\phi &= \frac{\pi}{4} \text{ radian} \\ \text{when } \phi &= \frac{\pi}{4} \text{ rad.,} \quad \cos \phi = \sin \phi = \frac{1}{\sqrt{2}}\end{aligned}$$

Putting these values in eqn. (2.11), we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{\sqrt{2} xy}{ab} = \frac{1}{2} \quad (2.13)$$

Eqn. (2.13) represents the equation of an oblique ellipse inscribed in a rectangle whose length parallel to the X-axis is  $2a$  and breadth  $2b$ . The ellipse touches the rectangle at points  $\left(\pm a, \pm \frac{b}{\sqrt{2}}\right)$  and  $\left(\pm \frac{a}{\sqrt{2}}, \pm b\right)$  [Fig. 2.1 (ii)]

case III :

$$\phi = \frac{\pi}{2} \text{ radian.}$$

when  $\phi = \frac{\pi}{2}$  rad.,  $\sin \phi = 1$  and  $\cos \phi = 0$ .

Hence eqn. (2.11) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (2.14)$$

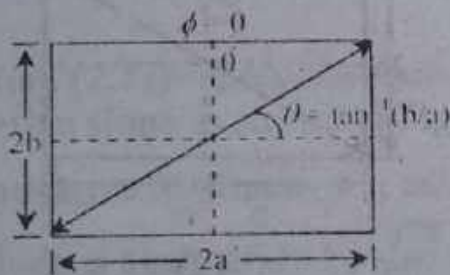
Eqn. (2.14) represents a symmetrical ellipse whose centre coincides with the origin. The semi-major and semi-minor axes of length  $2a$  and  $2b$  respectively coincide with the co-ordinate axes (Fig. 2.11 (iii))

If in addition  $a = b$ , i.e., the amplitudes of the two vibrations are equal; then eqn. (2.14) reduces to

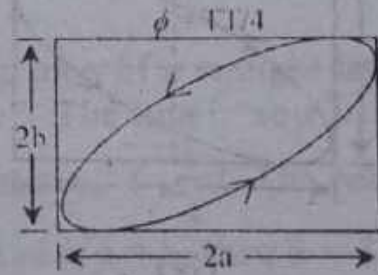
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

or,  $x^2 + y^2 = a^2$  (2.15)

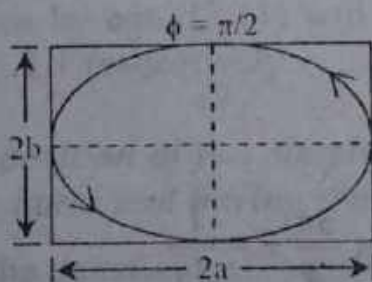
which is the equation of a circle of radius  $a$ . The locus of the particle becomes a circle [Fig. 2.1 (iv)].



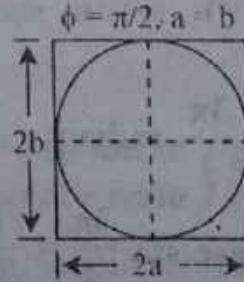
(i)



(ii)

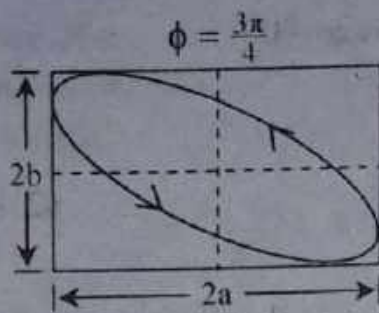


(iii)

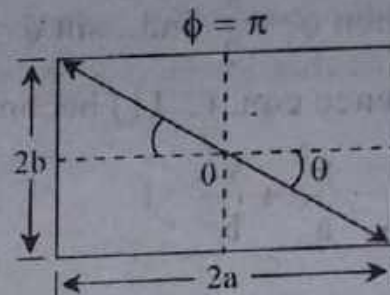


(iv)

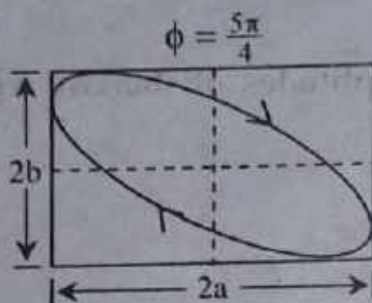




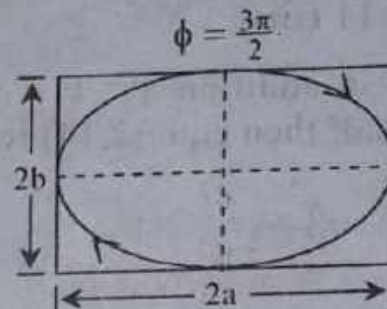
(v)



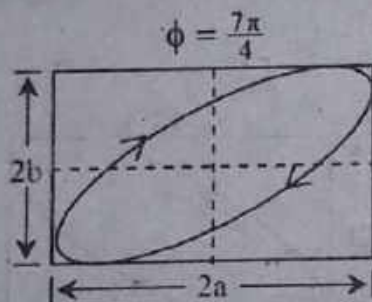
(vi)



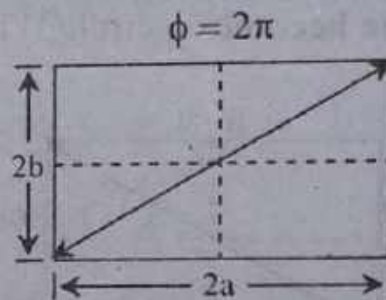
(vii)



(viii)



(ix)



(x)

Fig. 2.1

Case IV :

$$\phi = \frac{3\pi}{4} \text{ radian}$$

$$\text{In that case } \sin \frac{3\pi}{4} = \frac{1}{\sqrt{2}} \text{ and } \cos \frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$$

Hence eqn. (2.11) become

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{\sqrt{2} xy}{ab} = \frac{1}{2} \quad (2.16)$$

Eqn. (2.16) represents again the equation of an oblique ellipse with its axes rotated by  $\frac{\pi}{2}$  with respect to that in case II. [Fig. 2.1(v)].

case V :

$$\phi = \pi \text{ radian}$$

when  $\phi = \pi$  radian,  $\sin \phi = 0$  and  $\cos \phi = -1$ . Hence eqn. (2.11) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{2xy}{ab} = 0$$

$$\text{or, } \left( \frac{x}{a} + \frac{y}{b} \right)^2 = 0$$

$$\text{or, } \pm \left( \frac{x}{a} + \frac{y}{b} \right) = 0$$

$$\text{or, } y = -\frac{b}{a}x \quad (2.17)$$

Eqn. (2.17) represents again the equation of a straight line with a negative slope as shown in Fig. 2.1 (vi). The line is now inclined with negative X-direction at an angle of  $\tan^{-1} \left( -\frac{b}{a} \right)$ . The resultant amplitude is again given by  $\sqrt{a^2 + b^2}$ . And if  $a = b$ , then  $\theta = 45^\circ$ .

When  $\phi = \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$  and  $2\pi$  respectively the shapes of the curve given by eqn. (2.11) will be as shown in (vii), (viii), (ix) and (x) of Fig. 2.1 respectively.

#### **2.4 Composition of two simple harmonic motion at right angles to each other and having time periods in the ratio 1 : 2.**

Let the equations of the two simple harmonic motion be

$$x = a \sin (2\omega t + \phi) \quad (2.18)$$



$$\text{and } y = b \sin \omega t \quad (2.19)$$

where  $a$  is the amplitude for the motion along the  $X$ -axis and  $b$  is the amplitude for the motion along the  $Y$ -axis. The phase difference between the two vibrations is  $\phi$ .

From eqn. (2.18)

$$\begin{aligned} \frac{x}{a} &= \sin(2\omega t + \phi) \\ &= \sin 2\omega t \cos \phi + \cos 2\omega t \sin \phi \\ &= 2 \sin \omega t \cos \omega t \cos \phi + (1 - 2 \sin^2 \omega t) \sin \phi \quad (2.20) \end{aligned}$$

And from eqn. (2.19) we have

$$\begin{aligned} \frac{x}{a} &= \sin \omega t \\ \therefore \cos \omega t &= \sqrt{1 - \sin^2 \omega t} \\ &= \sqrt{1 - \frac{y^2}{b^2}} \end{aligned}$$

Substituting these value of  $\sin \omega t$  and  $\cos \omega t$  in eqn. (2.20) we get,

$$\begin{aligned} \frac{x}{a} &= 2 \cdot \frac{y}{b} \cdot \sqrt{1 - \frac{y^2}{b^2}} \cdot \cos \phi + \left(1 - 2 \cdot \frac{y^2}{b^2}\right) \sin \phi \\ \text{or, } \left[ \frac{x}{a} - \left(1 - \frac{2y^2}{b^2}\right) \sin \phi \right] &= \frac{2y}{b} \cos \phi \sqrt{1 - \frac{y^2}{b^2}} \\ \text{or, } \left[ \left( \frac{x}{a} - \sin \phi \right) + \frac{2y^2}{b^2} \sin \phi \right] &= \frac{2y \cos \phi}{b^2} \sqrt{1 - \frac{y^2}{b^2}} \end{aligned}$$

Squaring both sides,

$$\begin{aligned} \left( \frac{x}{a} - \sin \phi \right)^2 + \frac{4y^4}{b^4} \sin^2 \phi + 2 \left( \frac{x}{a} - \sin \phi \right) \frac{2y^2}{b^2} \sin \phi \\ = \frac{4y^2 \cos^2 \phi}{b^2} \left( 1 - \frac{y^2}{b^2} \right) \end{aligned}$$

$$\begin{aligned}
 \text{or, } \left( \frac{x}{a} - \sin \phi \right)^2 + \frac{4y^4}{b^4} (\sin^2 \phi + \cos^2 \phi) \\
 - \frac{4y^2}{b^2} (\sin^2 \phi + \cos^2 \phi) + \frac{4y^2}{b^2} \cdot \frac{x}{a} \sin \phi = 0 \\
 \text{or, } \left( \frac{x}{a} - \sin \phi \right)^2 + \frac{4y^2}{b^2} \left( \frac{y^2}{b^2} + \frac{x}{a} \sin \phi - 1 \right) = 0 \quad (2.21)
 \end{aligned}$$

Eqn. (2.21) represents the general equation of a curve having two loops, for any difference in phase and amplitude; the actual shape of the curve will of course depend upon the phase difference  $\phi$  between the two vibrations. The resulting curves for different values of  $\phi$  are shown in Fig. 2.2. Some of these cases are discussed below :

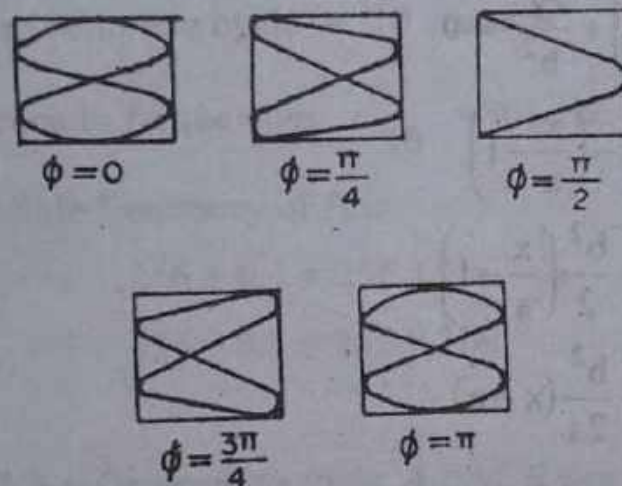


Fig. 2.2

case (1) :

If  $\phi = 0, \pi, 2\pi$ , etc.,  $\sin \phi = 0$ .

Eqn. (2.21) then becomes

$$\frac{x^2}{a^2} + \frac{4y^2}{b^2} \left( \frac{y^2}{b^2} - 1 \right) = 0 \quad (2.22)$$

The above equation represents the figure of eight and has two loops (Fig. 2.2).



case (II) :

$$\text{If } \phi = \frac{\pi}{2}, \sin \phi = +1$$

Then eqn. (2.21) becomes

$$\left(\frac{x}{a} - 1\right)^2 + \frac{4y^2}{b^2} \left(\frac{y^2}{b^2} + \frac{x}{a} - 1\right) = 0$$

$$\text{or, } \left(\frac{x}{a} - 1\right)^2 + \frac{4y^2}{b^2} \left(\frac{x}{a} - 1\right) + \frac{4y^4}{b^4} = 0$$

$$\text{or, } \left[\left(\frac{x}{a} - 1\right)^2 + \frac{2y^2}{b^2}\right]^2 = 0$$

$$\text{or, } \left(\frac{x}{a} - 1\right) + \frac{2y^2}{b^2} = 0$$

$$\text{or, } \frac{2y^2}{b^2} = -\left(\frac{x}{a} - 1\right)$$

$$\text{or, } y^2 = -\frac{b^2}{2} \left(\frac{x}{a} - 1\right)$$

$$\text{or, } y^2 = -\frac{b^2}{2a} (x - a)$$

(2.23)

Eqn. (2.23) represents the equation of a parabola with vertex at (a, 0).

## 2.5 Lissajous' figures

As can be seen from the discussion in articles 2.3 and 2.4 that the composition of two simple harmonic vibrations in mutually perpendicular directions gives rise to an *elliptical path*. The actual shape of the curve will, however, depend upon the phase difference  $\phi$  between the two vibrations – and also on the ratio of the frequencies of the component vibrations. These figures or curves were first produced optically by Lissajous by reflecting a beam of light from two mirrors, in turn attached to two forks vibrating at right angles to one another. These figures are now known as *Lissajous' figures*.

When two rectangular simple harmonic vibrations whose periods are nearly but not exactly equal act simultaneously on a particle, then the pattern of Lissajous' figures migrates slowly through the sequence shown in Figs. 2.1 and 2.2. The number of such complete sequences gone through per second is equal to the difference of frequencies of the component simple harmonic vibrations.

---

**Example 2.2.** *In an experiment to obtain Lissajous' figures, one tuning fork is of frequency 256 Hz and a circular figure occurs after every ten seconds. What deductions may be made about the frequency of the other tuning fork?*

**Soln.**

Frequency of A = 256 Hz.

Time for one complete cycle = 10 seconds

$$\therefore \text{difference in frequencies} = \frac{1}{10} = 0.1 \text{ Hz.}$$

So, the possible frequency of B is

either  $256 + 0.1 = 256.1 \text{ Hz}$

or  $256 - 0.1 = 255.9 \text{ Hz.}$

**Example 2.3.** *Two tuning forks A and B are used to produce Lissajous' figures. The frequency of A is slightly greater than that of B and is 200 Hz. It is found that the figure completes its cycle in 5 seconds. What is the frequency of B?*

**Soln.**

Frequency of A = 200 Hz.

Time for one complete cycle = 5 seconds

$$\therefore \text{the difference in frequencies} = \frac{1}{5} = 0.2 \text{ Hz.}$$

Since the frequency of A is larger than that of B, the frequency of B

$$= 200 - 0.2 = 199.8 \text{ Hz.}$$



**Example 2.4.** Two tuning forks A and B are of nearly equal frequencies. Frequency of A is 256 Hz. When the two tuning forks are used to obtain Lissajous' figures, it is found that the complete cycle of changes takes place in 20 seconds. When the tuning fork is loaded with a little wax, the time taken for one complete cycle of change is 10 seconds. Calculate the original frequency of B.

**Soln.**

Frequency of A = 256 Hz.

Time for one complete cycle = 20 secs.

$$\therefore \text{Difference in frequencies} = \frac{1}{20} = 0.05 \text{ Hz.}$$

So, possible frequency of B is

either  $256 + 0.05 = 256.05 \text{ Hz.}$

or,  $256 - 0.05 = 255.95 \text{ Hz.}$

After loading, time for a complete change of cycle is 10 seconds, i.e., the time decreases. Suppose the frequency of B is 256.05 Hz. After loading, the frequency of B will be lowered and its difference with the frequency of A becomes less. Therefore, the time taken for a complete change of cycle will be more than 20 seconds. Hence the frequency of B cannot be 256.05 Hz.

Suppose the frequency of B = 255.95 Hz. After loading its difference with the frequency of A is increased. Therefore, the cycle of change will take place in less time.

Hence the original frequency of B = 255.95 Hz.

## 2.6 Experimental determination of Lissajous' figures

A and B are two tuning forks with frequencies in the ratio of 1:2. The prongs of A vibrate in a horizontal plane while the prong of B vibrates in a vertical plane. A strong beam of light from an electric arc is made to converge on a mirror  $M_1$  attached to one of the prongs of the fork A with the help of a convergent lens. The arrangement is such that after reflection from the mirror  $M_1$ , the light is again reflected from the mirror  $M_2$  attached to one of the prongs of B. It is further adjusted so that after reflection from  $M_1$  and  $M_2$ , the spot of light is obtained at O, the centre

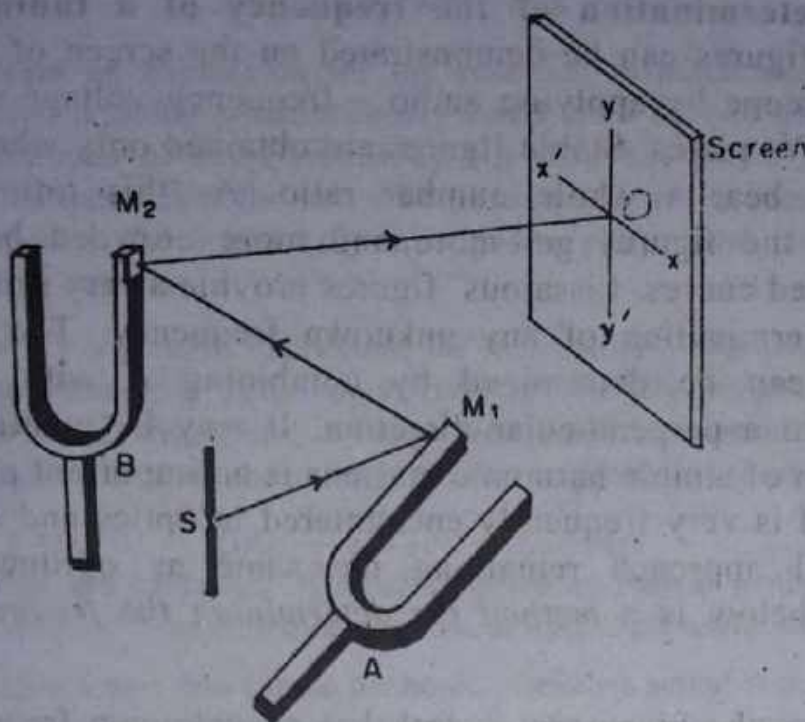


Fig. 2.3

of a white opaque screen (Fig. 2.3). When only the tuning fork A vibrates, the spot of light moves along  $XX'$ . The spot of light moves along  $YY'$  when only the tuning fork B vibrates. When both the forks vibrate in directions at right angles to one another, the resulting motion of the beam of light will trace out a figure of eight on the screen. However, the shape of the figure will be different for different phase difference of the vibrations of the tuning forks and for different frequency ratio of the forks.

### 2.7 Uses of Lissajous' figures

(1) **To determine the ratio of time periods :** Lissajous' figures can be used to determine the ratio of the time period of two constituent vibrations. The number of times the curve touches the horizontal and the vertical sides of a rectangle bounding the Lissajous' figures is found. If the curve touches the horizontal side  $m$  times and the vertical side  $n$  times, the ratio of the time period is  $m:n$ . In the figure of eight in Fig. 2.2, the curve touches the horizontal side once and the vertical side twice. Hence the time periods of the two component vibrations are in the ratio of 1:2.



**(2) Determination of the frequency of a tuning fork :**  
 Lissajous' figures can be demonstrated on the screen of a cathode ray oscilloscope by applying audio - frequency voltage signals on the X- and Y- plates. Stable figures are obtained only when the two frequencies bear a whole number ratio. As this ratio goes on increasing, the figures get more and more crowded but always remain closed curves. Lissajous' figures provide a very sensitive test for the determination of any unknown frequency. The unknown frequency can be determined by combining it with a known frequency in a perpendicular direction. It may be mentioned that composition of simple harmonic motions is an important problem in physics and is very frequently encountered in optics and acoustics; the general approach remaining the same as outlined above. Described below is a *method for determining the frequency of a tuning fork* :

Suppose the given tuning fork has an unknown frequency  $n_1$ . Let B be another tuning fork of frequency  $n$  which is nearly the same as that of A. If the two tuning forks are set into vibration in perpendicular planes in the manner described above, Lissajous' figures are obtained on the screen. As the tuning forks differ slightly in frequency, the phase difference between the two changes with time. As a result the shapes of the Lissajous' figures changes continuously with the phase changing from 0 to  $2\pi$ . Suppose the complete cycle of changes take place in  $t$  seconds. Then the difference in frequencies of A and B =  $\frac{1}{t}$ .

Therefore, the frequency of A =  $n \pm \frac{1}{t}$ .

Now attach a little wax to the tuning fork A. The experiment is repeated and the time taken to complete one cycle of operation is noted. Let it be  $t_1$ . If  $t_1$  is greater than  $t$ , then the frequency of A

$$= (n - \frac{1}{t}).$$

If  $t_1$  less than  $t$ , then the frequency of A

$$= (n + \frac{1}{t}).$$

Thus the frequency of unknown tuning fork can be determined.

## EXERCISES

- [1] Obtain an expression for the resultant displacement of a particle which is being simultaneously acted upon by two simple harmonic vibrations of same frequency but different phase and amplitude. What happens if the two vibrations are (i) in the same phase, (ii) in opposite phase and (iii) in opposite phase with their amplitudes being equal.
- [2] Derive a general expression for the resultant vibration of a particle simultaneously acted upon by two initially perpendicular simple harmonic vibrations, having the same time period but different amplitudes and phase angles. What happens if the phase difference is (i) 0, (ii)  $\frac{\pi}{4}$ , (iii)  $\frac{3\pi}{4}$  and (iv)  $\pi$  radians?
- [3] What are Lissajous' figures? Describe an optical method of obtaining Lissajous' figures. Explain how these figures are useful in the laboratory.
- [4] Explain how two simple harmonic vibrations acting simultaneously on a particle in mutually perpendicular directions can be compounded. Deduce an expression for the resultant vibrations.
- [5] Two mutually perpendicular simple harmonic vibrations of amplitudes  $a$  and  $b$  are acting simultaneously on a particle. Their time periods are in the ratio of 1:2 and their phase difference is  $\phi$ . Derive an expression for the resultant vibration of the particle.
- [6] Find the resultant of two simple harmonic motions of equal periods when they act at right angles to one another. Analytically discuss the different important cases. How can these be demonstrated?
- [7] Find the resultant of two simple harmonic motions of periods in the ratio 1:2 and having (i) zero phase difference and (ii) phase difference of  $45^\circ$ .
- [8] Two simple harmonic motions acting simultaneously on a particle are given by

$$y_1 = \sin \left( \omega t + \frac{\pi}{3} \right)$$

$$y_2 = 2 \sin \omega t$$

Find the equation of the resultant vibration.

- [9] The lower of the two tuning forks has a frequency of 380 Hertz. If the time required for a complete cycle of changes is 10 seconds, calculate the frequency of the other fork.
- [10] In an experiment to obtain Lissajous' figures, one tuning fork is of frequency 250 Hz. and a circular figure appears after every five



seconds. What deductions may be made about the frequency of the other tuning fork ?

- [11] Two tuning forks A and B are nearly of equal frequencies. Frequency of A is 256. When the two tuning forks are used to obtain Lissajous' figures, the complete cycle of changes takes place in 10 seconds. When the tuning fork B is loaded with a little wax, the time taken is 20 seconds. Calculate the original frequency of B.
- [12] Two tuning forks A and B are nearly of equal frequencies. The frequency of A is 288. When the two tuning forks are used to obtain Lissajous' figures, the complete cycle of changes takes place in 20 seconds. When the tuning fork B is loaded with a little wax, the time taken for one complete change of cycle is 10 seconds. Calculate the original frequency of B.

## CHAPTER III

### DAMPED AND FORCED HARMONIC OSCILLATION

*Motion in a resisting medium-Logarithmic decrement of damped oscillatory system-Power dissipation in damped harmonic oscillation : Quality factor-Example of damped harmonic oscillation : the LCR circuit-Forced vibration : Quality factor-Sharpness of resonance-Phase of the driven oscillator-Power absorption-Driven LCR circuit-Solved problems-Exercises.*

#### 3.1 Motion in a resisting medium

When the bob of a simple pendulum is displaced from its mean position and then released, it will execute simple harmonic motion. The lone force acting on the system is the restoring force trying perpetually to bring it back to its mean position. The time period of oscillation depends only on the length of the pendulum and the acceleration due to gravity of the place. As there is no loss of energy by friction or otherwise, the pendulum will go on oscillating with the same time period and amplitude for any length of time. These types of vibrations will be undamped and are referred to as *undamped free vibration*.

In actual practice a simple harmonic oscillator almost always vibrates in a resisting medium, like air, oil, etc. Consequently when the oscillator vibrates in such a medium, energy is dissipated in each vibration in overcoming the opposing frictional or viscous forces. The amplitude of vibration, therefore, goes on decreasing progressively with time. Such forces, which are *non-conservative* in nature, have thus a damping effect on the oscillations and are aptly referred to as *damping, resistive or dissipative* forces. In the absence of any such forces, the oscillations will continue indefinitely, without any change in amplitude (Fig. 3.1). But in the presence of these forces, the amplitude of vibration decreases continuously with time and finally the oscillations die out. Such vibrations are called *free damped vibrations*.



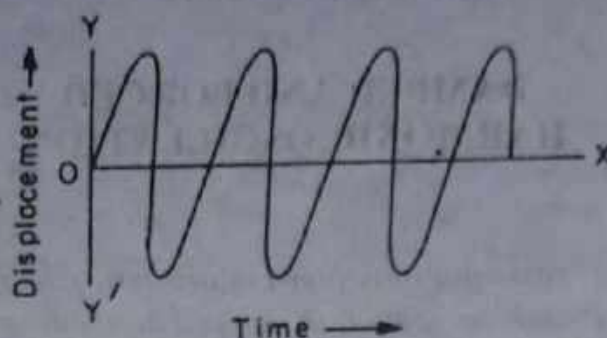


Fig. 3.1

Thus a body executing simple harmonic oscillations in a damping medium will be simultaneously subjected to the following two opposing forces :

(i) the restoring force acting on the body which is proportional to the displacement of the body and acts in a direction opposite to the displacement. Let this force be  $-ay$  where  $a$  is the force constant.

(ii) a resistive or damping force. It has been shown by *Mayevski* that at ordinary velocities, the opposing, resistive or damping force is, to a first approximation, proportional to the velocity of the oscillating body.

As most cases of interest to us fall in the category of ordinary velocities, the damping or resistive force may thus be represented by

$$F = -bv = -b \frac{dy}{dt}$$

where  $b$  is a constant of proportionality.  $b$  is a positive constant called *damping coefficient* of the medium and may be looked upon as the resistive or dissipative force per unit velocity. The negative sign signifies a restraining influence on the vibration of the particle.

Thus, the differential equation of motion of a body executing damped harmonic oscillations may be written as

$$m \frac{d^2y}{dt^2} = -ay - b \frac{dy}{dt}$$

$$\text{or, } m \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + ay = 0$$

$$\text{or, } \frac{d^2 y}{dt^2} + \frac{b}{m} \frac{dy}{dt} + \frac{a}{m} y = 0$$

$$\text{or, } \frac{d^2 y}{dt^2} + 2\lambda \frac{dy}{dt} + w^2 y = 0 \quad (3.1)$$

$$\text{where } 2\lambda = \frac{b}{m} \text{ and } w^2 = a/m.$$

Eqn. (3.1) is referred to as the *differential equation of a damped harmonic oscillator*.

**Solution of eqn. (3.1) :** Eqn. (3.1) which is a *homogeneous linear type differential equation of the second order*, must have at least one solution of the form  $y = Ae^{kt}$  where  $A$  and  $k$  are both arbitrary constants. Let this be used as a trial solution,

Differentiating  $y = Ae^{kt}$  with respect to time, we get

$$\frac{dy}{dt} = kAe^{kt} \text{ and } \frac{d^2 y}{dt^2} = k^2 Ae^{kt}$$

Substituting these values in eqn. (3.1), we have

$$k^2 Ae^{kt} + 2\lambda kAe^{kt} + w^2 Ae^{kt} = 0$$

Dividing throughout by  $Ae^{kt}$ , we have

$$k^2 + 2\lambda k + w^2 = 0 \quad (3.2)$$

Eqn. (3.2) is clearly a *quadratic equation* in  $k$ , the solution of which is

$$k = -\lambda \pm \sqrt{\lambda^2 - w^2}$$

The differential equation (3.1) is, therefore, satisfied by two values of  $y$ , viz,

$$y = Ae^{(-\lambda + \sqrt{\lambda^2 - w^2})t}$$

and

$$y = Ae^{(-\lambda - \sqrt{\lambda^2 - w^2})t}$$



The equation being a linear one, the linear sum of the two linearly independent solutions of the equation is also a – and, indeed, the most general solution. Thus the general solution is

$$y = A_1 e^{(-\lambda + \sqrt{\lambda^2 - w^2})t} + A_2 e^{(-\lambda - \sqrt{\lambda^2 - w^2})t} \quad (3.3)$$

where  $A_1$  and  $A_2$  are two arbitrary constants.

The values of the arbitrary constants  $A_1$  and  $A_2$  may be determined as follows:

Differentiating eqn. (3.3) with respect to  $t$ , we get

$$\begin{aligned} \frac{dy}{dt} &= (-\lambda + \sqrt{\lambda^2 - w^2}) A_1 e^{(-\lambda + \sqrt{\lambda^2 - w^2})t} \\ &+ (-\lambda - \sqrt{\lambda^2 - w^2}) A_2 e^{(-\lambda - \sqrt{\lambda^2 - w^2})t} \end{aligned} \quad (3.4)$$

Let the maximum value of the displacement  $y$  be  $y_{\max} = a_0$ , say at time  $t = 0$ . Then we have from eqn. (3.3)

$$y_{\max} = a_0 = A_1 + A_2 \quad (3.5)$$

Again, the velocity is zero at maximum displacement, i.e.,  $\frac{dy}{dt} = 0$  at  $t = 0$ .

Hence from eqn. (3.4), we have

$$\begin{aligned} &(-\lambda + \sqrt{\lambda^2 - w^2}) A_1 + (-\lambda - \sqrt{\lambda^2 - w^2}) A_2 = 0 \\ \text{or, } &-\lambda(A_1 + A_2) + \sqrt{\lambda^2 - w^2} (A_1 - A_2) = 0 \\ \text{or, } &\sqrt{\lambda^2 - w^2} (A_1 - A_2) = \lambda(A_1 + A_2) = \lambda a_0 \\ \text{or, } &A_1 - A_2 = \frac{\lambda a_0}{\sqrt{\lambda^2 - w^2}} \end{aligned} \quad (3.6)$$

Adding eqns. (3.5) and (3.6), we have

$$2A_1 = a_0 + \frac{\lambda a_0}{\sqrt{\lambda^2 - w^2}}$$

$$\text{or, } A_1 = \frac{1}{2} \left( a_0 + \frac{\lambda a_0}{\sqrt{\lambda^2 - w^2}} \right) = \frac{1}{2} a_0 \left( 1 + \frac{\lambda}{\sqrt{\lambda^2 - w^2}} \right)$$

$$\text{and } A_2 = a_0 - \frac{1}{2} a_0 \left( 1 + \frac{\lambda}{\sqrt{\lambda^2 - w^2}} \right) = \frac{1}{2} a_0 \left( 1 - \frac{\lambda}{\sqrt{\lambda^2 - w^2}} \right)$$

Substituting these values of  $A_1$  and  $A_2$  in eqn. (3.3), we have,

$$y = \frac{1}{2} a_0 e^{-\lambda t} \left[ \left( 1 + \frac{\lambda}{\sqrt{\lambda^2 - w^2}} \right) e^{\left( \sqrt{\lambda^2 - w^2} \right) t} + \left( 1 - \frac{\lambda}{\sqrt{\lambda^2 - w^2}} \right) e^{\left( -\sqrt{\lambda^2 - w^2} \right) t} \right] \quad (3.7)$$

Three important cases now arise :

(i)  $\lambda^2 > w^2$  : Damping is large;  $\sqrt{\lambda^2 - w^2}$  is clearly a real quantity with a *positive value, less than  $\lambda$* . Thus each of the two terms on the right hand side of eqn. (3.7) has an exponential term with a negative power and hence each decreases exponentially with time. In this case the particle does not vibrate. The displacement, after attaining its maximum value falls off asymptotically to zero [curve (i) Fig. 3.2]. There is thus no oscillation and the motion is, therefore, called *overdamped* or *aperiodic* or *dead beat*. Examples are a dead beat galvanometer or a pendulum oscillating in a viscous fluid like oil.

(ii)  $\lambda^2 = w^2$  : In this case,  $\sqrt{\lambda^2 - w^2}$  is obviously equal to zero, so that each of the two terms on the right hand side of eqn. (3.7) becomes infinite and the solution breaks down.

Let us, however, consider that  $\lambda^2$  is not quite equal to  $w^2$ , but very nearly so, so that  $\sqrt{\lambda^2 - w^2} = h$ , a *very small quantity but not zero*. Then from eqn. (3.3), we have

$$y = A_1 e^{(-\lambda + h) t} + A_2 e^{(-\lambda - h) t}$$



$$= e^{-\lambda t} (A_1 e^{ht} + A_2 e^{-ht})$$

$$\text{or, } y = e^{-\lambda t} \left[ A_1 \left( 1 + ht + \frac{h^2 t^2}{2!} + \frac{h^3 t^3}{3!} + \dots \right) \right. \\ \left. + A_2 \left( 1 - ht + \frac{h^2 t^2}{2!} + \frac{h^3 t^3}{3!} + \dots \right) \right]$$

Neglecting terms containing second and higher powers of  $h$ , we get

$$y = e^{-\lambda t} [A_1 (1 + ht) + A_2 (1 - ht)] \\ = e^{-\lambda t} [(A_1 + A_2) + (A_1 - A_2) ht] \quad (3.8)$$

Putting  $(A_1 + A_2) = M$  and  $(A_1 - A_2) h = N$ , we have

$$y = e^{-\lambda t} (M + Nt)$$

Recalling that  $y = y_{\max} = a_0$  and  $\frac{dy}{dt} = 0$  at  $t = 0$ , we have

$$M = y_{\max} = a_0 \text{ and } N = \lambda a_0.$$

$$[A_1 - A_2 = \frac{\lambda a_0}{\sqrt{\lambda^2 - w^2}} = \frac{\lambda a_0}{h} ; \text{ or, } (A_1 - A_2) h = N = \lambda a_0].$$

Eqn. (3.8), therefore, becomes

$$y = e^{-\lambda t} (a_0 + \lambda a_0 t) \\ = a_0 e^{-\lambda t} (1 + \lambda t) \quad (3.9)$$

The second term in eqn. (3.9), viz,  $a_0 \lambda t e^{-\lambda t}$  decays less rapidly than the first term, viz,  $a_0 e^{-\lambda t}$  and the displacement of the oscillator first increases but as  $t$  increases the exponential factor  $e^{-\lambda t}$  becomes more important and the displacement decreases rapidly reaching the value zero for a finite value of  $t$ . The oscillator just ceases to oscillate and its motion just becomes aperiodic or non-oscillatory. This is called the case of *critical damping* [curve (ii) in Fig. 3.2] the necessary condition for which is that  $\lambda^2$  should be almost, but not quite, equal to  $w^2$  (i.e.,  $\lambda^2 \rightarrow w^2$ ).

This principle finds application in many pointer-type instruments like galvanometers where the pointer moves at once to, and stays at, the correct position, without any annoying oscillations.

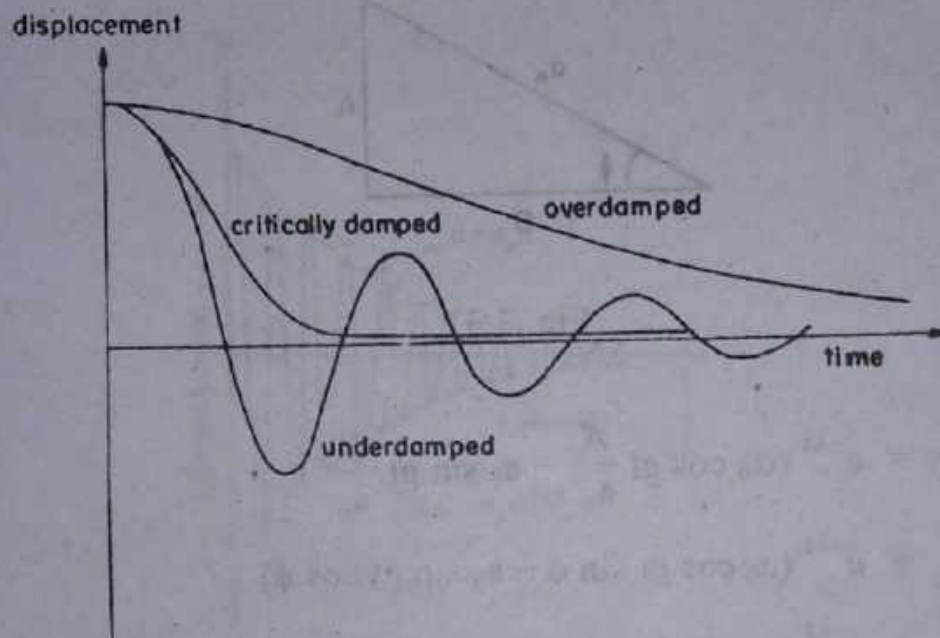


Fig. 3.2

It may be seen that although the motion is non-oscillatory both in case of critical damping and overdamping the time taken to reach near the equilibrium position from a given displacement becomes greater and greater with the increase of damping.

(iii)  $\lambda^2 < w^2$  : the quantity  $\sqrt{\lambda^2 - w^2}$  is clearly imaginary, say, equal to  $ig$  where  $i = \sqrt{-1}$  and  $g = \sqrt{w^2 - \lambda^2}$  a real quantity.

Eqn. (3.3) then becomes

$$y = A_1 e^{(-\lambda + ig)t} + A_2 e^{(-\lambda - ig)t}$$

$$\text{or, } y = e^{-\lambda t} (A_1 e^{igt} + A_2 e^{-igt})$$

$$= e^{-\lambda t} [A_1 (\cos gt + i \sin gt) + A_2 (\cos gt - i \sin gt)]$$

$$= e^{-\lambda t} [(A_1 + A_2) \cos gt + i (A_1 - A_2) \sin gt]$$

Putting  $(A_1 + A_2) = A$  and

$i (A_1 - A_2) = B$ , we have



$$y = e^{-\lambda t} (A \cos gt + B \sin gt)$$

if A, B and  $a_0$  be related as shown in Fig. 3.3.

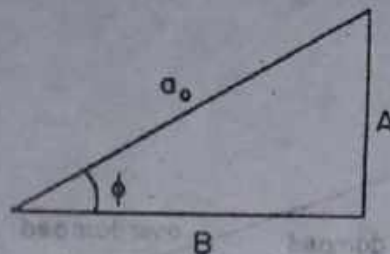


Fig. 3.3

$$\begin{aligned} y &= e^{-\lambda t} \left( \alpha_0 \cos gt \frac{A}{a_0} + a_0 \sin gt \frac{B}{a_0} \right) \\ &= e^{-\lambda t} (a_0 \cos gt \sin \phi + a_0 \sin gt \cos \phi) \\ &= e^{-\lambda t} a_0 \sin (gt + \phi) \\ &= a_0 e^{-\lambda t} \sin (gt + \phi) \end{aligned} \quad (3.10)$$

Eqn. (3.10) is the equation of a damped harmonic oscillator with amplitude  $a_0 e^{-\lambda t}$  and frequency  $g/2\pi = (\sqrt{w^2 - \lambda^2}) / 2\pi$ . It is so called because the sine term in the equation suggests the oscillatory character of the motion and the exponential term, the gradual damping out of the oscillations [curve (iii) Fig. 3.2].

Thus damping clearly produces two effects :

(i) The frequency of the damped harmonic oscillator,  $g/2\pi$  is smaller than its natural or undamped frequency. This means that damping somewhat decreases the frequency or increases the time-period of the oscillator. In actual practice, the damping is small in majority of cases (particularly in the case of musical instruments) and its effect on the time-period or frequency of oscillation is, therefore, quite negligible.

(ii) The amplitude of the oscillation does not remain constant at  $a_0$  which represents the amplitude in the absence of any damping, but decays exponentially with time, to zero, in accordance with the

term  $e^{-\lambda t}$ . The term  $e^{-\lambda t}$  is called the *damping factor*. The time displacement curve of a damped harmonic oscillator is illustrated in Fig. 3.4.

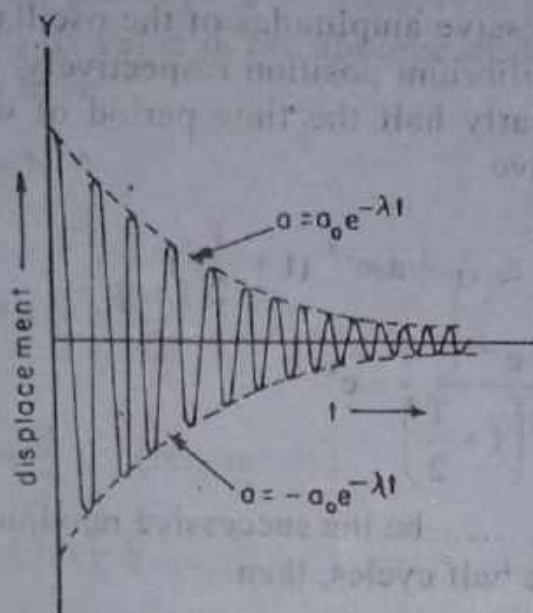


Fig. 3.4

Since the maximum values of  $\sin(gt + \phi)$  alternate between  $+1$  and  $-1$ , the time displacement curve of the oscillating particle is bound by the dotted curves  $a = a_0 e^{-\lambda t}$  and  $a = -a_0 e^{-\lambda t}$ .

Thus although the amplitude of an underdamped harmonic oscillator decreases exponentially with time, it does perform a sort of oscillatory motion. The motion does not, of course, repeat itself and is thus not periodic in the usual sense of the term. However, it still has a time period  $2\pi/g = 2\pi/\sqrt{\omega^2 - \lambda^2}$  which is the time interval between its successive passages in the same direction past the equilibrium point. It is obviously also the time interval between successive maximum displacements on the same side of the position of equilibrium.

**Note :** Even if the body vibrates in vacuum, thereby eliminating external force due to the friction of air, still the vibration is not free. Viscosity or internal friction arises within the material of the body during its vibration. This internal friction ultimately brings the vibration to a stop.



### 3.2 Logarithmic decrement of damped oscillatory system

As can be seen from eqn. (3.10), the amplitude in the case of damped harmonic motion goes on decreasing progressively. If  $a_n$  and  $a_{n+1}$  be the successive amplitudes of the oscillating body on the two sides of the equilibrium position respectively, the time interval between them is clearly half the time period of oscillation and is thus  $T/2$ . Thus we have

$$a_n = a_0 e^{-\lambda t} \quad \text{and} \quad a_{n+1} = a_0 e^{-\lambda \left(t + \frac{T}{2}\right)}$$

$$\text{Then } \frac{a_n}{a_{n+1}} = \frac{e^{-\lambda t}}{e^{-\lambda \left(t + \frac{T}{2}\right)}} = e^{\lambda T/2}$$

If  $a_n, a_{n+1}, a_{n+2}, \dots$  be the successive maximum amplitudes at the end of successive half cycles, then

$$\frac{a_n}{a_{n+1}} = \frac{a_{n+1}}{a_{n+2}} = \frac{a_{n+2}}{a_{n+3}} = \dots = e^{\lambda T/2} = d$$

$d$  is a constant for the oscillating system and is called the *decrement* or *damping ratio* of the swing. The same naturally applies also to angular amplitudes. If  $\theta_1, \theta_2, \dots, \theta_n, \theta_{n+1}, \dots$  be the successive angular amplitudes (deflections from the zero position), then

$$\frac{\theta_n}{\theta_{n+1}} = d$$

The logarithm of the decrement i.e.,

$$\log_e d = \frac{\lambda T}{2} = \mu; \quad \text{or, } d = e^\mu \quad (3.12)$$

The constant  $\mu$  which is the natural logarithm of the decrement is referred to as the logarithmic decrement for the oscillation.

Now the interval that elapses between the start of oscillation and observing of the first deflection or the first throw  $\theta_1$ , is a half swing or one quarter of oscillation. Hence  $\theta_1$ , the deflection observed after quarter of a period, is smaller than the true deflection  $\theta_0$ .

From eqn. (3.11) we see that  $\theta_1 = d\theta_2$ ;  $\theta_2 = d\theta_3$ ; hence  $\theta_1 = d^2\theta_3$ . Since the ratio of the amplitudes or the decrement, at the end of each successive half-cycle of oscillations is  $d$ , the decrement in half swing (or quarter of a period)  $= e^{\mu/2}$ . Thus, if  $\theta_0$  be the true value of this first throw, *i.e.*, its value in the absence of damping, and  $\theta_1$ , its observed value, we have

$$\begin{aligned}\frac{\theta_0}{\theta_1} &= e^{\mu/2} \\ &= \left( 1 + \frac{\mu}{2} + \frac{(\mu/2)^2}{2!} + \dots \right) \\ &= 1 + \frac{\mu}{2} \quad (\text{very nearly})\end{aligned}$$

$$\text{Hence } \theta_0 = \theta_1 \left( 1 + \frac{\mu}{2} \right).$$

Thus knowing the logarithmic decrement ( $\mu$ ) for a given galvanometer, necessary corrections for damping to the first throw  $\theta_1$  can be easily made.

### 3.3 Power dissipation in damped harmonic oscillation

When a simple harmonic oscillator vibrates in a resistive medium, energy is dissipated in each vibration in overcoming this opposing force. It is, therefore, clear that its energy gets continuously dissipated during its oscillations and, hence, the amplitude of oscillations will go on decreasing progressively with time. Let us calculate this *rate of dissipation of energy* or its *power dissipation*, as it is more commonly called.

The oscillation of a damped harmonic oscillator is given by

$$y = a_0 e^{-\lambda t} \sin(gt + \phi)$$

Hence its velocity at a given instant  $t$  is

$$\frac{dy}{dt} = a_0 e^{-\lambda t} [-\lambda \sin(gt + \phi) + g \cos(gt + \phi)]$$

and the kinetic energy of oscillation of the particle at that instant,  $t$



$$\begin{aligned}
 \frac{1}{2} m \left( \frac{dy}{dt} \right)^2 &= \frac{1}{2} m a_0^2 e^{-2\lambda t} [ -\lambda \sin(gt + \phi) + g \cos(gt + \phi) ]^2 \\
 &= \frac{1}{2} m a_0^2 e^{-2\lambda t} [ \lambda^2 \sin^2(gt + \phi) + g^2 \cos^2(gt + \phi) \\
 &\quad - 2\lambda g \sin(gt + \phi) \cos(gt + \phi) ]
 \end{aligned}$$

When the damping is small as is usually the case, the amplitude does not change appreciably over a time-period; so that the factor  $e^{-2\lambda t}$  may be taken to be constant over the period. Further, the average value of both  $\sin^2(gt + \phi)$  and  $\cos^2(gt + \phi)$  over a complete time period is  $\frac{1}{2}$  while the average value of  $2g \sin(gt + \phi) \cos(gt + \phi) = \sin 2(gt + \phi)$  over the time period is zero. Hence the average kinetic energy of the damped harmonic oscillator over a complete cycle at the given instant  $t$  is

$$\frac{1}{2} m a_0^2 e^{-2\lambda t} \left[ \frac{1}{2} \lambda^2 + \frac{1}{2} g^2 - 0 \right]$$

Since damping is usually very small,  $\lambda^2$  may be neglected when compared to  $g^2$ . The above expression then becomes

$$\frac{1}{4} m a_0^2 e^{-2\lambda t} g^2 \quad (3.14)$$

The potential energy of the oscillator at the given instant  $t$  when the displacement is  $y$  is

$$\begin{aligned}
 \int_0^y m w^2 y \cdot dy &= \frac{1}{2} m w^2 y^2 \\
 &= \frac{1}{2} m w^2 [a_0 e^{-\lambda t} \sin(gt + \phi)]^2 \\
 &= \frac{1}{2} m a_0^2 e^{-2\lambda t} [w^2 \sin^2(gt + \phi)]
 \end{aligned}$$

Since the average value of  $\sin^2(gt + \phi)$  over a complete cycle is  $\frac{1}{2}$ , the average P.E. of the oscillator over a complete cycle at the given instant  $t$

$$\begin{aligned}
 &= \frac{1}{2} m a_0^2 e^{-2\lambda t} \cdot \frac{1}{2} \omega^2 \\
 &= \frac{1}{4} m a_0^2 \omega^2 e^{-2\lambda t} \\
 &= \frac{1}{4} m a_0^2 g^2 e^{-2\lambda t}
 \end{aligned}
 \tag{3.15}$$

Since, when damping is small,  $g \simeq \omega$ .

Hence, average total energy of the oscillator at the given instant

$E = \text{average P.E.} + \text{average K.E.}$

$$\begin{aligned}
 &= \frac{1}{4} m a_0^2 g^2 e^{-2\lambda t} + \frac{1}{4} m a_0^2 g^2 e^{-2\lambda t} \\
 &= \frac{1}{2} m a_0^2 g^2 e^{-2\lambda t} \\
 &= E_0 e^{-2\lambda t}
 \end{aligned}
 \tag{3.16}$$

where  $\frac{1}{2} m a_0^2 g^2 = E_0 = \text{the average total energy of the undamped oscillator.}$

$\therefore$  Average power dissipation (P)

$$\begin{aligned}
 &= \text{rate of loss of energy} = - \frac{dE}{dt} \\
 &= \left( -\frac{1}{2} m a_0^2 g^2 \right) (-2\lambda) e^{-2\lambda t} = m a_0^2 g^2 \lambda e^{-2\lambda t} \\
 &= 2\lambda E.
 \end{aligned}
 \tag{3.17}$$

The loss of energy is obviously due to the work done against the dissipative force and appears in the form heat.

### 3.4 Quality factor

The *quality factor*,  $Q$ , also referred to as the *figure of merit*, of a harmonic oscillator is defined as  $2\pi$  times the ratio between the energy stored and the energy lost per period. Being a ratio, it is a dimensionless quantity — a pure number.



Thus,

$$Q = 2\pi \frac{\text{energy stored}}{\text{energy lost per period}}$$

$$= \frac{2\pi E}{P.T.}$$

where  $P$  is the rate of loss of energy  $= 2\lambda E$ .

Now  $\frac{2\pi}{T} = g$  where  $g$  is the frequency of the damped harmonic oscillator. Then

$$Q = \frac{Eg}{P} = \frac{Eg}{2\lambda E} = \frac{g}{2\lambda}$$

In the case of low damping  $g \simeq w$ ; therefore,

$$Q = \frac{w}{2\lambda}. \quad (3.18)$$

The quality factor, as the name suggests, measures the quality of a harmonic oscillator in so far as damping is concerned. The less the damping, the better the quality of the harmonic oscillator as an oscillator and, therefore, the higher its quality factor  $Q$ .

We also know that

$$w = \sqrt{\frac{a}{m}} \text{ and } 2\lambda = \frac{b}{m}$$

Hence,

$$Q = \frac{w}{2\lambda} = \sqrt{\frac{a}{m}} \cdot \frac{m}{b} = \frac{\sqrt{am}}{b} \quad (3.19)$$

clearly indicating that lower the value of damping, i.e.,  $b$ , the higher the value of  $Q$ .

Or, as  $b \rightarrow 0$ ,  $Q \rightarrow \infty$

### 3.5 Example of damped harmonic oscillation – the LCR circuit

An electrical circuit which contains inductance ( $L$ ), capacitance ( $C$ ) and resistance ( $R$ ) renders an excellent example of damped harmonic

oscillation. The resistance by itself plays the part of a resistive or dissipative force analogous to that of friction or viscosity in the case of mechanical oscillations.

Fig. 3.5 shows a circuit containing a resistance, an inductance and a capacitance connected in series. A battery can be included into the circuit through the key  $K$ . When  $K$  is pressed, the capacitor is charged. On releasing  $K$ , the battery is thrown out of the circuit and the capacitance discharges through the resistance and the inductance.

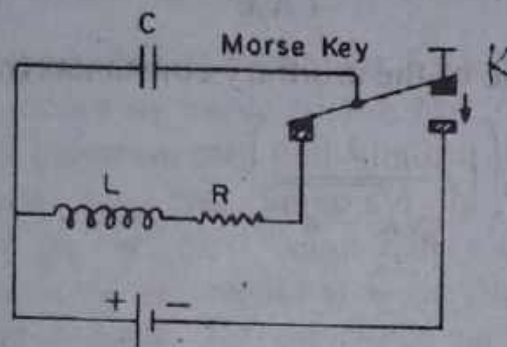


Fig. 3.5

Suppose that at a given instant  $t$ , the charge on the capacitor is  $Q$  and the current flowing through the circuit is  $i$ . Then the induced *emf*. across the inductance  $L$  is  $L \frac{di}{dt}$  and the potential difference across the resistance  $R$  is  $Ri$ . Since there is no external *emf*. in the circuit now (the battery being disconnected), then according to Kirchhoff's law,

$$Ri + \frac{Q}{C} = -L \frac{di}{dt}$$

$$\text{or, } L \frac{di}{dt} + Ri + \frac{Q}{C} = 0 \quad (3.20)$$

But  $i = \frac{dQ}{dt}$  ; therefore eqn. (3.20) becomes

$$L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C} = 0$$



$$\text{or, } \frac{d^2Q}{dt^2} + \frac{R}{L} \frac{dQ}{dt} + \frac{Q}{LC} = 0 \quad (3.21)$$

Eqn. (3.21) is very similar to eqn. (3.1), the differential equation of damped harmonic oscillation, with the difference that the displacement  $y$  is replaced by charge  $Q$ ,  $2\lambda$  by  $R/L$  and  $w^2$  by  $1/LC$ .

Proceeding in the same manner as described in Art. 3.1, the solution of eqn. (3.21) is

$$Q = A_1 e^{\left(-\lambda + \sqrt{\lambda^2 - w^2}\right)t} + A_2 e^{\left(-\lambda - \sqrt{\lambda^2 - w^2}\right)t}$$

where the value of the arbitrary constants  $A_1$  and  $A_2$  are

$$A_1 = \frac{1}{2} Q_0 \left( 1 + \frac{\lambda}{\sqrt{\lambda^2 - w^2}} \right)$$

and

$$A_2 = \frac{1}{2} Q_0 \left( 1 - \frac{\lambda}{\sqrt{\lambda^2 - w^2}} \right)$$

Thus,

$$Q = \frac{1}{2} Q_0 e^{-\lambda t} \times \left[ \left( 1 + \frac{\lambda}{\sqrt{\lambda^2 - w^2}} \right) e^{\left(\sqrt{\lambda^2 - w^2}\right)t} + \left( 1 - \frac{\lambda}{\sqrt{\lambda^2 - w^2}} \right) e^{\left(-\sqrt{\lambda^2 - w^2}\right)t} \right] \quad (3.22)$$

Again the following three cases arise :

(i) when the damping is large, i.e.,

$$\lambda^2 > w^2 \left( \text{or, } \frac{R^2}{4L^2} > \frac{1}{LC} \text{ or, } R^2 > \frac{4L}{C} \right)$$

both the terms on the right hand side of eqn. (3.22) have exponential terms with a negative power. The charge ( $Q$ ), therefore, decays exponentially with time without changing sign i.e., without

making any oscillation. Thus the discharge of the capacitor is non-oscillatory or damping [curve (i) Fig. 3.6].

(ii) when the damping is critical *i.e.*, when  $\lambda^2 = \omega^2$  (or,  $R^2 = 4L/C$ ). Here the discharge of the condenser just becomes aperiodic and dies down in the shortest possible time as shown by curve (ii), Fig. (3.6).

(iii) when the damping is small, *i.e.*,  $\lambda^2 < \omega^2$  (or,  $R^2 < \frac{4L}{C}$ ). The equation for the discharge of charge takes the form

$$Q = Q_0 e^{-\lambda t} \sin (gt + \phi) \quad (3.23)$$

*viz.* that of a damped oscillator (see Art. 3.1). The discharge curve of the capacitor is shown by curve (iii) in Fig. 3.6. Thus although the charge decreases exponentially with time, it does perform a sort of oscillatory motion, *i.e.*, the charge on the condenser alternately acquires positive and negative values before ultimately decaying to zero. The discharge of the capacitor is oscillatory in character, the oscillations being damped. The discharge curve is identical in form with the displacement curve of underdamped harmonic oscillator (curve (iii) Fig. 3.2) and is similarly bound by the curves

$$Q = +Q_0 e^{-\lambda t} \text{ and } Q = -Q_0 e^{-\lambda t}$$

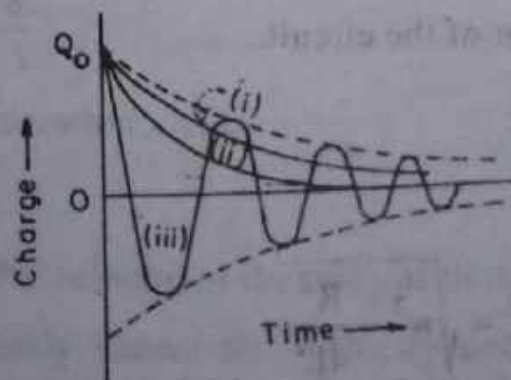


Fig. 3.6

Differentiating eqn. (3.23) with respect to time, we have

$$\frac{dQ}{dt} = \frac{dQ_0}{dt} e^{-\lambda t} \sin (gt + \phi)$$



Since  $\frac{dQ}{dt} = i$  and  $\frac{dQ_0}{dt} = i_0$ , the maximum value or the amplitude of the current, we have

$$i = i_0 e^{-\lambda t} \sin(gt + \phi)$$

The frequency of oscillation is

$$\begin{aligned} n &= \frac{g}{2\pi} = \frac{\sqrt{w^2 - \lambda^2}}{2\pi} \\ &= \frac{1}{2\pi} \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \end{aligned} \quad (3.24)$$

Thus the resistance  $R$  alone is responsible for the damping of the oscillations in this case (the energy dissipated appearing as heat) as can be seen from eqn (3.24). In the absence of  $R$  i.e., if  $R = 0$ , the frequency of oscillation becomes

$$n = \frac{1}{2\pi} \sqrt{\frac{1}{LC}}$$

The same as that of an LC circuit, with no damping of the oscillation.

The quality factor of the circuit,

$$Q = \frac{g}{2\lambda} = \frac{Lg}{R}$$

where

$$g = \sqrt{w^2 - \lambda^2} = \sqrt{w^2 - \frac{R^2}{4L^2}}$$

Since  $R$  is small,  $g \simeq w$ , and, therefore, the

$$\text{Quality factor, } Q = \frac{Lw}{R} \quad (3.25)$$

**Some useful relations for solving problems :**

From Art. 3.1, we know that for ordinary velocities, the opposing resistive force is given by

$$F = -bv = -b \frac{dy}{dt}$$

If there is no other force acting on the body, then according to Newton's second law of motion

$$F = m \frac{d^2y}{dt^2} = m \frac{dv}{dt} = -bv$$

$$\text{or, } \frac{dv}{dt} + \frac{b}{m}v = 0 \quad (i)$$

where  $\frac{b}{m}$ , denoted by  $2\lambda$  in the earlier articles, is the resistive force per unit mass per unit velocity.  $\lambda$  is the damping constant of the medium. The term  $\frac{m}{b}$  (reciprocal of  $\frac{b}{m}$ ) is also denoted by a constant  $\tau$  ( $= \frac{1}{2\lambda}$ ).

Eqn. (i) may be rewritten

$$\frac{dv}{v} = -\frac{dt}{\tau}$$

which on integration yields

$$v = v_0 e^{-t/\tau} \quad (ii)$$

where  $v_0$  is the velocity of the body at time  $t = 0$ .

Eqn. (ii) clearly shows that velocity decreases exponentially with time. This is expressed by saying that velocity is damped with time constant  $\tau$ .

When  $t = \tau$ , then

$$v = v_0 e^{-\tau/\tau} = v_0 e^{-1} = 0.368 v_0$$

Thus the *time constant*  $\tau$  may be defined as the time in which the velocity of the oscillating particle falls to  $1/e^{\text{th}}$  (i.e., 0.368 or



roughly, one-third) of its initial value.  $\tau$ , therefore, has the dimension of time and is also referred to as the relaxation time.

Now the kinetic energy of the oscillating particle is given by

$T = \frac{1}{2} mv^2$ . Substituting the value of  $v$  from eqn. (ii), we have

$T = \frac{1}{2} mv_0^2 e^{-2t/\tau}$ . If  $\frac{1}{2} mv_0^2 = T_0$ , the initial kinetic energy of the particle, then  $T = T_0 e^{-2t/\tau}$ . This indicates that like velocity, K.E. of the oscillating particle also falls exponentially with time, but with a relaxation time half that for velocity, i.e.,  $\tau/2$ , which, of course is expected since  $KE \propto (\text{velocity})^2$ .

**Example 3.1.** An underdamped harmonic oscillator has its amplitude reduced to  $1/10^{\text{th}}$  of its initial value after 100 oscillations. Its time period is 1.15 sec (i) Calculate the damping constant and the relaxation time, (ii) If the observed value of the first amplitude of the oscillator be 2 cm, what would be its value in the absence of damping?

(iii) Calculate the time in which its amplitude and its energy falls to  $1/e^{\text{th}}$  of its undamped value. If the mass of the oscillator be 1.127 gm, calculate its average rate of loss of energy.

**Soln.**

(i) The first amplitude ( $a_1$ ) obviously occurs at  $t = T/4$  where  $T$  is the time-period of oscillation.

Now the amplitude of a damped harmonic oscillator is given by  $a = a_0 e^{-\lambda t}$  where  $a_0$  is its amplitude in the absence of damping and  $\lambda$ , the damping constant. Then for the first amplitude we have

$$a_1 = a_0 e^{-\lambda T/4}$$

successive amplitudes occur at intervals of  $T/2$  sec each, the  $201^{\text{th}}$  amplitude after 100 oscillations occurs after  $(200 T/2 + T/4) = (100 T + T/4)$  secs. Therefore, we have

$$a_{201} = a_0 e^{-\lambda(100 T + T/4)}$$

$$\text{Hence } \frac{a_{201}}{a_1} = \frac{a_0 e^{-\lambda(100T + T/4)}}{a_0 e^{-\lambda T/4}}$$

$$= e^{-100\lambda T} = \frac{1}{10}$$

$$\text{or, } e^{-100\lambda (1.15)} = 10^{-1}$$

$$\text{or, } e^{115\lambda} = 10$$

$$\text{or, } 115\lambda = \log_e 10$$

$$\text{or, } 115\lambda = 2.30$$

$$\text{or, } \lambda = 0.02.$$

Thus the damping constant = **0.02**.

$$\text{The relaxation time } \tau = \frac{1}{2\lambda} = \frac{1}{2 \times 0.02} = \mathbf{25 \text{ sec.}}$$

(ii) The first amplitude,

$$a_1 = a_0 e^{-\lambda T/4} = a_0 e^{-\left(0.02 \times \frac{1.15}{4}\right)}$$

$$\text{or, } 2 = a_0 e^{-\frac{0.023}{4}}$$

$$\text{or, } a_0 = 2.e^{\frac{0.023}{4}}$$

$$\text{or, } \log_{10} a_0 = \log_{10} 2 + \frac{0.023}{4} \log_{10} e$$

$$\text{or, } \log_{10} a_0 = 0.3010 + \frac{0.023}{4} \times \frac{1}{2.3}$$

$$\text{or, } a_0 = \mathbf{2.011 \text{ cm.}}$$

Thus the undamped amplitude of the oscillator would be **2.011 cm**.

(iii) Let the amplitude to be reduced to  $1/e^{\text{th}}$  of its undamped value after time  $t$  from the start. Then

$$a = \frac{a_0}{e} = a_0 e^{-1} = a_0 e^{-\lambda t}$$

$$\text{or, } \lambda t = 1 \text{ or, } t = \frac{1}{0.02} = \mathbf{50 \text{ sec.}}$$



The amplitude of the oscillator falls to  $1/e^{\text{th}}$  of its undamped value in **50 sec.**

Similarly from the relation  $E = E_0 e^{-2\lambda t}$ , we have

$$E = \frac{E_0}{e} = E_0 e^{-1} = E_0 e^{-2\lambda t}$$

$$\text{or, } 2\lambda t = 1$$

$$\text{or, } t = \frac{1}{2\lambda} = \frac{1}{2 \times 0.02} = 25 \text{ sec.}$$

Thus the energy of the oscillator falls to  $1/e^{\text{th}}$  of its value in the absence of damping in **25 sec.**

The average rate of loss of energy

$$= \frac{E}{\tau} = 2\lambda E.$$

$$\text{Now, } E = \frac{1}{2} m a_0^2 \omega^2 e^{-2\lambda t}$$

$$m = 1.127 \text{ gm}$$

$$a_0 = 2.011 \text{ cm}$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{1.15}$$

$$\lambda = 0.02$$

$$t = 25 \text{ sec.}$$

rate of loss of energy

$$= 2 \times 0.02 \times \frac{1}{2} \times 1.12 \times (2.011)^2 \times \left( \frac{2\pi}{1.15} \right)^2 \times e^{-2(0.02 \times 25)}$$

$$= \frac{0.02 \times 1.127 \times (2.011)^2 \times 4\pi^2}{(1.15)^2 \times e}$$

$$= \frac{0.02 \times 1.127 \times (2.011)^2 \times 4\pi^2}{(1.15)^2 \times 2.7813}$$

$$= 1 \text{ erg/sec.}$$

**Example 3.2.** A massless spring, suspended from a rigid support, carries a mass of 500 gm at its lower end and the system oscillates with a frequency of 5/sec. If the amplitude is reduced to half its undamped value in 20 sec, calculate (i) the force constant of the spring, (ii) the relaxation time of the system and (iii) its quality factor.

(iv) What would be the quality factor of the system if the suspended mass be reduced to 150 gm?

**Soln.**

(i) The force constant  $a$  of the mass-spring system is given by  $w$

$$= \sqrt{\frac{a}{m}}$$

where  $w$  = angular frequency

$$= 2\pi n = 2\pi \cdot 5 = 10\pi \text{ rad / sec.}$$

$m$  = mass of the system = 500 gm.

$$\text{Then, } w^2 = \frac{a}{m}$$

$$\begin{aligned} \text{or, } a &= m \cdot w^2 = 500 \times (10\pi)^2 \\ &= 4.934 \times 10^5 \text{ dynes/cm.} \end{aligned}$$

(ii) The amplitude is reduced to half its undamped value in 20 sec.

or, from  $a = a_0 e^{-\lambda t}$  we have

$$a = \frac{a_0}{2} = a_0 e^{-\lambda \times 20}$$

$$\text{or, } e^{-20\lambda} = \frac{1}{2}$$

$$\text{or, } 20\lambda = \log_e 2$$

$$\text{or, } 20\lambda = 0.693$$

$$\text{or, } \lambda = \frac{0.693}{20}$$

$$\text{The relaxation time } \tau = \frac{1}{2\lambda}$$



$$\therefore \tau = \frac{20}{2 \times 0.693} = 14.44 \text{ sec.}$$

(iii) The quality factor of the system,

$$Q = \frac{w}{2\lambda} = \frac{10\pi \times 20}{2 \times 0.693} = 453.7$$

(iv) The force constant  $a$  remains the same irrespective of the mass suspended from the spring and so does the damping factor  $b$ .

The damping factor  $b = 2\lambda m$

$$= 2 \times \frac{0.693}{20} \times 500$$

$$= 115 \times 0.3010$$

when the mass is reduced to 150 gm, the frequency  $w' = \sqrt{\frac{a}{m'}}$

$$= \sqrt{\frac{5\pi^2 \times 10^4}{150}} = 18.26\pi.$$

$$\text{The relaxation time now, } \tau' = \frac{1}{2\lambda} = \frac{m'}{b} = \frac{150}{115 \times 0.3010}$$

So the quality factor now,

$$Q' = \frac{w'}{2\lambda} = w' \cdot \tau'$$

$$= \frac{18.26\pi \times 150}{115 \times 0.3010} = 248.5.$$

**Example 3.3.** Deduce the frequency and quality factor for a circuit with  $L = 2 \text{ mH}$ ,  $C = 5 \mu\text{F}$  and  $R = 0.2 \text{ ohm}$ .

**Soln.**

The circuit is equivalent to a damped (or underdamped) harmonic oscillator. The angular frequency of oscillation is given by  $g = \sqrt{w^2 - \lambda^2}$  where  $w$  is the natural angular frequency of the oscillator and  $\lambda$  is the damping constant. Hence the frequency of oscillation

$$n = \frac{g}{2\pi} = \frac{\sqrt{w^2 - \lambda^2}}{2\pi}$$

$$\text{or, } n = \frac{1}{2\pi} = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

Here  $L = 2\text{mH} = 0.002$  henry

$C = 5\mu\text{F} = 5 \times 10^{-6}$  farad

$R = 0.2$  ohm.

$$\therefore n = \frac{1}{2\pi} \sqrt{\frac{1}{0.002 \times 5 \times 10^{-6}} - \frac{(0.2)^2}{4 \times (0.002)^2}}$$

$$= \frac{1}{2\pi} \sqrt{\frac{4 \times 10^8 - 10^4}{4}}$$

$$= \frac{10^2}{2\pi} \sqrt{10^4 - 1} = \frac{10^4}{2\pi}$$

$$= 1.592 \times 10^3 \text{ cps.}$$

Since the resistance is small, the quality factor  $Q = \frac{L\omega}{R} = \frac{Lg}{R}$

where  $g = 2\pi n$

$$= \frac{2\pi \times 10^4}{2\pi} = 10^4 \text{ rad/sec.}$$

$\therefore$  Quality factor for the circuit,

$$Q = \frac{0.002 \times 10^4}{0.2} = 0.01 \times 10^4 = 100.$$

**Example 3.4.** A particle of mass 3 gm is subjected to an elastic force of 48 dyne -  $\text{cm}^{-1}$  and a damping force of 12 dyne -  $\text{cm}^{-1} \text{ sec}$ . If the motion is oscillatory, find its period.

**Soln.**

$$\omega = \sqrt{\frac{a}{m}} = \sqrt{\frac{48 \text{ dyne} - \text{cm}^{-1}}{3 \text{ gm}}} = 4 \text{ sec}^{-1}$$



$$\text{and } 2\lambda = \frac{b}{m} ; \text{ or, } \lambda = \frac{b}{2m}$$

$$= \frac{12}{2 \times 3} = 2 \text{ sec}^{-1}.$$

since  $w > \lambda$ , the motion is oscillatory.

Frequency of the oscillatory motion,

$$\frac{g}{2\pi} = \frac{\sqrt{w^2 - \lambda^2}}{2\pi}$$

Hence the period of the motion

$$= \frac{1}{g/2\pi} = \frac{2\pi}{\sqrt{w^2 - \lambda^2}}$$

$$= \frac{2\pi}{\sqrt{4^2 - 2^2}} = \frac{2\pi}{\sqrt{12}}$$

$$= 1.81 \text{ sec. approx.}$$

**Example 3.5.** Suppose a tuning fork in air has a frequency  $g/2\pi = 200$  cps and its oscillations die away to  $1/e$  of its former amplitude in one second. Show that the reduction in frequency brought about by air damping is exceedingly small.

**Soln.**

For damped vibrations,

$$\frac{g}{2\pi} = \frac{\sqrt{w^2 - \lambda^2}}{2\pi}$$

Let the amplitude at time  $t$  be given by

$$y = Ae^{-\lambda t}$$

Then the amplitude after one second,

$$y' = \frac{y}{e} = Ae^{-\lambda(t+1)} \quad (\text{ii})$$

Dividing (ii) by (i)

$$\frac{1}{e} = e^{-\lambda}; \quad \text{or, } e^{-1} = e^{-\lambda}$$

$$\text{or, } \lambda = 1$$

$$\text{Hence, } \frac{g}{2\pi} = 200 = \frac{\sqrt{w^2 - 1}}{2\pi}$$

Therefore, the value of  $w$  is

$$w = [(2\pi \times 200)^2 + 1]^{1/2}$$

$$\text{Hence } \frac{w}{2\pi} = [(200)^2 + \frac{1}{4\pi^2}]^{1/2}$$

Since  $\frac{1}{4\pi^2}$  is negligible as compared to  $(200)^2$ , damping due to air has only a negligible effect on the frequency of the tuning fork.

### 3.6 Forced vibration

The time period of a body executing simple harmonic motion depends upon the dimensions of the body and its elastic properties. When the body oscillates in a medium like air, its oscillations, as we know get damped, i.e., its amplitude falls exponentially with time to zero due to dissipation of energy.

If, however, an external periodic force is constantly applied to the oscillator, of a frequency not necessarily the same as the natural frequency of the oscillator, a sort of tussle ensues between the damping force and the applied force. The damping force tends to retard the motion of the body and the applied force tends to maintain it. Initially, the amplitude of the oscillations increases, then decreases with time, becomes minimum and again increases. After some initial erratic movements, the body ultimately succumbs to the applied or driving force and settles down to oscillating with the frequency of the applied or driving force and a constant amplitude and phase so long as the applied force remains operative. Such vibrations of the body are called forced (or driven) vibrations. The amplitude of this forced vibration of the body depends on the difference between the natural frequency and the frequency of the



applied force. The amplitude will be large if difference in frequencies is small and vice versa.

Let the periodic force which is applied on a damped harmonic oscillator be  $F = F_0 \sin pt$ . This is obviously a sinusoidal force of amplitude  $F_0$  and frequency  $p/2\pi$ .

Now the damping and restoring forces on the oscillator are  $-b \frac{dy}{dt}$  and  $-ay$  respectively where  $b$  and  $a$  have the same meaning as in Art. 3.1. Hence the equation of motion for forced vibration may be written as

$$\begin{aligned} m \frac{d^2 y}{dt^2} &= -b \frac{dy}{dt} - ay + F \\ \text{or, } m \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + ay &= F_0 \sin pt \\ \text{or, } \frac{d^2 y}{dt^2} + 2\lambda \frac{dy}{dt} + w^2 y &= f_0 \sin pt \end{aligned} \quad (3.26)$$

$$\text{where } 2\lambda = \frac{b}{m}, \quad w^2 = \frac{a}{m} \text{ and } f_0 = \frac{F}{m}$$

After the initial tussle between the damping and the applied force dies down, a steady state is attained and the oscillator settles down to oscillate with the frequency  $p/2\pi$  of the applied force and a constant amplitude. Let a particular solution of the equation of motion of the body as given by eqn. (3.26), after the steady state has been attained, be

$$y = A \sin (pt - \theta) \quad (3.27)$$

where  $A$  is the amplitude and  $\theta$  is the possible phase difference between the applied force and the displacement of the oscillator.

Then, we have

$$\frac{dy}{dt} = Ap \cos (pt - \theta) \quad (3.28)$$

$$\text{and } \frac{d^2 y}{dt^2} = -Ap^2 \sin (pt - \theta) = -p^2 y \quad (3.29)$$

Substituting these values of  $\frac{dy}{dt}$  and  $\frac{d^2y}{dt^2}$  in eqn. (3.26), we get

$$\begin{aligned} -Ap^2 \sin(pt - \theta) + 2\lambda Ap \cos(pt - \theta) + w^2 A \sin(pt - \theta) &= f_0 \sin pt \\ &= f_0 \sin[(pt - \theta) + \theta] \\ &= f_0 \sin(pt - \theta) \cos \theta + f_0 \cos(pt - \theta) \sin \theta \\ \text{or, } A(w^2 - p^2) \sin(pt - \theta) + 2\lambda Ap \cos(pt - \theta) \\ &= f_0 \cos \theta \sin(pt - \theta) + f_0 \sin \theta \cos(pt - \theta) \quad (3.30) \end{aligned}$$

If this solution is to hold good for all values of  $t$ , the respective coefficients of  $\sin(pt - \theta)$  and  $\cos(pt - \theta)$  on either side of eqn. (3.30) must be equal. Thus we must have

$$A(w^2 - p^2) = f_0 \cos \theta \quad \dots\dots\dots (i)$$

$$\text{and } 2\lambda Ap = f_0 \sin \theta \quad \dots\dots\dots (ii)$$

Squaring and adding eqns. (i) and (ii), we have

$$A^2(w^2 - p^2)^2 + 4\lambda^2 A^2 p^2 = f_0^2 \cos^2 \theta + f_0^2 \sin^2 \theta$$

$$\text{or, } A^2[(w^2 - p^2)^2 + 4\lambda^2 p^2] = f_0^2$$

$$\text{or, } A^2 = \frac{f_0^2}{(w^2 - p^2)^2 + 4\lambda^2 p^2}$$

Thus the amplitude of the driven or forced oscillator is

$$A = \frac{f_0}{\sqrt{(w^2 - p^2)^2 + 4\lambda^2 p^2}} \quad (3.31)$$

Only the positive value of the square root has been considered in eqn. (3.31). Its negative value will mean opposite phase but then  $\theta$  too will change by  $\pi$ .

Thus there will be no effect on the value of  $A$ .

The value of  $\theta$  is given by the expression



$$\tan \theta = \frac{f_0 \sin \theta}{f_0 \cos \theta} = \frac{2\lambda A p}{A(w^2 - p^2)} = \frac{2\lambda p}{(w^2 - p^2)}$$

Thus the phase difference between the driven or forced oscillator and the applied force is

$$\theta = \tan^{-1} \left( \frac{2\lambda p}{(w^2 - p^2)} \right) \quad (3.32)$$

Since  $\sin \theta$  is positive, it follows that  $\theta$  must lie between 0 to  $\pi$ .

Substituting these value in eqn. (3.27) we get

$$y = \frac{f_0}{\sqrt{(w^2 - p^2)^2 + 4\lambda^2 p^2}} \times \sin \left[ pt - \tan^{-1} \left( \frac{2\lambda p}{(w^2 - p^2)} \right) \right] \quad (3.33)$$

Eqn. (3.33) represents a S.H.M. of frequency  $p/2\pi$  i.e., the same as that of the driving force, but lagging behind it in phase by  $\theta = \tan^{-1} [2\lambda p/(w^2 - p^2)]$  where  $\theta$  lies between 0 and  $\pi$ .

Now, because of the presence of the term  $f_0 \sin pt$  which contains neither the variable ( $y$ ) or its derivative; eqn. (3.26) is an inhomogeneous differential equation and hence  $y = A \sin (pt - \theta)$  is not really its complete solution. The solution will, therefore, be complete only if we add to it a complementary function which is a solution of the related homogeneous equation  $\frac{d^2 y}{dt^2} + 2\lambda \frac{dy}{dt} + w^2 y = 0$

One such solution, as we know, is

$$y = a_0 e^{-\lambda t} \sin (gt + \phi) \text{ which represents a damped harmonic oscillation}$$

of frequency  $\frac{g}{2\pi} = \frac{\sqrt{w^2 - \lambda^2}}{2\pi}$ . The addition of this term to the solution  $y = A \sin (pt - \theta)$  does not, in any way, impair the validity of the latter, since the term, taken by itself, reduces the left hand side of eqn. (3.26) to zero.

Thus the complete solution of eqn. (3.26) is

$$y = a_0 e^{-\lambda t} \sin(gt + \phi) + A \sin(pt - \theta)$$

where, the first term on the right hand side represents an initial damped oscillation of frequency  $g/2\pi$  with its amplitude decaying exponentially to zero while the second term represents a forced vibration of frequency  $p/2\pi$  and a constant amplitude  $A$ . The former (damped) oscillation dies out quickly and the latter alone then remains effective, so that one is left with  $y = A \sin(pt - \theta)$  as the equation of motion of the forced (or driven) oscillation, with its amplitude  $A$  and phase angle  $\theta$  as given by eqns. (3.31) and (3.32) respectively.

### Maximum displacement of a driven oscillator — Resonance.

Although the period of the forced vibration will eventually be same as that of the applied force, its amplitude, irrespective of the magnitude of the damping forces present, will depend on the ratio between its natural frequency and the frequency of the applied force. However, the amplitude will have a definite value for a particular amount of damping. This means that if the damping (resisting) forces acting on the body are changed without changing the ratio between its natural frequency and the frequency of the applied force, the amplitude will change due only to change in damping.

When the damping *i.e.*,  $\lambda$  has a finite value greater than zero, the value of the amplitude, as given by eqn. (3.31) will obviously be maximum when the denominator in the equation has its minimum value; *i.e.*, when

$$\frac{d}{dp} [(w^2 - p^2)^2 + 4\lambda^2 p^2] = 0$$

$$\text{or, } -2(w^2 - p^2) \cdot 2p + 4\lambda^2 2p = 0$$

$$\text{or, } (w^2 - p^2) = 2\lambda^2$$

$$\text{or, } p^2 = w^2 - 2\lambda^2$$

$$\text{or, } p = \sqrt{w^2 - 2\lambda^2}$$



Thus, the amplitude will be maximum when the driving frequency  $p/2\pi = \frac{\sqrt{w^2 - 2\lambda^2}}{2\pi}$ . Let us denote this particular value of the frequency by  $p_r/2\pi$ . This state of vibration when the amplitude of the driven oscillator becomes maximum is called *resonance* (or *amplitude resonance*). The particular forced frequency ( $p_r/2\pi$ ) for which resonance takes place is called *resonant frequency*.

In the absence of damping, resonance takes place when the natural frequency of the oscillator is equal to the frequency of the driving force. But when damping is present, the resonant frequency

$p_r/2\pi = \left(\sqrt{w^2 - 2\lambda^2}\right)/2\pi$  is smaller than the natural, undamped frequency  $w/2\pi$  or the natural, damped frequency  $g/2\pi = \left(\sqrt{w^2 - \lambda^2}\right)/2\pi$  of the oscillator.

Substituting  $p_r = \sqrt{w^2 - 2\lambda^2}$  for  $p$  in the expression for  $A$  as given by eqn. (3.31), we have

maximum amplitude,

$$\begin{aligned} A_{\max} &= \frac{f_0}{\sqrt{(w^2 - p^2) + 4\lambda^2 p^2}} \\ &= \frac{f_0}{\sqrt{[w^2 - (w^2 - 2\lambda^2)]^2 + 4\lambda^2 (w^2 - 2\lambda^2)}} \\ &= \frac{f_0}{\sqrt{4\lambda^4 + 4\lambda^2 w^2 - 8\lambda^4}} \\ &= \frac{f_0}{\sqrt{4\lambda^2 w^2 - 4\lambda^4}} \\ &= \frac{f_0}{\sqrt{4\lambda^2 (w^2 - \lambda^2)}} \\ &= \frac{f_0}{2\lambda\sqrt{w^2 - \lambda^2}} = \frac{f_0}{2\lambda(w^2 - \lambda^2)^{1/2}} \quad (i) \end{aligned}$$

If there is no damping, i.e., if  $\lambda = 0$ , then the amplitude  $A_{\max}$  tends to infinity. This, however, never happens since damping is never actually zero. When it is very small  $\lambda^2$  will be negligibly small compared to  $w^2$  and the maximum amplitude becomes  $A_{\max} = f_0/2\lambda w$ . Hence smaller the value of  $\lambda$  larger the value of the maximum amplitude. Fig. 3.7 illustrates this point.

Further, it is clear from the expression for  $A$  itself (eqn. 3.31) that the magnitude of  $A$  is controlled by the factor  $(w^2 - p^2)$  i.e.,  $A$  depends upon the relative values of  $p$  and  $w$ . Thus the value of  $A$  diminishes both when  $p > w$  and when  $p < w$ . Fig. 3.7 shows a number of curves plotted with amplitude  $A$  along the Y-axis and  $p$  along the X-axis. Each curve corresponds to a given value of damping ( $\lambda$ ). It will be noted that,

(i) The peak value of amplitude  $A_{\max} = f_0/2\lambda w$  occurs when  $(w^2 - p^2) = 0$  i.e., when  $p = w$ , which represents the condition for amplitude resonance. If, in addition there is no damping i.e.,  $\lambda = 0$ ,  $A_{\max}$  becomes infinity for this value of  $p$ , the curve being asymptotic to the Y- or the amplitude axis. Since damping is never zero, we do not obtain such a curve in actual practice but only those of the type shown for light and moderate damping.

(ii) The maximum value of  $A$  is different for different degree of damping, becoming less and less as damping increases. But, provided the damping is not too large, the maximum value of  $A$  always occurs at, or very nearly at  $(w^2 - p^2) = 0$ . This means that the resonant frequency  $p_r$  in all these cases is equal to or very nearly equal to  $w$ .

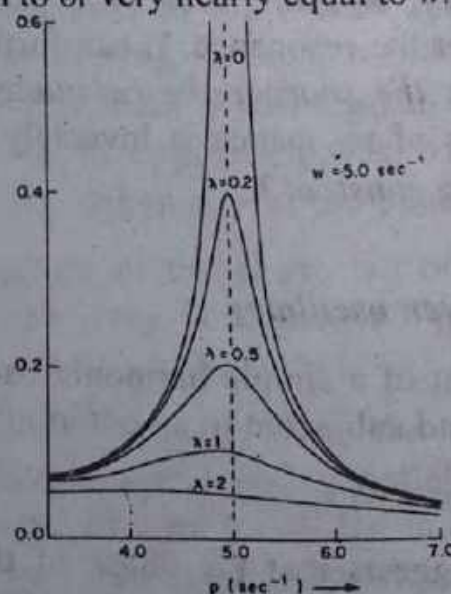


Fig. 3.7



### 3.7. Quality factor

The ratio of the response (or amplitude) of the oscillator when the driving frequency is equal to the resonant frequency to the response when the driving frequency is zero or negligibly small is called the quality factor of the oscillator and is denoted by  $Q$ . Hence

$$Q = \frac{f_0 / 2\lambda w}{f_0 / w^2} = \frac{w}{2\lambda} \quad (3.34)$$

### 3.8 Sharpness of resonance

Referring to Fig. 3.7 it can be seen that the curve between  $p$  and  $A$  fall more steeply on either side of the respective peak values of  $A$  when the damping is low than when it is high. Thus when the damping is large, the amplitude remains more or less at its peak value over an appreciable range of  $p$  on either side of  $(w^2 - p^2) = 0$  i.e. even when  $p \neq w$ . The oscillator thus responds to a number of frequencies near about  $w$  on either side of it. The response in this case is, therefore, said to be *flat*.

On the other hand, if the damping is low the steep fall of the curve on either side of the peak value of  $A$  shows that the oscillator responds only to the frequency exactly equal to its natural frequency  $w$  and to none others. The resonance in this case is said to be *sharp*.

Thus sharpness of resonance may be regarded, in a way, as a measure of the rate of fall of amplitude from its maximum value at the resonant frequency, on either side of it. The sharper the fall in amplitude, the sharper the resonance. It can further be seen, that the smaller the damping, the sharper the resonance. In fact it can be shown that sharpness of resonance is inversely proportional to the square of the damping constant  $\lambda$ .

### 3.9 Phase of the driven oscillator

The displacement of a simple harmonic oscillator vibrating in a damping medium and subjected to an external force is given by

$$y = A \sin (pt - \theta)$$

The equation suggests that the phase of the driven oscillator lags behind the driven force by an angle given by

$$\theta = \tan^{-1} \left( \frac{2\lambda p}{w^2 - p^2} \right)$$

Obviously the phase angle  $\theta$  depends upon damping and the relative values of  $w$  and  $p$ . The following three cases arise :

(i) when  $p < w$  i.e., driving frequency is less than the natural, undamped frequency of the oscillator :

$\tan \theta$  will then be a positive quantity and the value of  $\theta$  lies between 0 and  $\pi/2$  for all values of  $\lambda$ .

(ii) when  $p > w$  i.e., the driving frequency is greater than the natural, undamped frequency of the oscillator :

$\tan \theta$  will then be a negative quantity and the value of  $\theta$  lies between  $\pi/2$  and  $\pi$  for all values of  $\lambda$ .

(iii) when  $p = w$  i.e., the driving frequency is equal to the natural undamped frequency of the oscillator :

In this case  $\tan \theta = \infty$  for all values of  $\lambda$  and therefore  $\theta = \pi/2$ , i.e., the driven oscillator will differ in phase from the driving force by  $\pi/2$ .

It is clear from the above discussion that the phase angle  $\theta$  changes from 0 to  $\pi$  but remains *positive* throughout. Further if the damping is negligible and  $p < w$ ,  $\theta$  will tend to be equal to zero and the displacement of the driven oscillator will be in the same phase with the driving force. If on the other hand damping is negligible but  $p > w$ , then  $\theta$  will tend to be equal to  $\pi$  and the displacement will be in opposite phase with the driving force. During resonance, i.e., when  $p = w$ , the phase difference is  $\pi/2$ .

The general nature of the phase lag of the driven oscillator with the driving frequency is shown in Fig. 3.8. The following points emerge :

(i) when the damping is zero, the curve runs along the frequency axis from 0 to  $w$  and again parallel to it from  $w$  to  $2w$  but removed from first part by (curve ABCDE in Fig. 3.8). Thus for zero damping  $\theta = 0$  when  $p < w$  but suddenly jumps to  $\theta = \pi$  when  $p > w$ .



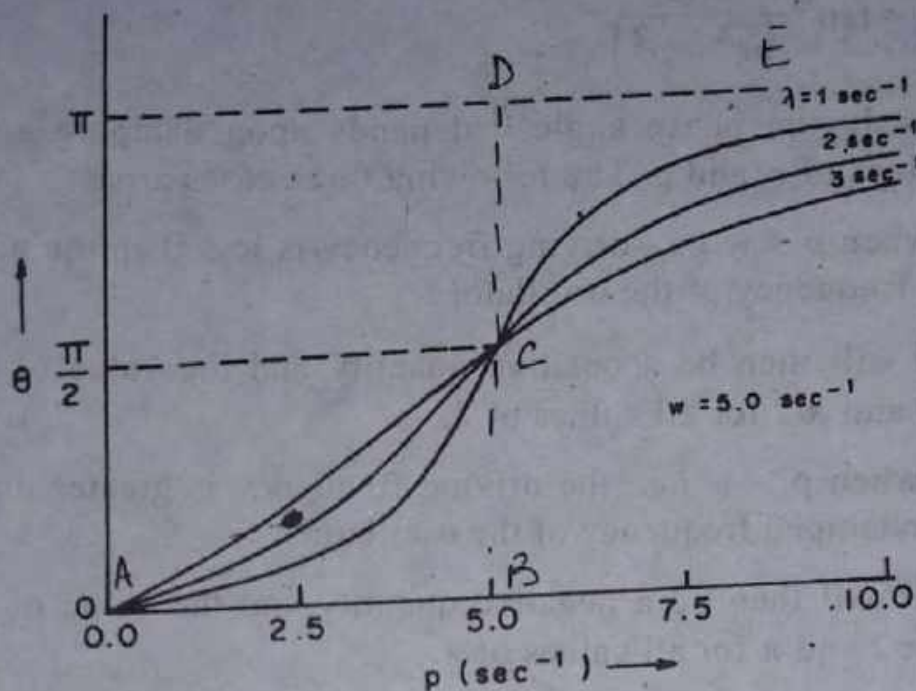


Fig. 3.8

(ii) when damping is present, the phase lag ( $\theta$ ) increases from 0 to  $\pi/2$  as the driving frequency  $p$  increases from 0 to  $w$ , and then approaches  $\pi$  as  $p$  continues to increase beyond  $w$ .

(iii) The phase angle  $\theta$  varies more rapidly when the damping is low than when it is high.

(iv) Except when the damping is very high, all curves pass through  $\pi/2$  when  $p = w$ . This means that amplitude resonance occurs in all these cases when the driving frequency is equal to the natural undamped frequency of the oscillator, i.e.,  $p = w$ .

### 3.10 Power absorption

In many important problem on driven harmonic oscillators, one is interested not so much in the displacement of the oscillator from its equilibrium position as in the energy absorbed by the oscillator to keep itself in motion. We shall, therefore, calculate the average power absorbed per cycle by the driven oscillator to offset its loss of power in overcoming the frictional or resistive force and thus to maintain its oscillations.

When a driven harmonic oscillator settles down to a steady state of oscillations, the average power absorbed by it will obviously be just equal to the average power dissipated.

Let  $F = F_0 \sin pt$  be the applied or the driving force and  $dy$ , the displacement of the oscillator in time  $dt$ . Then the work done or the energy supplied by the applied force to the oscillator is given by

$$dE = F \cdot dy = F \cdot \frac{dy}{dt} \cdot dt$$

$$\text{or, } \frac{dE}{dt} = F \cdot \frac{dy}{dt} \quad (3.35)$$

$$\text{Now } y = A \sin (pt - \theta)$$

$$= \frac{f_0}{\sqrt{(w^2 - p^2)^2 + 4\lambda^2 p^2}} \sin (pt - \theta)$$

Hence,

$$\frac{dy}{dt} = \frac{pf_0}{\sqrt{(w^2 - p^2)^2 + 4\lambda^2 p^2}} \cos (pt - \theta)$$

$$\text{or, } v = v_0 \cos (pt - \theta) \quad (3.36)$$

where  $v_0 = \frac{f_0}{\sqrt{(w^2 - p^2)^2 + 4\lambda^2 p^2}}$  is referred to as the *velocity amplitude*.

[ Eqn. 3.36. can be written as

$$v = v_0 \sin \left[ (pt - \theta) + \frac{\pi}{2} \right]$$

showing that the velocity leads the displacement in phase by  $\pi/2$ . Further, the velocity amplitude  $v_0$  varies with  $p$ , being zero when  $p = 0$  and maximum when  $p = w$ , the maximum value being  $f_0/2\lambda$ .

It may be mentioned here that in analogy with amplitude resonance, the acquiring of maximum velocity by a driven oscillator is referred to as velocity resonance. Obviously the velocity resonance



occurs when the driving frequency  $p$  is equal to  $w$ , the natural undamped frequency of the oscillator,  $p$  is then referred to as  $p_r$ , the resonant frequency.]

From eqn. 3.26,

$$F = F_0 \sin pt = mf_0 \sin pt \quad (3.37)$$

Substituting these values of  $F$  and  $\frac{dy}{dt}$  in the expression for  $P$  as given by eqn. (3.35), we have

Power absorbed by the oscillator,

$$\begin{aligned} P &= \frac{dE}{dt} = F \cdot \frac{dy}{dt} \\ &= mf_0 \sin pt \cdot \frac{pf_0 \cos(pt - \theta)}{\sqrt{(w^2 - p^2)^2 + 4\lambda^2 p^2}} \\ &= \frac{mf_0^2 p}{\sqrt{(w^2 - p^2)^2 + 4\lambda^2 p^2}} \sin pt \cos (pt - \theta) \end{aligned}$$

The average power absorbed ( $P_{av}$ ), over a complete time-period or full cycle is

$$\begin{aligned} &\frac{1}{T} \int_0^T \frac{mf_0^2 p}{\sqrt{(w^2 - p^2)^2 + 4\lambda^2 p^2}} \times \sin pt \cos (pt - \theta) dt \\ &= \frac{mf_0^2 p}{\sqrt{(w^2 - p^2)^2 + 4\lambda^2 p^2}} \cdot \frac{1}{T} \times \int_0^T \sin pt \cos (pt - \theta) dt \quad (3.38) \end{aligned}$$

$$\text{Now } \frac{1}{T} \int_0^T \sin pt \cos (pt - \theta) dt$$

$$= \frac{1}{T} \int_0^T \frac{1}{2} [\sin (2pt - \theta) - \sin \theta] dt$$

$$= \frac{1}{2} \sin \theta$$

Thus the average power,  $P_{av}$ , over a complete full cycle is given by

$$P_{av} = \frac{mf_0^2 p}{\sqrt{(w^2 - p^2)^2 + 4\lambda^2 p^2}} \cdot \frac{1}{2} \sin \theta \quad (3.39)$$

Again, from eqns. 3.30(ii) and 3.31 we get  $f_0 \sin \theta = 2\lambda Ap$ .

$$\text{and } A = \frac{f_0}{\sqrt{(w^2 - p^2)^2 + 4\lambda^2 p^2}}$$

Hence

$$\begin{aligned} \sin \theta &= \frac{2\lambda Ap}{f_0} \cdot \frac{f_0}{\sqrt{(w^2 - p^2)^2 + 4\lambda^2 p^2}} \\ &= \frac{2\lambda p}{\sqrt{(w^2 - p^2)^2 + 4\lambda^2 p^2}} \end{aligned}$$

Thus, the average power over a complete cycle,

$$\begin{aligned} P_{av} &= \frac{mf_0^2 p}{\sqrt{(w^2 - p^2)^2 + 4\lambda^2 p^2}} \times \frac{\lambda p}{\sqrt{(w^2 - p^2)^2 + 4\lambda^2 p^2}} \\ &= \frac{m\lambda f_0^2 p^2}{(w^2 - p^2)^2 + 4\lambda^2 p^2} \end{aligned}$$

$$\text{But } \frac{f_0^2 p^2}{(w^2 - p^2)^2 + 4\lambda^2 p^2} = v_0^2$$

where  $v_0$  is the velocity amplitude.

Therefore, the average power absorbed by the driven oscillator per cycle or per complete time period,

$$P_{av} = m\lambda v_0^2 \quad (3.40)$$



It is clear from relation (3.38) above, that the average power absorbed per cycle will have its maximum value, when  $p = w$ , so that,

$$P_{\max} = \frac{1}{4} \cdot \frac{mf_0^2}{\lambda} \quad (3.41)$$

Thus the resonant frequency for the average power absorbed *i.e.*, the frequency at which the value of the average power absorbed becomes maximum, turns out to be the same as the resonant frequency for velocity. Or,  $p = p_r = w$ .

It should be noted that the power resonance takes place at  $p_r = w$  whereas the amplitude resonance takes place at  $p_r = \sqrt{w^2 - 2\lambda^2}$

### 3.11 Driven LCR circuit

An LCR circuit, *i.e.*, a circuit containing an inductance  $L$ , a capacitance  $C$  and a resistance  $R$  connected in series acts as a damped oscillator (Art. 3.5). If an external source of alternating electro-motive force (*emf*) is included in the circuit, also in series, the circuit now acts as a driven oscillator. The source of *emf* supplies the necessary energy to maintain the oscillation.

Let  $i$  be the current in the circuit at a given instant and  $Q$  the charge on the capacitor at that instant. Then the potential drop across the resistance is  $Ri$ , that across the inductance is  $-L \frac{di}{dt}$  and that across the capacitance is  $\frac{Q}{C}$ .

If  $E = E_0 \sin pt$  be the external *emf*, impressed on the circuit, then we have

$$\begin{aligned} E - L \frac{di}{dt} &= \frac{Q}{C} + Ri \\ \text{or, } L \frac{di}{dt} + Ri + \frac{Q}{C} &= E = E_0 \sin pt \quad (3.41) \\ \text{or, } \frac{di}{dt} + \frac{R}{L} i + \frac{Q}{LC} &= \frac{E_0}{L} \sin pt \end{aligned}$$

Since  $i = \frac{dQ}{dt}$ , the above relation may be written as

$$\frac{d^2Q}{dt^2} + \frac{R}{L} \frac{dQ}{dt} + \frac{Q}{LC} = \frac{E_0}{L} \sin pt \quad (3.42)$$

Eqn. (3.41) is identical in form with eqn. (3.26), with  $Q$  replacing  $y$  and  $\frac{R}{L}$ ,  $\frac{Q}{LC}$  and  $\frac{E_0}{L}$  replacing  $2\lambda$ ,  $w^2$  and  $f_0$  respectively.

Similarly, the steady state solution of eqn. (3.42) is

$$Q = \frac{E_0/L}{\sqrt{\left(\frac{1}{LC} - p^2\right)^2 + \left(\frac{pR}{L}\right)^2}} \sin (pt - \theta)$$

$$\text{where } \theta = \tan^{-1} \frac{pR/L}{1/LC - p^2}$$

The value of the current flowing through the circuit can be easily obtained by differentiating this expression for  $Q$  with respect to  $t$ . Alternately, however, the value of the current can be obtained directly in the manner described below.

Let  $i = i_0 \sin (pt - \theta)$  be a trial solution of eqn. (3.41).

Then

$$\frac{di}{dt} = i_0 p \cos (pt - \theta)$$

$$\text{and } Q = \int i dt = \int i_0 \sin (pt - \theta) \cdot dt = -\frac{i_0}{p} \cos (pt - \theta)$$

Substituting these values in eqn. (3.40), we get

$$L p i_0 \cos (pt - \theta) + R i_0 \sin (pt - \theta) - \frac{i_0}{Cp} \cos (pt - \theta) = E_0 \sin pt$$

$$\text{or, } i_0 [R \sin (pt - \theta) + (Lp - 1/Cp) \cos (pt - \theta)] = E_0 \sin pt \quad (3.43)$$

Putting  $R = a \cos \theta$  and  $(Lp - 1/Cp) = a \sin \theta$ , we have



$$a = \sqrt{R^2 + \left(Lp - \frac{1}{Cp}\right)^2} \text{ and } \tan \theta = \frac{Lp - 1/Cp}{R}$$

Substituting these values of  $R$  and  $(Lp - 1/Cp)$  in relation (3.43) above, we have

$$i_0 a [\cos \theta \sin (pt - \theta) + \sin \theta \cos (pt - \theta)] = E_0 \sin pt$$

$$\text{or, } i_0 a \sin [(pt - \theta) + \theta] = i_0 a \sin pt = E_0 \sin pt$$

$$\text{or, } i_0 \cdot \sqrt{R^2 + (Lp - 1/Cp)^2} \cdot \sin pt = E_0 \sin pt$$

From the above relation we find that

$$i_0 \sqrt{R^2 + (Lp - 1/Cp)^2} = E_0$$

$$\text{or, } i_0 = \frac{E_0}{\sqrt{R^2 + (Lp - 1/Cp)^2}} \quad (3.44)$$

$$\therefore i = i_0 \sin (pt - \theta)$$

$$= \frac{E_0}{\sqrt{R^2 + (Lp - 1/Cp)^2}} \sin (pt - \theta) \quad (3.45)$$

$$\text{where } \theta = \tan^{-1} \frac{Lp - \frac{1}{Cp}}{R}$$

It can be readily seen from eqn. (3.44) that the denominator  $\sqrt{R^2 + (Lp - 1/Cp)^2}$  functions as the effective resistance in the circuit. It is called the *impedance* of the circuit and denoted by the letter  $Z$  and measured in *ohms*.

The impedance itself is made up to two parts, the ohmic resistance  $R$  and the quantity  $(Lp - 1/Lp)$ , called the *reactance*. The reactance usually denoted by  $X$ , is also measured in ohms although  $L$  and  $C$  are individually measured in henry and farad respectively. The relation between resistance, reactance, impedance and the phase angle  $\theta$  can best be illustrated as in Fig. 3.9.

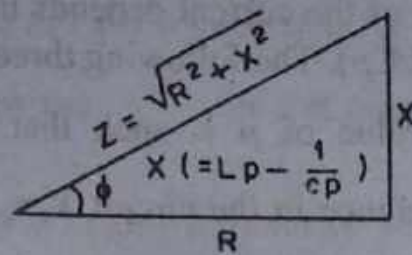


Fig. 3.9

The reactance too is made up of two parts :  $Lp$ , the reactance due to inductance (*inductive reactance*) and  $1/Cp$ , the reactance due to capacitance (*capacitive reactance*). Inductive and capacitive reactances are usually denoted by  $X_L$  and  $X_C$  respectively.

Hence  $X = X_L + X_C$

Thus, we have

$$Z = \sqrt{R^2 + X^2}$$

or, *Impedance*

$$= \sqrt{\text{Resistance}^2 + \text{Reactance}^2}$$

The maximum (or peak) value of the current is given by

$$i_0 = \frac{E_0}{Z} = \frac{E_0}{\sqrt{R^2 + X^2}}$$

Eqn. (3.45) can, therefore, be written as

$$i = \frac{E_0}{Z} \sin (pt - \theta) \\ = i_0 \sin (pt - \theta)$$

$$\text{and } \theta = \tan^{-1} \frac{Lp - 1/Cp}{R} = \tan^{-1} \frac{X_L - X_C}{R}$$

is the difference in phase between the current and the *emf* in the circuit. The ohmic resistance is quite independent of the frequency and, as such, has no effect on the phase angle. Hence the phase angle is determined by the reactance ( $X_L - X_C$ ). In an inductive circuit the *emf* leads the current whereas in a capacitive circuit the current leads the *emf*. When both inductance and capacitance are present in a circuit, the



lead or lag of the emf, or the current depends upon the relative values of  $X_L$  (or  $Lp$ ) and  $X_C$  (or  $1/Cp$ ). The following three cases then arise :

(i) When the value of  $p$  is such that  $X_L = X_C$ . In this case  $Lp = \frac{1}{Cp}$  ; or the reactance in the circuit  $X = X_L - X_C = 0$ , so that the impedance in the circuit = resistance  $R$ . This is the least value of the impedance and hence the value of the current in the circuit will be maximum.

$$\text{or, } i_0 = \frac{E_0}{R}$$

When current in the circuit becomes maximum, we have the case of resonance. From the condition of resonance we have

$$Lp = \frac{1}{Cp} ; \text{ or, } p^2 = \frac{1}{LC}$$

$$\text{or, } p = \frac{1}{\sqrt{LC}} = w, \text{ the natural frequency of the circuit.}$$

$$\begin{aligned} \text{Hence, the resonance frequency} &= \frac{X_L - X_C}{R} = 0 \\ &= \frac{p}{2\pi} = \frac{w}{2\pi} = \frac{1}{2\pi\sqrt{LC}} \end{aligned}$$

The circuit under this condition is said to be *resonance series circuit*.

Further. Since  $X_L = X_C$ ,

$$\tan \theta = \frac{X_L - X_C}{R} = 0$$

or,  $\theta = 0$ . Thus when  $X_L = X_C$ , the current in the circuit is in phase with the emf.

(ii) when  $p$  has a value such that  $X_L > X_C$  :

since  $Lp > \frac{1}{Cp}$ , the net reactance in the circuit is inductive,  $\tan \theta$  is

now a positive quantity and therefore,  $\theta = \frac{\pi}{2}$ . The applied emf thus

leads the current by  $\frac{\pi}{2}$  and the value of  $p$  is greater than  $w$ .

(iii) when the value of  $p$  is such that  $X_L < X_C$  :

since  $Lp$  is now less than  $\frac{1}{Cp}$ , the net reactance in the circuit is

capacitive ;  $\tan \theta$  is now a negative quantity and therefore,  $\theta = -\frac{\pi}{2}$ .

The current thus leads the applied emf by  $\frac{\pi}{2}$  and the value of  $p$  is less than  $w$ .

The changes in the maximum value of the current  $i_0$  and the value of the phase angle  $\theta$  with  $p$  are shown in Fig. 3.10 respectively, for different values of  $R$ . It can be seen that lower the value of  $R$ , the sharper the resonance, and the current lags or leads the emf according as  $p$  is greater than or less than  $w$ .  $R$  corresponds to  $\lambda$  in the case of mechanical oscillators.

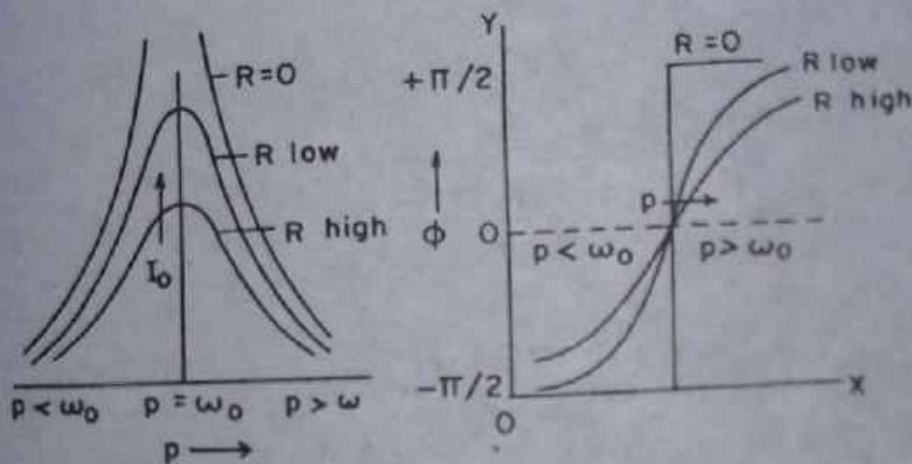


Fig. 3.10

**Example 3.6.** A particle of mass 2 gm is free to vibrate under the action of an elastic force of 128 dyne-cm<sup>-1</sup> and a damping force of 8 dyne-cm<sup>-1</sup> sec. A periodically varying outside force of maximum value 256 dynes is applied to the particle. Find the frequency for displacement resonance and velocity resonance.

**Soln.**

$$\text{frequency} = \frac{p}{2\pi} = \frac{1}{2\pi} \sqrt{w^2 - 2\lambda^2}$$



$$\text{Now } w^2 = \frac{a}{m} = \frac{128 \text{ dyne} - \text{cm}^{-1}}{2 \text{ gm}} \sqrt{w^2 - 2\lambda^2} = 64 \text{ sec}^{-2}$$

$$\text{and } \lambda = \frac{b}{2m} = \frac{8 \text{ dyne} - \text{cm}^{-1} \text{ sec}}{2 \times 2 \text{ gm}} = 2 \text{ sec}^{-2}$$

$$\begin{aligned} \therefore \text{frequency} &= \frac{1}{2\pi} \sqrt{64 - 4 \times 2} \\ &= \frac{1}{2\pi} \sqrt{64 - 8} \\ &= 1.19 \text{ sec}^{-1} \end{aligned}$$

For velocity resonance,

$$\text{frequency} = \frac{1}{2\pi} \sqrt{\frac{a}{m}}$$

$$= \frac{1}{2\pi} \sqrt{\frac{128 \text{ dyne} - \text{cm}^{-1}}{2 \text{ gm}}}$$

$$= \frac{1}{2\pi} \sqrt{64} = \frac{8}{2\pi} = 1.27 \text{ sec}^{-1}$$

**Example 3.7.** In an experiment on forced oscillations, the frequency of a sinusoidal driving force is changed while its amplitude is kept constant. It is found that the amplitude of vibrations is 0.01 mm at very low frequency of the driving force and goes upto a maximum of 5.0 mm at driving frequency 200 cps. Calculate the quality factor and the relaxation time of the system.

**Soln.**

The quality factor of the driven oscillator,

$$Q = \frac{A_{\max}}{A} = \frac{f_o / 2\lambda w}{f_o / w^2}$$

$$= \frac{w}{2\lambda} = w\tau \left( \tau = \frac{1}{2\lambda} \right)$$

$$A_{\max} = 5.0 \text{ mm and } A = 0.01 \text{ mm.}$$

$$\therefore Q = w\tau = \frac{5}{0.01} = 500$$

$$\text{or, } \tau = \frac{Q}{w}$$

$$\text{where } w = 2\pi n = 2\pi \times 200 = 400\pi.$$

$$\therefore \tau = \frac{500}{400\pi} \approx 0.40 \text{ sec.}$$

**Example 3.8.** A harmonic oscillator of quality factor 10 is subjected to a sinusoidal applied force of frequency one and a half times the natural frequency of the oscillator. If the damping be small, obtain (i) the amplitude of the forced oscillation in terms of its maximum amplitude and (ii) the angle  $\theta$  by which it will be out of phase with the driving force.

**Soln.**

The amplitude of a driven oscillator is given by

$$\begin{aligned} A &= \frac{f_0}{\sqrt{(w^2 - p^2)^2 + 4\lambda^2 p^2}} \\ &= \frac{f_0}{\sqrt{w^4 \left(1 - \frac{p^2}{w^2}\right)^2 + w^4 \cdot \frac{4\lambda^2 p^2}{w^4}}} \\ &= \frac{f_0}{w^2 \sqrt{\left(1 - \frac{p^2}{w^2}\right)^2 + 4 \frac{\lambda^2 p^2}{w^4}}} \end{aligned}$$

where  $w$  is the natural frequency of the oscillator and  $p$ , the angular frequency of the driving force.

Now, the quality factor,



$$Q = \frac{w}{2\lambda} = 10; \quad \text{or, } \frac{2\lambda}{w} = \frac{1}{10}$$

$$\text{and } \frac{p}{w} = \frac{3}{2}$$

$$\begin{aligned} \therefore A &= \frac{f_0}{w^2 \sqrt{\left(1 - \frac{9}{4}\right)^2 + \left(\frac{2\lambda}{w}\right)^2 \frac{p^2}{w^2}}} \\ &= \frac{f_0}{w^2 \sqrt{\frac{25}{16} + \frac{1}{100} \times \frac{9}{4}}} \\ &= \frac{f_0}{w^2 \left(\frac{634}{400}\right)^{1/2}} = \frac{20f_0}{w^2 \sqrt{634}} \end{aligned}$$

And, for low damping, amplitude is maximum when  $p = w$

$$\text{or, } \frac{p}{w} = 1. \text{ Then}$$

$$\begin{aligned} A_{\max} &= \frac{f_0}{w^2 \sqrt{(1-1)^2 + \frac{1}{100} \times 1}} \\ &= \frac{f_0}{w^2 \left(\frac{1}{100}\right)^{1/2}} = \frac{10f_0}{w^2} \end{aligned}$$

$$\begin{aligned} \therefore \frac{A}{A_{\max}} &= \frac{20f_0}{w^2 \sqrt{634}} \times \frac{w^2}{10f_0} \\ &= \frac{2}{\sqrt{634}} \approx 0.08 \end{aligned}$$

$$\text{Or, } A = 0.08 A_{\max}$$

Thus, the amplitude of the forced oscillation will be 0.08 times the maximum amplitude.

$$\begin{aligned}\text{Now, } \tan \theta &= \frac{2\lambda p}{w^2 - p^2} \\ &= \frac{p \cdot w / 10}{w^2 - p^2} = \frac{\frac{3w}{2} \cdot \frac{w}{10}}{w^2 - \frac{9w^2}{4}} \\ &= \frac{3w^2}{20} \times \frac{4}{5w^2} = \frac{3}{5} = 0.12.\end{aligned}$$

$$\text{or, } \theta = \tan^{-1} 0.12 = 6^\circ 51'$$

Therefore, the forced oscillation is  $(180^\circ - 6^\circ 51') = 173^\circ 9'$  out of phase with the driving force.

**Example 3.9.** A harmonic oscillator consisting of a 50 gm mass attached to a massless spring has a quality factor 200. If it oscillates with an amplitude of 2 cm in resonance with a periodic force of frequency 20 c.p.s., calculate (i) the average energy stored in it and (ii) the rate of dissipation of energy.

**Soln.**

In case of resonant vibration,

The average energy stored in the oscillator = maximum potential energy.

Now, the maximum potential energy is given by  $\frac{1}{2} a y^2$  where  $a$  is the force constant and  $y$  is the displacement of the oscillator. Since  $a = mw^2$ , the maximum P.E. =  $\frac{1}{2} mw^2 y^2$ .

$$\begin{aligned}\text{Here } m &= 50 \text{ gm, } w = 2\pi n = 2\pi \times 20 \\ &= 40\pi \text{ and } y = 2 \text{ cm.}\end{aligned}$$

$\therefore$  the average energy absorbed or stored up in the oscillator

$$= \frac{1}{2} \times 50 \times (40\pi)^2 \times 22$$



$$= 16 \times 10^4 \times \pi^2$$

$$= 1.58 \times 10^6 \text{ ergs.}$$

By definition, the quality factor

$$Q = 2\pi \cdot \frac{\text{average energy stored}}{\text{energy dissipated per cycle}}$$

$\therefore$  energy dissipated per cycle

$$= 2\pi \cdot \frac{\text{average energy stored}}{Q}$$

$$= 2\pi \cdot \frac{1.58 \times 10^6}{200}$$

$$= \pi \times 1.58 \times 10^4 \text{ ergs.}$$

Since there are 20 cycles to the second, energy dissipated per second or the rate of dissipation of energy

$$= 20\pi \times 1.58 \times 10^4$$

$$= 9.926 \times 10^5 \approx 10^6 \text{ erg/sec.}$$

**Example 3.10.** A resistance of 10 ohms is joined in series with an inductance of 0.5 henry. What capacitance should be put in series with the combination to obtain maximum current? What will be the potential difference across the resistance, inductance and capacitance? The current is being supplied by a 200 volts and 50 cycles A.C. mains.

**Soln.**

The impedance in a series LCR circuit,

$$Z = R^2 + \left( L_p - \frac{1}{C_p} \right)^2$$

For resonance or maximum current,

$$Z = R; \text{ or, } L_p - \frac{1}{C_p} = 0$$

$$\text{or, } L_p = \frac{1}{C_p}$$

$$\text{Therefore, } C = \frac{1}{Lp^2}$$

$$\text{Now } L = 0.5 \text{ henry, } p = 2\pi n$$

$$= 2\pi \times 50 = 100\pi.$$

$\therefore$  the required capacitance,

$$C = \frac{1}{0.5 \times (100\pi)^2}$$

$$= 20.27 \times 10^{-6} \text{ farad}$$

$$= 20.27 \mu\text{F}.$$

The maximum current will obviously be

$$i_0 = \frac{E_0}{Z} = \frac{E_0}{R} = \frac{200}{10} = 20 \text{ amp}$$

( $\because Z = R$  at resonance).

Potential difference across the resistance,

$$i_0 R = 20 \times 10 = 200 \text{ volts.}$$

Also at resonance,

$$X_L = Lp \text{ due to inductance} = X_C = \frac{1}{Cp} \text{ due to capacitance.}$$

Hence potential difference across  $X_L$

= potential difference across  $X_C$

$$\text{or, } i_0 Lp = i_0 / Cp$$

$$= 220 \times 0.5 \times 100 \pi = 3142 \text{ volts.}$$



**EXERCISES**

- [1] What is meant by a damping or a dissipative force ? Establish the differential equation of a damped harmonic oscillator and solve it to obtain an expression for the displacement of the oscillator.

Discuss in detail the conditions under which the oscillations become aperiodic, critically damped and oscillatory.

- [2] What is meant by the terms (i) logarithmic decrement and (ii) the quality factor of a damped harmonic oscillator. Obtain expressions for these terms.
- [3] Derive an expression for the velocity of a damped harmonic oscillator and hence obtain a definition for the relaxation time of the oscillator.
- [4] Show that both the velocity and the kinetic energy of a damped harmonic oscillator fall exponentially with time, the relaxation time in the latter case being half of that in the former.
- [5] Discuss an LCR circuit as an example of a damped harmonic oscillator and show that it is the resistance alone which is responsible for the damping of oscillations. Discuss the conditions under which the discharge of the capacitor is periodic, critically damped and oscillatory. What is the quality factor of such a circuit ?
- [6] What are damped vibrations ? Obtain an expression for the displacement in the case of a damped oscillatory motion. Discuss the effect of damping on the natural frequency.
- [7] What are free, damped and forced vibrations? A particle executing damped simple harmonic motion is subjected to an external periodic force. Establish the differential equation for the motion of the particle, explaining the physical meaning of each term and each constant in the equation. Solve the above equation and obtain expression for its maximum amplitude and quality factor.
- [8] Discuss analytically the theory of forced vibrations in the presence of damping force proportional to the velocity of the vibrating particle and obtain the condition of resonance. Explain what do you mean by sharpness of resonance.
- [9] Discuss resonance and sharpness of resonance and explain clearly the factors on which the sharpness depends.
- [10] Show that a driven oscillator is always behind the driving force in phase and that its displacement is maximum when the driving force is zero and vice versa.

Illustrate for zero, low and medium damping, the general nature of the phase lag of a driven oscillator as the driving frequency gradually increases and passes through the natural frequency of the oscillator.

[11] Obtain an expression for the current in a driven LCR circuit and discuss how the current lags or leads the applied *emf* in phase (a) when the net reactance in the circuit is capacitive and when it is inductive and (b) when the net reactance in the circuit is equal to resistance. Illustrate your answer with suitable graphs.

[12] Give the theory of oscillations in a LCR circuit with small resistance, deducing expressions for (i) frequency of oscillations, (ii) power dissipation and (iii) quality factor of the circuit. Explain the terms impedance and reactance.

[13] An under damped harmonic oscillator has its amplitude reduced to  $1/10^{\text{th}}$  of its initial value after 100 oscillations. If its time-period be 1.15 sec, calculate the time in which (i) its amplitude and (ii) its energy falls to  $1/e^{\text{th}}$  of its undamped value. If the mass of the oscillator be 1.127 gm, calculate its average rate of loss of energy.

[ (i) 50 sec. (ii) 25 sec. ; 1 erg/sec ]

[14] If the damping in the case of a damped harmonic oscillator be very small, obtain expressions for (i) average total energy of the oscillator and (ii) average rate of energy dissipation.

Show that the average rate of energy dissipation is equal to the work done by the damping force per second. [ (i)  $E = \frac{1}{C_p} m \omega^2 a_0 e^{-2\lambda t}$ , (ii)  $2\lambda E$  (iii)  $-b \frac{dy}{dt} = p$  ].

[15] Show that the relaxation time ( $\tau$ ) of a damped harmonic oscillator has the dimensions of time. Also show that (i) the amplitude of the oscillator falls to 0.6084 of its undamped value in time  $\tau$  and that (ii) it is reduced to half its undamped value in time  $1.3846 \tau$ .

[16] If the quality factor of an underdamped harmonic oscillator of frequency 512 be  $8 \times 10^4$ , calculate the time in which its energy is reduced to  $1/e^{\text{th}}$  of its energy in the absence of damping. How many oscillations does the oscillator make in this time?

Calculate the percentage reduction in the frequency of the oscillator due to damping. What inference do you draw from it. [24.87 sec, 12740,  $1.954 \times 10^{-9}$ , damping has little effect on the frequency of a harmonic oscillator if its quality factor is large].

[17] If the quality factor of a sonometer wire of frequency 300 cps be  $2 \times 10^3$ , in what time will its energy be reduced to  $1/e^{\text{th}}$  of its energy in the absence of damping? [1.061 sec]



- [18] The amplitude of a simple pendulum, with a bob of mass 400 gm and oscillating in air, falls to half its undamped value on completion of 50 complete oscillations. (i) If its time-period be 2 sec, calculate its quality factor. (ii) What should be the mass of the bob, its size and shape remaining unchanged, so that the damping may become critical? [(i) 453.7 (ii) 0.44 gm]
- [19] A capacitor of capacitance  $1 \mu\text{F}$  is discharged through a resistance of 2 ohms and an inductance of 1 henry. Is the discharge of the capacitor oscillatory? If so, obtain its frequency and calculate the time in which the amplitude of the oscillations falls to 10% of its undamped value. [Discharge oscillatory;  $n = 159$ ; 2.3 sec]
- [20] Find whether the discharge of a condenser for the following inductive circuit is oscillatory;  $C = 0.1 \mu\text{F}$ ,  $L = 10\text{mH}$ ,  $R = 200$  ohms. If the circuit is oscillatory, calculate its frequency. [Discharge oscillatory;  $n = 4772$  cps]
- [21] A damped harmonic oscillator is subjected to a sinusoidal driving force whose frequency is altered but amplitude kept constant. It is found that the amplitude of the oscillator increases from 0.02 mm at very low driving frequency to 8.0 mm at a frequency of 100 cps. Obtain the value of (i) the quality factor, (ii) the relaxation time and (iii) the damping factor. [(i) 400 (ii) 0.4245 sec (iii) 1.178]
- [22] A damped harmonic oscillator of quality factor 20 is subjected to a sinusoidal driving force of frequency twice the natural frequency of the oscillator. If the damping be small, what fraction will the amplitude of the oscillator be of its maximum value and by what angle will it differ in phase from the driving force? [ $A = 0.017 A_{\text{max}}$ ;  $178.5^\circ$  out of phase with the driving force]
- [23] An alternating potential of 110 volts and 50 cycles frequency is applied to a circuit containing a resistance of 200 ohms, inductance of 5 henry and a capacitance of  $2\mu\text{F}$ . Calculate the maximum current in the circuit. [0.78 amp]
- [24] An alternating voltage of 4 volts is applied at the resonant frequency to a coil of inductance 0.004 H and resistance 20 ohms in series with a  $0.001 \mu\text{F}$  capacitor. Find (i) the resonant frequency (ii) the current flowing, (iii) the voltage across the capacitor.
- [25] Find the capacitance of the capacitor which must be placed in series with a resistance of 5 ohms and an inductance of 200 millihenry to bring the current in phase with the voltage if the frequency be 60 cycles per second. What current will flow if 100 volts be imposed on the circuit? [ $35.20\mu\text{F}$ ; 20 amp]

## CHAPTER IV

### WAVE MOTION

*Wave motion : types of waves-transverse and longitudinal wave motion-some definitions connected with wave motion-Expression for a plane progressive wave-Differential equation of wave motion-Particle velocity-Phase or wave velocity-Energy density of a plane progressive wave-Energy current-intensity of a wave.*

#### 4.1 Wave motion-types of waves

Wave motion can be thought of as the transport of energy and momentum from one point in space to another without the transport of matter. For example, water waves, sound waves, light waves and radio waves are all known to carry energy of one form or the other from one place to another. In the case of water and sound wave, although a medium is necessary, there is no bulk motion of the intervening medium. In the case of light and radio waves, no intervening medium is at all necessary. Thus wave motion may be divided into two broad categories.

(i) **Mechanical wave motion** : This sort of wave motion is possible only in media (solid, liquid or gas) which possess inertia as well as elasticity. Water waves and sound waves are examples of this type of wave motion and are, therefore, referred to as mechanical waves.

(ii) **Electromagnetic (or non-mechanical) wave motion** : No material medium is necessary for the propagation of this sort of wave motion. Light and radio waves which can travel through empty space, belong to this category and are, therefore, referred to as non-mechanical or electromagnetic waves.

As sound waves are mechanical waves, only mechanical type of wave motion – to be referred simply as wave motion from now on, will be discussed in this chapter.

#### Production and propagation of wave motion

The production and propagation of wave motion through a medium which possesses elasticity and inertia will now be investigated. No particle of an elastic medium can be disturbed without affecting its immediate neighbour and, tending to recover its original position; it first stores up



potential energy and then converts it back into kinetic energy. The neighbouring particle which has thus been disturbed then performs a similar motion, so that each successive particle repeats, in turn, the movements of its predecessor a little later than it and then hands down the same on to its successor. This results in a transference of energy from particle to particle all along the line. One complete oscillation of a particle of the medium obviously produces one single wave or a pulse and its repeated periodic motion, a succession of waves or a wavetrain.

A wave motion may thus be defined as a disturbance or a condition that travels onwards through a medium due to the repeated periodic motion of its particles about their mean or equilibrium positions, each particle repeating the movements of its predecessor a little later than it and handing it on to its successor, so that there is a regular phase difference between one particle and the next.

It must be clearly understood that what is propagated in a wave motion is only a state of motion of the matter – not the matter itself. The wave motion is a form of dynamic condition, arising out of the vibration of the particles of the medium about their mean positions, that is propagated from one point to the other point in the medium. According to the laws of physics, any dynamic condition is related to momentum and energy. Thus, in conclusion, *it may be said that in wave motion momentum and energy are transferred or propagated.* It is not a case of propagation of matter as a whole.

The simplest type of periodic motion performed by a particle is, of course, the simple harmonic motion. The corresponding wave motion is, therefore, called a simple harmonic or sinusoidal wave motion. This is the most general type of wave motion and will be dealt with in the following discussion.

It may be emphasized again that, *but for the properties of elasticity and inertia, it would not have been possible for a wave motion to be produced in or propagated through a medium.* As will be seen later, these two properties in fact determine the velocity of propagation of the wave motion through the medium. A wave may travel through a medium over fairly large distances. In order that the wave may do so without any attenuation (*i.e.*, without any decrease in its amplitude), a third property is also necessary, *viz.*, that *the medium should offer the least frictional resistance* so as not to unduly damp the periodic motion of the particles.

A wave motion which progresses onwards through a medium, with energy transferred across every section of it, is called a *travelling or a progressive wave motion* to distinguish it from what is called a *standing or stationary wave motion* in which there is no onward movement of the wave motion through the medium and hence no transference of energy across any section of it.

#### 4.2 Transverse and longitudinal wave motion

There are two distinct types of wave motion :

(i) transverse and (ii) longitudinal.

(i) *Transverse wave motion* : In transverse wave motion, the particles of the medium oscillate up and down about their mean or equilibrium position in a direction at right angles to the direction of propagation of the wave motion itself. This form of wave motion therefore travels in the form of crests and troughs with one crest and an adjoining trough making up one wave (Fig. 4.1). A succession of waves make up a wavetrain.

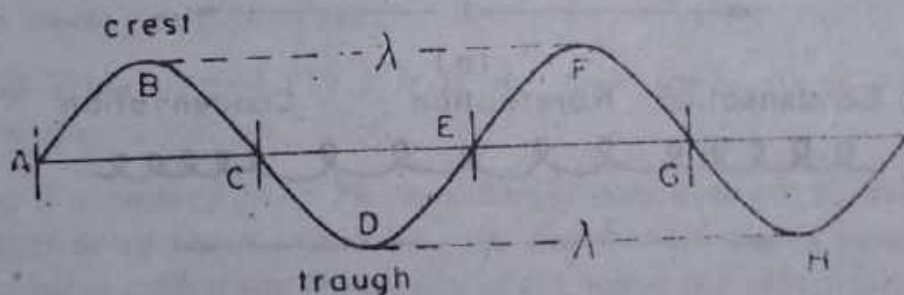


Fig. 4.1

Transverse wave motion is possible in media which possess *elasticity of shape or rigidity* i.e., solids. Waves travelling along a stretched string is an example of transverse wave motion. Although water and liquids do not possess the property of rigidity, transverse wave motion is still possible in them because they possess another equally effective property of resisting any vertical displacement of their particles (or keeping their level). Gases, however, possess neither rigidity nor do they resist any vertical displacement of their particles (or keep their levels). *A transverse wave motion is, therefore, not possible in a gaseous medium.*



(ii) *Longitudinal wave motion* : In this type of wave motion, the particles of the medium oscillate to and fro about their mean or equilibrium position, along the direction of propagation of the wave motion itself. The wave motion, therefore, travels in the form of *compressions* (or *condensations*) and *rarefactions* i.e., in the particles of the medium getting closer together and further apart alternately. This type of wave motion is possible in media possessing elasticity of volume, i.e., in solids, liquids as well as gases. Waves produced in a spring or helix when one end of it is suddenly compressed or pulled out and then released or sound waves in air are examples of this type of wave motion (Fig. 4.2). As in the case of transverse wave motion, one compression and the adjoining rarefaction constitute a wave or pulse and a succession of them, a wavetrain

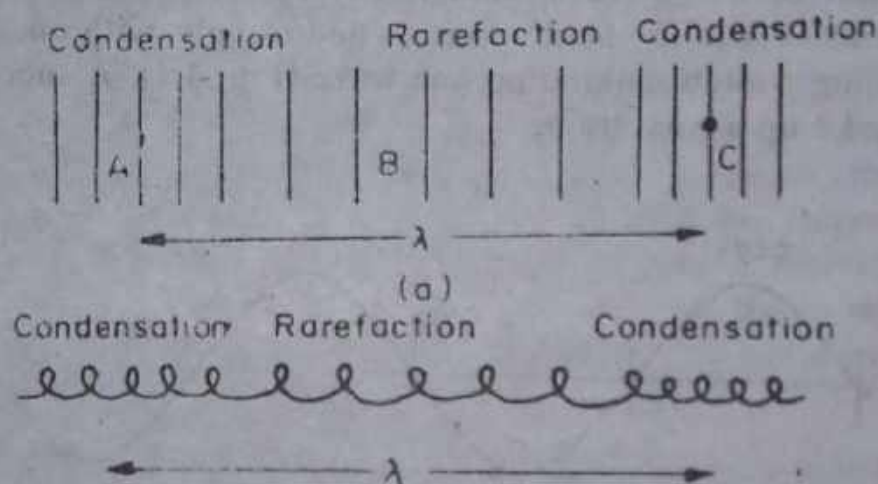


Fig. 4.2

### Characteristics of wave motion

Before proceeding further, the important characteristics of wave motion, whether transverse or longitudinal, may be summarized below :

(i) Wave motion is a disturbance produced in a medium by the repeated periodic motion of the particles of the medium. It is only this disturbance which travels forward through the medium as the wave while the particles of the medium vibrate about their mean positions – they are not propagated through the medium.

(ii) The velocity with which the wave propagates through the medium and the velocity with which the particles of the medium vibrate about their mean positions are different. *While the velocity of the wave is constant, the velocity of the particles is different at different positions.* The velocity of the particle is maximum at the mean position and zero at the extreme position of the particle.

(iii) There is a phase difference between the particles of the medium. The particle ahead starts vibrating a little later than a particle just preceeding it.

### 4.3 Some definitions connected with wave motion

(i) **Wavelength** : Since a wave or a pulse is produced in the time taken by a particle of the medium to complete one full oscillation about its mean position, *wavelength may be defined as the distance travelled by the wave in the time in which the particle completes one vibration.* It may also be defined as *the distance between any two nearest particles of the medium which are in the same phase.*

(ii) **Amplitude** : *It is the maximum displacement of the particle from its mean position of rest.*

(iii) **Time period (T)** : *It is the time taken by a particle to complete one vibration.*

(iv) **Frequency (n)** : *The number of complete oscillations made by a particle of the medium i.e., the number of waves produced, in one second is called the frequency of the wave (or vibration).*

Suppose frequency =  $n$

Time taken to complete  $n$  vibrations = 1 second

Time taken to complete 1 vibration =  $\frac{1}{n}$  second

By definition time taken to complete one vibration is the time period (T).

$$\therefore T = \frac{1}{n} ; \text{ or } nT = 1$$

$$\text{Frequency} \times \text{Time period} = 1$$



(v) **Angular frequency ( $\omega$ )** : The rate of change of phase with time is called the angular frequency and is designated by  $\omega$ . Since in one complete cycle, a phase change of  $2\pi$  occurs in a time  $T$  (the time period of the cycle), angular frequency

$$\omega = \frac{2\pi}{T} = 2\pi n$$

and has the unit of *radian per second*, the same as that of angular velocity, also designated  $\omega$  (also see Art. 1.3).

(vi) **Angular wave number ( $k$ )** : the angular wave number,  $k$ , is defined as the rate of change of phase with distance. Since in one complete cycle, a phase change of  $2\pi$  takes place within a distance  $\lambda$  (the wavelength),  $k = 2\pi/\lambda$ .

(vii) **Wave number ( $k$ )** : The wave number is the number of waves in a unit length of the wave pattern and is given by

$$k = \frac{1}{\lambda} = \frac{k}{2\pi}$$

(viii) **Velocity ( $v$ )** : Velocity of the wave is the distance traveled by the wave in one time period ( $T$ ).

$$\therefore \text{Velocity (v)} = \frac{\text{wavelength}}{\text{time period}} = \frac{\lambda}{T}$$

$$\text{or } \lambda = vT$$

But, frequency  $\times$  time period = 1

$$\text{or, } n \times T = 1$$

$$\therefore T = \frac{1}{n}$$

$$\therefore v = \frac{\lambda}{T} = \frac{\lambda}{1/n}$$

$$\text{or } v = n \lambda$$

(ix) **Phase** : The phase of a vibrating particle is defined as *the ratio of the displacement of the vibrating particle at any instant to the amplitude of the vibrating particle ( $y/a$ )*. It is also defined as *the fraction of the time interval that has elapsed since the particle crossed the mean position of rest in the positive direction*. The phase is equal to the *angle swept by the radius vector since the vibrating particle last crossed its mean position of rest*.

(x) **Wave front** : According to origin of wave motion a vibrating particle placed at a point in a homogeneous medium, extending in all directions, communicates its motion to all its neighbouring particles. The neighbouring particles which have thus been disturbed then perform, in turn, the motion of the vibrating particle. Due to this periodic vibration of the particles a wave motion is produced which travel in every direction with equal velocity. The wave motion, therefore, reach all particles which are at equal distances from the point simultaneously. The position of all these particles can be represented by the surface of a sphere drawn with the position of the vibrating particle as the centre. With time the wave advances into spheres of gradually increasing radius. Such a sphere is known as a wavefront. A wavefront at any instant of time may, therefore, be defined as *the loci of all the neighbouring particles in the medium which are just being disturbed at that instant of time and are consequently in the same state of vibration*.

In a homogeneous medium, the wavefronts are always actually spherical. But if a wavefront is considered at a considerable distance from the source, then any small portion of the wavefront can be considered *plane*.

#### 4.4 Expression for a plane progressive wave

*A plane progressive wave is one which travels onward through the medium in a given direction without attenuation i.e., with its amplitude constant.*

A progressive wave may be either transverse or longitudinal. In either case, there exists a regular phase difference between any two successive particles of the medium. A typical waveform is shown in Fig. 4.3. Let a wave originating at O, travel to the right along the x-axis. If we start counting the time at the moment when the particle at



O just passes through its mean position in the positive direction (i.e. upwards in the case of transverse wave and forward in the case of longitudinal wave), the equation of motion of this particle at O is obviously

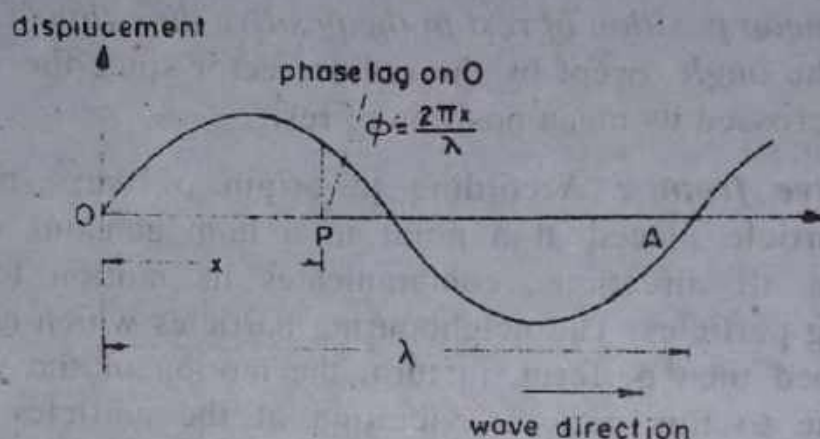


Fig. 4.3

$$y = a \sin \omega t$$

where  $y$  is the displacement of the particle at time  $t$ ,  $a$  its amplitude and  $\omega$  its angular velocity. Since the motion of O is received and repeated by the successive particles to the right of O, the phase lag goes on increasing as we proceed away from O towards the right. Thus for a particle at P which is at a distance  $x$  away from O, let this phase difference be  $\phi$ . Hence the equation of motion of the particle at P is

$$y = a \sin (\omega t - \phi).$$

For a difference in path of  $\lambda$ , i.e., one wavelength, the corresponding difference in phase is  $2\pi$ . Hence for a distance  $x$ , the corresponding phase difference is  $\frac{2\pi}{\lambda} \cdot x$  i.e.,  $\phi = \frac{2\pi}{\lambda} x$ . Substituting this value of  $\phi$  in the above expression for  $y$ , we get

$$y = a \sin \left( \omega t - \frac{2\pi}{\lambda} x \right)$$

$$\text{or, } y = a \sin (\omega t - kx) \quad (4.1)$$

where  $k = \frac{2\pi}{\lambda}$  is referred to as the propagation constant or the angular wave number.

Now  $\omega = \frac{2\pi}{T} = 2\pi n$ . Again  $n = \frac{v}{\lambda}$  (from  $v = n\lambda$ ) where  $n$  is the frequency of the particle or the wave. Hence  $\omega = \frac{2\pi v}{\lambda}$ . Eqn. (4.1) then becomes

$$y = a \sin \frac{2\pi v}{\lambda} t - \frac{2\pi}{\lambda} x$$

$$\text{or, } y = a \sin \frac{2\pi}{\lambda} (vt - x) \quad (4.2)$$

$$\text{or, } y = a \sin k (vt - x) \quad (4.3)$$

Any one of the expressions written above or any one of their variations, such as those given below [eqns. 4.4 – 4.7], is referred to as the equation of a plane progressive wave motion in the positive direction of  $x$ .

Eqn. (4.2) can be written as

$$y = a \sin \frac{2\pi v}{\lambda} \left( t - \frac{x}{v} \right) \quad (4.4)$$

$$y = a \sin 2\pi n \left( t - \frac{x}{v} \right) \quad (4.5)$$

[since  $n\lambda = v$ ; or,  $v/\lambda = n$ ]

$$y = a \sin \frac{2\pi v}{\lambda} \left( t - \frac{x}{v} \right) \quad (4.6)$$

( $\because n = 1/T$ )

$$y = a \sin 2\pi \left( \frac{t}{T} - \frac{x}{\lambda} \right) \quad (4.7)$$

(since  $v = n\lambda = \lambda/T$ )

The expression most commonly used is given by equation (4.2)

Similarly, if the wave travels towards the left,  $x$  becomes negative and we have

$$y = a \sin \frac{2\pi}{\lambda} (vt + x) \quad (4.8)$$



$$\text{or, } y = a \sin (\omega t + kx) \quad (4.9)$$

The phase or the wave velocity is also given by

$$v = \lambda/T = \lambda / \frac{2\pi}{\omega} \quad \left( \text{from } \omega = \frac{2\pi}{T} \right)$$

$$= \frac{\lambda\omega}{2\pi} = \omega / \frac{2\pi}{\lambda} = \omega/k \quad (4.10)$$

$$\left( \text{since } k = \frac{2\pi}{\lambda} \right)$$

It should be emphasized that in deducing the above expression it has been assumed that at  $t = 0$ , the particle O just passes through its mean position in the positive direction, *i.e.*, at  $t = 0$ ,  $y = 0$ . If this is not the case and the particle is said to have an initial phase  $\theta$ , then the equation of the wave becomes

$$y = a \sin \frac{2\pi}{\lambda} (vt - x + \theta) \quad (4.11)$$

$$\text{and } y = a \sin (\omega t - kt + \theta) \quad (4.12)$$

where  $\theta$  is referred to as the initial phase or phase constant.

If  $\theta = 90^\circ$ , we have  $y = a$ , at  $x = 0$  and  $t = 0$ .

**The following points emerge from discussion of the wave motion**

(i) For a constant value of  $x$  (*i.e.*, at a given point  $x$ ) the displacement  $y$  varies simple harmonically with time, completing one full cycle in time  $\lambda/v$ , which, therefore, gives the periodic time  $T$  of the wave. This also gives the frequency  $n$  of the wave since frequency  $n = 1/T = v/\lambda$ .

(ii) For a constant value of  $t$  (*i.e.*, at a given instant  $t$ ), the displacement  $y$  varies simple harmonically with distance from the origin. For  $x = \lambda$ , the displacement ( $y$ ) is restored to its original value, so that the wavelength  $= \lambda$ .

(iii) The phase lag for a distance  $x$  is given by  $\frac{2\pi}{\lambda} \cdot x$ . Hence for a distance  $\lambda$ , the phase lag is  $\frac{2\pi}{\lambda} \cdot \lambda = 2\pi$  which, in effect, is the same thing as zero. Thus particles separated by a distance  $\lambda$  or an integral multiple of  $\lambda$  are in the same phase.

(iv) If the time  $t$  is increased by  $\delta t$  and the distance  $x$  by  $v\delta t$ , the value of  $y$  remains the same. This shows that a disturbance at one point is repeated after a time  $\delta t$  at a point  $v\delta t$  further away. This means, in other words, that the disturbance or the wave travels with a velocity  $v$  without any attenuation.

#### 4.5 Phase (or wave) velocity

The compressions and rarefactions of longitudinal wave or crests and trough of transverse wave advances through a medium with a constant velocity. In other words, advance of phase through a medium takes place with same velocity. This velocity of advance is known as *phase velocity*.

The equation of a plane progressive wave, travelling in a medium along the direction of positive  $x$ -axis, is given by

$$y = a \sin \frac{2\pi}{\lambda} (vt - x) \quad (i)$$

where  $v$  is referred to as the wave velocity and is equal to  $\lambda/T$ .

Rearranging, eqn. (i) can be written in the form

$$\begin{aligned} y &= a \sin \frac{2\pi vt}{\lambda} - \frac{2\pi}{\lambda} x \\ &= a \sin (2\pi nt - kx) \end{aligned} \quad (ii)$$

since  $v/\lambda = 1/T = n$  is the frequency of the wave and  $2\pi/\lambda = k$  is the propagation constant of the wave.

Further,  $2\pi n = \omega$  is the angular frequency of the wave. Hence eqn. (ii) reduces to

$$y = a \sin (\omega t - kx) \quad (iii)$$



Now  $(\omega t - kx)$  in eqn. (iii) is the constant phase of the wave which travels along the positive direction of the  $x$ -axis and its velocity, i.e., the phase velocity of the wave should be given by  $dx/dt$ .

Since  $(\omega t - kx)$  is a constant quantity, we have  $\frac{d}{dt}(\omega t - kx) = 0$

$$\text{or, } \omega - k \frac{dx}{dt} = 0$$

$$\text{or, } \omega = k \cdot \frac{dx}{dt}$$

Thus the phase velocity of the wave,

$$\frac{dx}{dt} = \frac{\omega}{k} \quad (4.13)$$

But  $\omega/k = \frac{2\pi n}{2\pi/\lambda} = n\lambda = \lambda/T = \text{wave velocity } v$ . Thus, for a single wave, in any given medium, wave velocity = phase velocity =  $v = \lambda/T = \omega/k$ .

**Exaple 4.1** The displacement (in metres) of a particle executing simple harmonic motion at any instant of time is given by

$$y = 0.1 \sin 2\pi (340t - 0.15).$$

Calculate (i) the amplitude of the vibrating particle (ii) wave velocity (iii) wave length (iv) frequency and (v) time period.

**Soln.**

The general equation of a simple harmonic wave is given by

$$y = a \sin \frac{2\pi}{\lambda} (vt - x) \quad (i)$$

$$\text{Here } y = 0.1 \sin 2\pi (340t - x) \quad (ii)$$

Comparing eqns. (i) and (ii)

$$a = 0.1 \text{ metre}$$

$$\lambda = 1 \text{ metre}$$

$$\text{wave velocity } v = 340 \text{ m/sec.}$$

$$\text{Frequency } n = \frac{v}{\lambda} = \frac{340}{1} = 340 \text{ Hz.}$$

$$\text{Time period, } T = \frac{1}{n} = \frac{1}{340} \text{ second.}$$

**Example 4.2** A simple harmonic wave of amplitude 8 units traverses a line of particles in the direction of positive x-axis. At any given instant of time, the displacement of a particle at a distance of 10 cm from the origin is +6 units while that for a particle at a distance of 25 cm from the origin is +4 units. Calculate the wavelength.

**Soln.**

The equation of a simple harmonic wave can be written as

$$y = a \sin 2\pi \left( \frac{t}{T} - \frac{x}{\lambda} \right)$$

$$\text{or, } y = a \sin 2\pi \left( \frac{t}{T} - \frac{x}{\lambda} \right)$$

In the first case,

$$y = +6, a = 8 \text{ and } x = 10 \text{ cms.}$$

$$\therefore \frac{6}{8} = \sin 2\pi \left( \frac{t}{T} - \frac{25}{\lambda} \right) \quad \text{(i)}$$

In the second case,

$$y = 4, a = 8 \text{ and } x = 25 \text{ cms.}$$

$$\therefore \frac{4}{8} = \sin 2\pi \left( \frac{t}{T} - \frac{25}{\lambda} \right) \quad \text{(ii)}$$

From eqn. (i)

$$0.75 = \sin 2\pi \left( \frac{t}{T} - \frac{10}{\lambda} \right)$$

$$\text{But } \sin \left( \frac{48.6\pi}{180} \right) = 0.75.$$

$$\therefore 2\pi \left( \frac{t}{T} - \frac{10}{\lambda} \right) = \frac{48.6}{180}$$



And from (ii)

$$0.5 = \sin 2\pi \left( \frac{t}{T} - \frac{25}{\lambda} \right)$$

But  $\sin \frac{\pi}{6} = 0.5$

$$\therefore 2\pi \left( \frac{t}{T} - \frac{25}{\lambda} \right) = \frac{\pi}{6} \quad (\text{v})$$

$$\text{or, } \frac{t}{T} - \frac{25}{\lambda} = \frac{1}{12} \quad (\text{vi})$$

Subtracting (vi) from (iv)

$$\frac{25}{\lambda} - \frac{10}{\lambda} = \frac{48.6}{360} - \frac{1}{12}$$

or,  $\lambda = 290.8 \text{ cm.}$

**Example 4.3** The velocity of a simple harmonic wave is 30 cm/s. At a time  $t = 0$  the displacement of a particle is given by

$$y = 4 \sin 2\pi \left( \frac{x}{100} \right)$$

Find the equation for the displacement at a time  $t = 2 \text{ sec.}$

**Soln.**

The general equation of a simple harmonic wave is

$$\begin{aligned} y &= a \sin \frac{2\pi}{\lambda} (vt - x) \\ &= a \sin \left( \frac{2\pi t}{T} - \frac{2\pi x}{\lambda} \right) \end{aligned}$$

when  $t = 0$ ,

$$\begin{aligned} y &= a \sin \left( -\frac{2\pi x}{\lambda} \right) \\ &= -a \sin \frac{2\pi x}{\lambda} \quad (\text{i}) \end{aligned}$$

At  $t = 0$ , the given equation is

$$y = 4 \sin 2\pi \left( \frac{x}{100} \right) \quad \text{(ii)}$$

Comparing eqns. (i) and (ii), we get

$$a = -4 \text{ and } \lambda = 100 \text{ cm.}$$

At  $t = 2 \text{ sec.}$

$$y = a \sin \frac{2\pi}{\lambda} (vt - x).$$

here  $a = -4 \text{ cm.}$   $\lambda = 100 \text{ cm}$  and  $t = 2 \text{ sec.}$

$$\therefore y = -4 \sin \frac{2\pi}{100} (30 \times 2 - x)$$

$$= -4 \sin \left[ \frac{6\pi}{5} - 2\pi \left( \frac{x}{100} \right) \right]$$

$$= 4 \sin \left[ 2\pi \left( \frac{x}{100} \right) - \frac{6\pi}{5} \right].$$

**Example 4.4** A plane progressive wave travelling along the  $+x$  - direction has the following characteristics :  $a = 0.2 \text{ cm}$ ,  $v = 360 \text{ cm/sec}$  and  $\lambda = 60 \text{ cm}$ .

(a) Write down the equation for it (i) when displacement is zero at  $x = 0$  and  $t = 0$  and (ii) when displacement is maximum at  $x = 0$  and  $t = 0$

(b) Obtain the displacement in either case at  $x = 120 \text{ cm}$  and  $t = 2 \text{ sec.}$

**Soln.**

(a) The general equation of a plane progressive wave

$$y = a \sin 2\pi \left( \frac{t}{T} - \frac{x}{\lambda} + \theta \right)$$

$$= a \sin 2\pi \left( nt - \frac{x}{\lambda} + \theta \right)$$



where  $\theta$  is the initial phase.

$$= 0.2 \sin 2\pi \left( 6t - \frac{x}{60} + \theta \right)$$

where  $n = \frac{v}{\lambda} = \frac{360}{60} = 6$

(i) Now  $y = 0$  at  $x = 0$  and  $t = 0$

$$\therefore 0 = 0.2 \sin 2\pi (0 - 0 + \theta)$$

or,  $\theta = 0$ .

$$\therefore y = 0.2 \sin 2\pi \left( 6t - \frac{x}{60} \right)$$

(ii) again  $y = 0.2$  at  $x = 0$  and  $t = 0$

$$\therefore 0.2 = 0.2 \sin 2\pi (0 - 0 + \theta)$$

or,  $\theta = \frac{\pi}{2}$

$$\therefore y = 0.2 \sin 2\pi \left( 6t - \frac{x}{\lambda} + \frac{\pi}{2} \right)$$

$$= 0.2 \cos 2\pi \left( 6t - \frac{x}{\lambda} \right)$$

(b) when  $y = 0$  at  $x = 0$  and  $t = 0$ , the equation of a progressive wave is

$$y = 0.2 \sin 2\pi \left( 6t - \frac{x}{\lambda} \right)$$

Here  $t = 2$  sec,  $x = 120$  cm,  $\lambda = 60$  cm

$$\therefore y = 0.2 \sin 2\pi \left( 6 \times 2 - \frac{120}{60} \right)$$

$$= 0.2 \sin 2\pi \times 10$$

$$= 0$$

Again when  $y = 0.2$  at  $x = 0$  and  $t = 0$ , the equation is

$$y = 0.2 \cos 2\pi \left( 6t - \frac{x}{60} \right)$$

$$= 0.2 \cos 2\pi \left( 6 \times 2 - \frac{120}{60} \right)$$

$$= 0.2 \cos 2\pi \times 10$$

$$= 0.2 \text{ cm.}$$

**Example 4.5.** A plane progressive wave train of frequency 400 cycles per second has a phase velocity of 480 m/sec. (i) How far apart are two points  $30^\circ$  out of phase? (ii) What is the phase difference between two displacements at a given point at times  $10^{-3}$  sec apart?

**Soln.**

The equation of a plane progressive wave is given by

$$y = a \sin \frac{2\pi}{\lambda} (vt - x)$$

where  $\frac{2\pi}{\lambda} (vt - x)$  is the phase angle of a point at a distance  $x$  from the origin at time  $t$ .

$\therefore$  phase angle of a point at a distance  $x_1$  from the origin at time  $t = \frac{2\pi}{\lambda} (vt - x_1)$

and phase angle of point at a distance  $x_2$  from the origin at time  $t = \frac{2\pi}{\lambda} (vt - x_2)$ .

Hence phase difference between the two points

$$\begin{aligned} &= \frac{2\pi}{\lambda} (vt - x_1) - \frac{2\pi}{\lambda} (vt - x_2) \\ &= \frac{2\pi}{\lambda} (x_2 - x_1) = \frac{2\pi v}{\lambda} \cdot \frac{(x_2 - x_1)}{v} \\ &= 2\pi n \frac{(x_2 - x_1)}{v} \end{aligned}$$

since  $\frac{v}{\lambda} = n$ , the frequency of the wave. The phase difference between the two points



$$= 30^\circ = \frac{30 \times \pi}{180} \text{ rad} = \frac{\pi}{6} \text{ rad.}$$

$$\therefore 2\pi n \frac{(x_2 - x_1)}{v} = \frac{\pi}{6}$$

$$\text{or, } 2n \frac{(x_2 - x_1)}{v} = \frac{6}{1}$$

Here  $n = 400$ ,  $v = 480 \text{ m/s} = 480 \times 10^2 \text{ cm/sec.}$

$$\therefore \frac{2 \times 400 \times (x_2 - x_1)}{480 \times 10^2} = \frac{1}{6}$$

$$\text{or, } x_2 - x_1 = \frac{480 \times 10^2}{2 \times 400 \times 6}$$

$$= 10 \text{ cm} = 0.1 \text{ m.}$$

(ii) Again, the phase angle at a point  $x$  from the origin at time

$$t_1 = \frac{2\pi}{\lambda} (vt_1 - x)$$

and the phase at the same point at time  $t_2$

$$= \frac{2\pi}{\lambda} (vt_2 - x)$$

$$\therefore \text{phase difference at the point at times } (t_2 - t_1) \text{ sec apart}$$

$$= \frac{2\pi}{\lambda} (vt_2 - x) - \frac{2\pi}{\lambda} (vt_1 - x)$$

$$= \frac{2\pi v}{\lambda} (t_2 - t_1) = 2\pi n (t_2 - t_1)$$

Here  $t_2 - t_1 = 10^{-3} \text{ sec.}$

$$\therefore \text{phase difference} = 2 \times \pi \times 400 \times 10^{-3}$$

$$= 0.8\pi \text{ rad.} = 144^\circ.$$

**Example 4.6.** A sinusoidal wave traveling along a string is described by

$$y(x, t) = 0.00327 \sin(72.1x - 2.72t)$$

in which the numerical constants are in SI units (0.00327 m, 72.1 rad/m, and 2.72 rad/s). What are (i) the amplitude, (ii) wavelength, (iii) period, (iv) wave number, (v) frequency, and (vi) speed of the wave?

**Soln :** The expression for a sinusoidal wave can also be written as

$$y(x, t) = y_m \sin(kx - \omega t)$$

where  $y_m$  is the amplitude of the wave.

Comparison with the given equation gives

(i) amplitude = 0.00327 m = **3.27 mm.**

(ii)  $k = 72.1 \text{ rad/m}$  and  $\omega = 2.72 \text{ rad/s}$ .

$$k = \frac{2\pi}{\lambda}; \quad \text{or } \lambda = \frac{2\pi}{k} = \frac{2\pi \text{ rad}}{72.1 \text{ rad/m}}$$

$$= 0.0871 \text{ m} = \mathbf{8.71 \text{ cm.}}$$

(iii)  $T = \frac{2\pi}{\omega} = \frac{2\pi \text{ rad}}{2.72 \text{ rad/s}} = \mathbf{2.31 \text{ s.}}$

(iv) wave number,  $k = \frac{1}{\lambda} = \frac{1}{0.0871 \text{ m}} = \mathbf{11.5 \text{ m}^{-1}}.$

(v) frequency,  $n = \frac{1}{T} = \frac{1}{2.31 \text{ s}} = \mathbf{0.433 \text{ Hz.}}$

(vi) velocity,  $v = \frac{\omega}{k} = \frac{2.72 \text{ rad/s}}{72.1 \text{ rad/m}} = \mathbf{0.377 \text{ m/s}}$   
 $= \mathbf{3.77 \text{ cm/s.}}$

#### 4.6 Differential equation of wave motion

Differentiating the most general form of a simple harmonic wave

$$y = a \sin \frac{2\pi}{\lambda} (vt - x)$$

with respect to time, we get



$$\frac{dy}{dt} = \frac{2\pi av}{\lambda} \cos \frac{2\pi}{\lambda} (vt - x)$$

Differentiating the above expression again with respect to time, we get

$$\begin{aligned} \frac{d^2y}{dt^2} &= -\frac{4\pi^2 v^2}{\lambda^2} a \sin \frac{2\pi}{\lambda} (vt - x) \\ &= -\frac{4\pi^2 v^2}{\lambda^2} \cdot y \end{aligned} \quad (4.13)$$

Similarly, differentiating eqn. (4.2) with respect to  $x$ , we get the slope of the displacement curve (also referred to as strain or compression)

$$\frac{dy}{dx} = -\frac{2\pi}{\lambda} a \cos \frac{2\pi}{\lambda} (vt - x)$$

Differentiating the above expression again with respect to  $x$ , we get the *rate of change of compression with distance*

$$\begin{aligned} \frac{d^2y}{dx^2} &= -\frac{4\pi^2}{\lambda^2} a \sin \frac{2\pi}{\lambda} (vt - x) \\ &= -\frac{4\pi^2}{\lambda^2} \cdot y \end{aligned} \quad (4.14)$$

From relations (4.13) and (4.14), we have

$$\frac{d^2y}{dt^2} = v^2 \left( -\frac{4\pi^2}{\lambda^2} \cdot y \right) = v^2 \frac{d^2y}{dx^2} \quad (4.15)$$

Eqn. (4.15) is referred to as the *differential equation of a plane or one-dimensional progressive wave*. The general differential equation of wave motion can be written as

$$\frac{d^2y}{dt^2} = K \frac{d^2y}{dx^2}$$

$$\text{where } K = v^2; \quad \text{or } v = \sqrt{K}$$

Any equation of this form can unhesitatingly be declared to represent a plane, progressive harmonic wave, the velocity of which is given by the square root of the co-efficient of  $d^2y/dx^2$ .

Now  $d^2y/dx^2$  gives the rate of change of compression with distance i.e., the *curvature of the displacement curve*. Hence, the differential equation as given by eqn. (4.15) may be interpreted to mean that

$$\begin{aligned} &\text{Particle acceleration at a point } \left[ \frac{d^2y}{dt^2} \right] \\ &= (\text{wave velocity})^2 [v^2] \times \text{curvature of the displacement curve at} \\ &\text{the point } \left[ \frac{d^2y}{dx^2} \right]. \end{aligned}$$

#### 4.7 Particle velocity and wave velocity

The equation of a harmonic plane progressive wave is given by

$$y = a \sin \frac{2\pi}{\lambda} (vt - x)$$

where

$y$  = displacement of a particle of the medium at a distance  $x$  from the origin and at an instant of time  $t$ .

$a$  = amplitude

$v$  = wave (or phase) velocity.

Differentiating the above equation with respect to time, we have particle velocity,

$$U = \frac{dy}{dt} = \frac{2\pi av}{\lambda} \cos \frac{2\pi}{\lambda} (vt - x) \quad (4.16)$$

The maximum value of the particle velocity is  $U_{\max} = \frac{2\pi a}{\lambda} \cdot v$ ,  
or, *maximum particle velocity* =  $\frac{2\pi a}{\lambda} \times (\text{wave velocity})$

The acceleration of the particle is given by

$$\begin{aligned} \frac{d^2y}{dt^2} &= \frac{4\pi^2 av^2}{\lambda^2} \sin \frac{2\pi}{\lambda} (vt - x) \\ &= -\frac{4\pi^2 v^2}{\lambda^2} \left[ a \sin \frac{2\pi}{\lambda} (vt - x) \right] \end{aligned}$$



$$= - \left( \frac{4\pi^2 v^2}{\lambda^2} \right) \cdot y$$

The acceleration is maximum when

$$y = a$$

Hence the maximum acceleration

$$= - \frac{4\pi^2 v^2}{\lambda^2} \cdot a \quad (4.17)$$

The minus sign indicates that the acceleration is directed towards its mean position.

Now differentiating eqn. (4.2) with respect to  $x$ , we get

the *slope of the displacement curve* (also referred to as strain or compression).

$$\frac{dy}{dx} = - \frac{2\pi a}{\lambda} \cos \frac{2\pi}{\lambda} (vt - x) \quad (4.18)$$

From eqns. (4.16) and (4.18), we get

$$U = \frac{dy}{dt} = -v \cdot \frac{dy}{dx} \quad (4.19)$$

Thus,

particle velocity at a point = - (wave velocity)  $\times$  (slope of the displacement curve at that point)

**Example 4.7** A train of simple harmonic waves is travelling in a gas along the positive direction of the  $x$ -axis, with an amplitude equal to 2 cm, velocity 300 metres/sec and frequency 400. Calculate the displacement, particle velocity and particle acceleration at a distance of 4 cm from the origin after an interval of 5 seconds.

**Soln.**

(i) displacement ( $y$ )

$$y = a \sin \frac{2\pi}{\lambda} (vt - x)$$

Here  $a = 2$  cm,  $v = 300$  m/s =  $3 \times 10^4$  cm/s,  $\lambda = v/n = 3 \times 10^4/400 = 75$  cm,  $x = 4$  cm,  $t = 5$  sec.

$$\begin{aligned}
 \therefore y &= 2 \sin \frac{2\pi}{75} (3 \times 10^4 \times 5 - 4) \\
 &= 2 \sin \left( \frac{2\pi}{75} \times 149996 \right) \\
 &= 2 \sin (2\pi \times 1999.9) \\
 &= 2 \sin (1999 \times 2\pi + 0.9 \times 2\pi) \\
 &= 2 \sin (1.8\pi) \\
 &= 2 \sin (\pi + 0.8) \\
 &= -2 \sin (0.8\pi) \\
 &= -2 \sin \left( 0.8\pi \times \frac{180}{\pi} \right)^\circ = -2 \sin 144^\circ \\
 &= -2 \sin (180 - 144)^\circ \\
 &= -2 \sin 36^\circ \\
 &= -2 \times 0.5878 = -1.1756 \text{ cm.}
 \end{aligned}$$

Thus, the displacement of the particle at a distance of 4 cm from the origin, after an interval of 5 seconds is **-1.1756 cm.**

(ii) particle velocity (U)

$$U = \frac{dy}{dt} = \frac{2\pi v}{\lambda} a \cos \frac{dy}{dt} (vt - x)$$

As we have seen in (i)

$$\sin \frac{2\pi}{\lambda} (vt - x) = \sin 36^\circ.$$

$$\text{Hence } \frac{2\pi}{\lambda} (vt - x) = 36^\circ$$

$$\therefore U = \frac{2\pi \times 3 \times 10^4}{75} \times 2 \times 0.809$$

$$= \frac{2\pi \times 3 \times 10^4}{75} \times 2 \times 0.809$$

$$= 4068 \text{ cm/sec} = \mathbf{40.68 \text{ m/sec.}}$$



(iii) particle acceleration  $\left(\frac{d^2y}{dt^2}\right)$

$$\frac{d^2y}{dt^2} = -\frac{4\pi^2v^2}{\lambda^2} a \sin \frac{2\pi}{\lambda} (vt - x)$$

$$= -\frac{4\pi^2v^2}{\lambda^2} \cdot y$$

$$= -\frac{4\pi^2(3 \times 10^4)^2}{(75)^2} \times (-1.1756)$$

$$(\because y = 1.1756)$$

$$= 7.429 \times 10^6 \text{ cm/sec}^2.$$

**Example 4.8.** Which of the following are solutions of the one-dimensional wave equation? (i)  $y = x^2 + v^2 t^2$ , (ii)  $y = x^2 - v^2 t^2$ , (iii)  $y = (x - vt)^2$ , (iv)  $y = 7x - 10t$ , (v)  $y = 2 \sin x \cos vt$  and (vi)  $y = \sin 2x \cos vt$

(i)  $y = x^2 + v^2 t^2$

Differentiating with respect to  $t$ , we get

$$\frac{dy}{dt} = 2v^2 t$$

Differentiating again,  $\frac{d^2y}{dt^2} = 2v^2$

Now differentiating with respect to  $x$ ,

$$\frac{dy}{dx} = 2x.$$

Differentiating again,  $\frac{d^2y}{dx^2} = 2.$

Clearly,  $2v^2 = v^2 \cdot (2)$

or,  $\frac{d^2y}{dt^2} = v^2 \cdot \frac{d^2y}{dx^2}$

which is the differential equation of a one-dimensional wave.

**Expression (i) is, therefore, a solution of the one-dimensional wave equation,**

$$(ii) \ y = x^2 - v^2 \cdot t^2$$

**differentiating with respect to t**

$$\frac{dy}{dt} = -2v^2 \cdot t$$

$$\text{and } \frac{d^2y}{dt^2} = -2v^2.$$

**differentiating with respect to x**

$$\frac{dy}{dx} = 2x \text{ and } \frac{d^2y}{dx^2} = 2$$

$$\text{Now } -2v^2 \neq v^2 \cdot (2)$$

$$\text{Hence } \frac{d^2y}{dt^2} \neq v^2 \cdot \frac{d^2y}{dx^2}$$

**Expression (ii) is, therefore, not a solution of the one-dimensional wave equation.**

$$(iii) \ y = (x - vt)^2 \\ = x^2 - 2xvt + v^2t^2$$

**differentiating with respect to t**

$$\frac{dy}{dt} = -2xv + 2v^2 \cdot t$$

$$\text{and } \frac{d^2y}{dt^2} = 2v^2.$$

**differentiating with respect to x**

$$\frac{dy}{dx} = 2x - 2vt$$

$$\text{and } \frac{d^2y}{dx^2} = 2$$

$$\text{Now } 2v^2 = v^2 \cdot (2)$$



$$\text{or, } \frac{d^2 y}{dt^2} = v^2 \cdot \frac{d^2 y}{dx^2}$$

Expression (iii) is, therefore, a solution of the one-dimensional wave equation.

$$(iv) \ y = 7x - 10t$$

differentiating with respect to  $t$

$$\frac{dy}{dt} = -10 \text{ and } \frac{d^2 y}{dt^2} = 0$$

differentiating with respect to  $x$

$$\frac{dy}{dx} = 7 \text{ and } \frac{d^2 y}{dx^2} = 0$$

$$\text{Now, } 0 = v^2 \times (0)$$

$$\text{or, } \frac{d^2 y}{dt^2} = v^2 \times \frac{d^2 y}{dx^2}$$

Expression (iv) is, therefore, a solution of one-dimensional wave equation.

$$(v) \ y = 2 \sin x \cos vt$$

differentiating with respect to  $t$

$$\frac{dy}{dt} = -2 \sin x \cdot v \cdot \sin vt$$

$$\text{and } \frac{d^2 y}{dt^2} = -2 \sin x \cdot v^2 \cdot \cos vt = -v^2 y$$

differentiating with respect to  $x$

$$\frac{dy}{dx} = 2 \cos x \cos vt$$

$$\text{and } \frac{d^2 y}{dx^2} = -2 \sin x \cos vt = -y.$$

$$\text{Now } -v^2 \cdot y = v^2 \cdot (-y)$$

$$\text{or, } \frac{d^2 y}{dt^2} = v^2 \frac{d^2 y}{dx^2}$$

Expression (v) is therefore, a solution of one-dimensional wave equation.

$$(vi) \ y = \sin 2x \cos vt$$

differentiating with respect to t

$$\frac{dy}{dt} = \sin 2x \cdot v \cdot \sin vt$$

$$\text{and } \frac{d^2 y}{dt^2} = \sin 2x \cdot v^2 \cdot \cos vt = -v^2 y$$

differentiating with respect to x

$$\frac{dy}{dx} = 2 \cos 2x \cos vt$$

$$\text{and } \frac{d^2 y}{dx^2} = -4 \sin 2x \cos vt$$

$$= -4 \cdot y$$

$$\text{Now } -v^2 \cdot y \neq -4 \cdot y$$

$$\text{or, } \frac{d^2 y}{dt^2} \neq v^2 \frac{d^2 y}{dx^2}$$

Expression (vi) is therefore not a solution of one dimensional wave equation.

#### 4.8 Distribution of velocity and pressure in a plane progressive wave

The general equation of a plane progressive wave is given by

$$y = a \sin \frac{2\pi}{\lambda} (vt - x) \quad (4.20)$$

and the particle velocity

$$U = \frac{dy}{dt} = \frac{2\pi av}{\lambda} \cos \frac{2\pi}{\lambda} (vt - x)$$



The rate of change of displacement with distance i.e.,  $\frac{dy}{dx}$  gives the strain in the medium. If  $\frac{dy}{dx}$  is positive, it represents a region of rarefaction. If  $\frac{dy}{dx}$  is negative, it is a region of compression. The bulk modulus of elasticity of the medium is given by

$$K = \frac{\text{volume stress}}{\text{volume strain}} \\ = \frac{\text{change in pressure}}{\text{volume strain}} \\ = \frac{dp}{dy/dx}$$

$$\text{or, } dp = -K \left( \frac{dy}{dx} \right)$$

$$\text{or, } dp = K \left( -\frac{dy}{dx} \right)$$

If  $\frac{dy}{dx}$  is negative,  $dp$  is +ve, i.e., it is a region of compression. If  $\frac{dy}{dx}$  is positive,  $dp$  is negative, i.e., it is a region of rarefaction.

Differentiating the general equation of a plane progressive wave with respect to distance

$$\frac{dy}{dx} = -\frac{2\pi a}{\lambda} \cos \frac{2\pi}{\lambda} (vt - x)$$

$$\therefore dp = K \left( -\frac{dy}{dx} \right)$$

$$= \frac{2\pi Ka}{\lambda} \cos \frac{2\pi}{\lambda} (vt - x)$$

Fig. 4.4 shows the graphs for displacement, velocity and change in pressure. In Fig. 4.4 (iii)  $dp$  represents the change in pressure while  $P_0$  is the normal pressure which the medium possesses in the absence of the propagation of the wave.

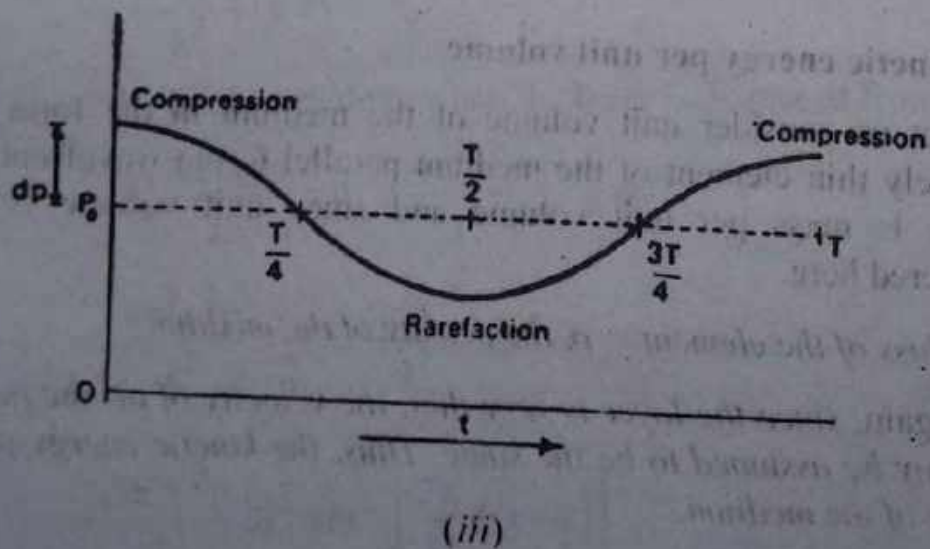
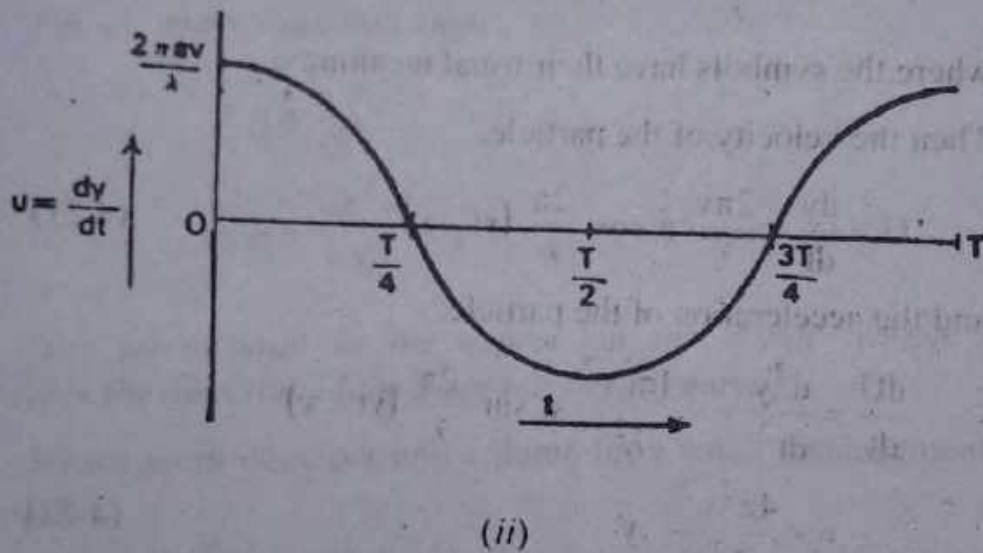
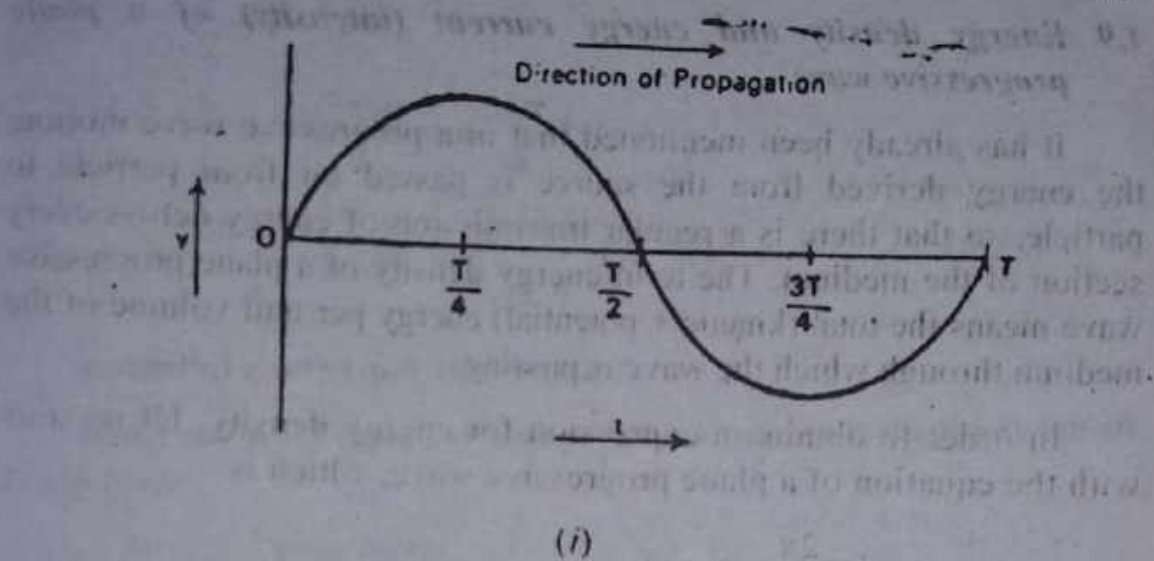


Fig. 4.4



#### 4.9 Energy density and energy current (intensity) of a plane progressive wave

It has already been mentioned that in a progressive wave motion, the energy derived from the source is passed on from particle to particle, so that there is a regular transmission of energy across every section of the medium. The term energy density of a plane progressive wave means the total (kinetic + potential) energy per unit volume of the medium through which the wave is passing.

In order to obtain an expression for energy density, let us start with the equation of a plane progressive wave, which is

$$y = a \sin \frac{2\pi}{\lambda} (vt - x)$$

where the symbols have their usual meanings.

Then the velocity of the particle,

$$U = \frac{dy}{dt} = \frac{2\pi v}{\lambda} a \cos \frac{2\pi}{\lambda} (vt - x) \quad (4.21)$$

and the acceleration of the particle,

$$\begin{aligned} \frac{dU}{dt} &= \frac{d^2 y}{dt^2} = -\frac{4\pi^2 v^2}{\lambda^2} a \sin \frac{2\pi}{\lambda} (vt - x) \\ &= -\frac{4\pi^2 v^2}{\lambda^2} y \end{aligned} \quad (4.22)$$

##### kinetic energy per unit volume

Let us consider unit volume of the medium in the form of an extremely thin element of the medium parallel to the wavefront. Now density is mass per unit volume and since unit volume is being considered here,

Mass of the element =  $\rho$ , the density of the medium.

Again, since the layer is very thin, the velocity of all the particles in it may be assumed to be the same. Thus, the kinetic energy per unit volume of the medium

$$= \frac{1}{2} (\text{mass}) (\text{velocity})^2$$

$$\begin{aligned}
 &= \frac{1}{2} \cdot \rho \cdot U^2 \\
 &= \frac{1}{2} \cdot \rho \cdot \left[ \frac{2\pi a v}{\lambda} \cos \frac{2\pi}{\lambda} (vt - x) \right]^2 \\
 &= \frac{2\pi^2 a^2 v^2}{\lambda^2} \cdot \rho \cdot \cos^2 \left[ \frac{2\pi}{\lambda} (vt - x) \right] \quad (4.23)
 \end{aligned}$$

### potential energy per unit volume

Now, the work done per unit volume for a small displacement  $dy$  of the layer

force  $\times$  displacement.

force = mass  $\times$  acceleration

$$\begin{aligned}
 &= \rho \times \frac{d^2 y}{dt^2} \\
 &= \rho \times \frac{4\pi^2 v^2}{\lambda^2} \cdot y
 \end{aligned}$$

[the minus sign in the expression for  $d^2 y/dt^2$  which merely indicates the direction of the force has been ignored].

Hence work done per unit volume for a small displacement  $dy$  of the layer,

$$= \frac{4\pi^2 v^2 \rho}{\lambda^2} y dy$$

Then, the total work done when the layer is displaced from 0 to  $y$ ,

$$\begin{aligned}
 &= \int_0^y \frac{4\pi^2 v^2 \rho}{\lambda^2} y dy = \frac{4\pi^2 v^2 \rho}{\lambda^2} \int_0^y y dy \\
 &= \frac{4\pi^2 v^2 \rho}{\lambda^2} \cdot \frac{y^2}{2} = \frac{2\pi^2 v^2 \rho}{\lambda^2} \cdot y^2 \\
 &= \frac{2\pi^2 v^2 \rho}{\lambda^2} \cdot a^2 \sin^2 \left[ \frac{2\pi}{\lambda} (vt - x) \right]
 \end{aligned}$$



Obviously, this work must be stored up in the medium in the form of potential energy.

Hence,

*Potential energy (P.E.) per unit volume of the medium*

$$= \frac{2\pi^2 v^2 \rho}{\lambda^2} a^2 \sin^2 \left[ \frac{2\pi}{\lambda} (vt - x) \right] \quad (4.24)$$

**energy density**

Thus, the total energy per unit volume of the medium or the *energy density of the plane progressive wave*,  $E = K.E. + P.E.$

$$\begin{aligned} \text{or, } E &= \frac{2\pi^2 a^2 v^2 \rho}{\lambda^2} \left[ \cos^2 \frac{2\pi}{\lambda} (vt - x) + \sin^2 \frac{2\pi}{\lambda} (vt - x) \right] \\ &= \frac{2\pi^2 v^2 \rho}{\lambda^2} \cdot a^2 \end{aligned} \quad (4.25)$$

$$\begin{aligned} &= 2\pi^2 \left( \frac{v}{\lambda} \right)^2 \cdot \rho \cdot a^2 \\ &= 2\pi^2 n^2 a^2 \rho \end{aligned} \quad (4.26)$$

where  $n = v/\lambda$  is the frequency of the wave.

It is interesting to note that although both kinetic and potential energies of the wave depend upon the values of  $x$  and  $t$ , its total energy or the energy density is quite independent of either.

#### 4.10 Energy current – intensity of a wave

In case the cross-section of the beam be unity, expressions (4.25) and (4.26) give the total energy of the beam or the wave *per unit length*.

If  $v$  is the velocity of the wave, then a new length  $v$  of the medium is set into motion every second; therefore, the energy transferred per second must be the energy contained in length  $v$ . This rate of flow of energy per unit area of cross-section of the wavefront along the direction of wave propagation is called the *energy current (C) or the energy flux of the wave* and is obviously equal to  $E \times v$ .

Thus, the energy current or the energy flux of a plane progressive wave,

$$C = 2\pi^2 n^2 a^2 \rho v \text{ ergs/sec. cm}^2. \quad (4.27)$$

Now the intensity of the wave ( $I$ ) is defined as the *quantity of incident energy per unit area of the wavefront per unit time* and is the same as the energy current or the energy flux of the wave.

Hence

$$I = 2\pi^2 n^2 a^2 \rho v \quad (4.28)$$

As can be seen from relation (4.27), intensity of a wave is proportional to the square of the amplitude of the wave, i.e.,  $I \propto a^2$ .

*Since in a medium with little or no frictional resistance, a plane wave travels without attenuation, i.e., with its amplitude undiminished, the intensity of the wave remains the same throughout.*

#### 4.11 Distribution of energy in a plane progressive wave

The manner in which the total energy of a plane progressive wave, as calculated in Art. 4.4, is distributed over a complete wavelength will now be investigated.

The average  $K$ ,  $E$ , over a complete wavelength is given by

$$\begin{aligned} & \frac{1}{\lambda} \int_0^\lambda \frac{2\pi^2 v^2 \rho}{\lambda^2} a^2 \cos^2 \frac{2\pi}{\lambda} (vt - x) dx \\ &= \frac{2\pi^2 n^2 a^2 \rho}{\lambda} \int_0^\lambda \frac{1}{2} \left[ 1 + \cos \frac{4\pi}{\lambda} (vt - x) \right] dx \\ & \quad [ \because v^2/\lambda^2 = n^2 ] \end{aligned}$$

$$\text{Now } \int_0^\lambda \cos \frac{4\pi}{\lambda} (vt - x) dx$$

$$= -\frac{\lambda}{4\pi} \left[ \sin \frac{4\pi}{\lambda} (vt - x) \right]_0^\lambda$$

$$= -\frac{\lambda}{4\pi} \left[ \sin \frac{4\pi}{\lambda} (vt - \lambda) - \sin \frac{4\pi}{\lambda} vt \right]$$

$$= \frac{\lambda}{4\pi} \left[ \sin \left( \frac{4\pi}{\lambda} vt - 4\pi \right) - \sin \frac{4\pi}{\lambda} vt \right] = 0$$

Therefore, the average K. E. of the wave over a complete wavelength

$$\begin{aligned} &= \frac{2\pi^2 n^2 a^2 \rho \lambda}{2\lambda} \\ &= \pi^2 n^2 a^2 \rho \quad (4.29) \\ &= \frac{1}{2} \text{ total energy.} \end{aligned}$$

Similarly, it can be shown that the average P.E. of the wave over a complete wavelength

$$\begin{aligned} &= \frac{1}{\lambda} \int_0^\lambda \frac{2\pi^2 v^2 a^2 \rho}{\lambda^2} \sin^2 \frac{2\pi}{\lambda} (vt - x) dx \\ &= \pi^2 n^2 a^2 \rho = \frac{1}{2} \text{ total energy.} \quad (4.30) \end{aligned}$$

Thus, at any given instant, the energy of a plane progressive harmonic wave is, on the average, half kinetic and half potential in form.

**Example 4.9.** A source of sound has an amplitude of 0.25 cm and a frequency of 512 Hz. If the velocity of sound in air is 340 m/s and the density of air is 0.00129 gm/cm<sup>3</sup>, what is the rate of flow of energy per square cm?

**Soln.**

Total energy per unit volume	$n = 512 \text{ Hz}$
= energy density = $2\pi^2 n^2 a^2 \rho$	$a = 0.25 \text{ cm}$
Rate of flow of energy per sq. cm.	$v = 340 \text{ m/s}$
= Energy density $\times v$	$= 34000 \text{ cm/s}$
= $2\pi^2 n^2 a^2 \rho v$	$\rho = 0.00129 \text{ gm/cm}^3$
$= (2) (3.14)^2 (512)^2 (0.25)^2 (0.00129) (34000)$	



$$= 1.417 \times 10^{-7} \text{ ergs/cm}^2 \cdot \text{s}$$

$$= 1.417 \text{ J/cm}^2 \cdot \text{s}.$$

**Example 4.10.** The excess pressure in a travelling sound wave is given by the equation  $p = 1.5 \sin \{[(2\pi/\lambda)(x - 330t)]\}$ , where  $x$  and  $\lambda$  are in metres,  $t$  is in seconds, and  $p$  is in pascals. (i) What is the velocity of the wave? (ii) If  $\lambda = 2 \text{ m}$ , what is the frequency of the wave? (iii) what is the maximum pressure (pressure amplitude)? (iv) What is the pressure at  $x = \frac{1}{6} \text{ m}$  and  $t = 0$ ?

**Soln.** The standard equation of a wave is

$$y = a \sin \left[ 2\pi n \left( t - \frac{x}{v} \right) \right]$$

The given equation can be written in this form as

$$p = -1.5 \sin \left[ 2\pi \left( \frac{330}{\lambda} \right) (t - x/330) \right]$$

(the minus sign is only a phase difference)

Comparing the two equations, we get

$$(i) \quad v = 330 \text{ m/s.}$$

$$(ii) \quad n = \frac{v}{\lambda} = \frac{330}{2} \text{ Hz} = 165 \text{ Hz.}$$

$$(iii) \quad p_0 = 1.5 \text{ Pa (pressure amplitude)}$$

$$(iv) \quad p = -1.5 \sin \left[ 2\pi \frac{330}{2} \left( 0 - \frac{\frac{1}{6}}{330} \right) \right]$$

$$= 1.5 \sin \frac{\pi}{6} = 0.75 \text{ Pa.}$$

## EXERCISES

1. What are mechanical and non-mechanical waves? Clearly explain the term wave motion and hence distinguish between longitudinal and transverse wave motions.
2. Obtain an expression for a plane progressive wave travelling in the positive x-direction in the form

$$y = a \sin \frac{2\pi}{\lambda} (vt - x)$$

and write it in as many different forms as you can.

3. Distinguish between particle velocity and wave velocity and obtain a relation between the two.

4. Show that  $U = -v \frac{dy}{dx}$

where  $U$  is the particle velocity at a point,  $\frac{dy}{dx}$  is the slope of the displacement curve at that point and  $v$  is the wave velocity.

5. Show that in the case of progressive longitudinal waves,

$$\text{particle velocity} = \text{wave velocity} \times \text{compression.}$$

6. Derive the relation

$$\frac{d^2 y}{dt^2} = v^2 \frac{d^2 y}{dx^2}$$

7. Derive the differential equation of one dimensional wave motion.
8. Define energy density and energy current of a plane progressive wave. Obtain expressions for both.
9. Show that the energy density of a plane progressive wave is given by  $E = 2\pi^2 n^2 a^2 \rho$ , where the symbols have their usual meanings.
10. Show that for a plane progressive wave, on the average half the energy is kinetic and half potential.
11. What is meant by intensity of a plane progressive wave? Obtain an expression for the intensity of a wave.
12. Show that for a plane progressive wave, particle acceleration at a point  $\left[ \frac{d^2 y}{dt^2} \right] = (\text{wave velocity})^2 [v^2] \times \text{curvature of the displacement}$

$$\text{curve at the point } \left[ \frac{d^2 y}{dx^2} \right].$$

13. For the wave  $y = 5 \sin 30\pi [t - (x/240)]$ , where  $x$  and  $y$  are in centimetres and  $t$  is in seconds, find (i) the displacement at  $t = 0$  and  $x = 2$  cm (ii) wavelength (iii) velocity of the wave and (iv) frequency of the wave. [ $-3.535$  cm,  $16$  cm,  $240$  cm/s,  $15$  Hz]

14. A wave along a string is given by the relation

$$y = 0.02 \sin (30\pi t - 4.0\pi x)$$

where  $x$  is in metres and  $t$  is in seconds. Find its amplitude, frequency, speed and wavelength. [ $0.02$  m,  $4.78$  Hz,  $7.5$  m/s,  $1.5$  m]

15. When a simple harmonic wave is propagated through a medium, the displacement through a medium, the displacement of the particle (in cm) at any instant of time is given by

$$y = 10 \sin \frac{2\pi}{100} (36000t - 20)$$

Calculate (i) the amplitude of the vibrating particle, (ii) wave velocity, (iii) wave length, (iv) frequency and (v) time period.

- [(i)  $10$  cm (ii)  $36000$  cm/sec (iii)  $100$  cm (iv)  $360$  Hz (v)  $\frac{1}{360}$  second]

16. A longitudinal harmonic wave train is travelling along a string in the positive direction of the  $x$ -axis. If the frequency of the wave be  $50$  cps and the distance between a condensation and an adjacent rarefaction in the string,  $10$  cm, obtain the velocity of the wave.

Assuming the amplitude of the wave to be  $2.0$  cm, write down the wave equation, taking  $y = 0$  at  $x = 0$  and  $t = 0$ .

[ $1000$  cm/sec,  $2 \sin 0.1\pi (1000t - x)$ ].

17. A wave of frequency  $500$  cycles per second has a phase velocity of  $360$  metres per second (a) How far apart are two points  $60^\circ$  out of phase? (b) What is the phase difference between two displacements at a certain point at times  $10^{-5}$  sec. apart?

[(a)  $12.0$  cm (b)  $\pi$  radian or  $180^\circ$ ]

18. A simple harmonic wave is represented by

$$y = 10 \sin \frac{2\pi t}{T} + \alpha$$

The time period is  $30$  sec. At time  $t = 0$ , the displacement is  $5$  cm. Calculate (i) phase angle at  $t = 7.5$  sec, and (ii) phase difference between two positions at a time interval of  $6$  sec.

[ $\alpha = \frac{\pi}{6}$  rad.; phase angle at  $t = 7.5$  s,  $\frac{2\pi}{3}$  rad. phase difference =  $\frac{2\pi}{5}$  rad]



19. A simple harmonic wave of amplitude 0.08 m traverses a line of particles in the direction of positive x-axis. At any given instant of time, for a particle at a distance 0.1 m from the origin, the displacement is 0.08 m and for a particle at a distance of 0.25 m from the origin, the displacement is 0.04 m. Calculate the wavelength.
20. A source of sound has a frequency of 256 Hz and an amplitude of  $5 \times 10^{-3}$  m. What is the flow of energy across a square metre per second if the velocity of sound in air is 350 m/s and density of air =  $1.29 \text{ kg/m}^3$ .
21. A musical instrument of frequency 250 is sending out waves of amplitude  $10^{-3}$  cm. Find (i) energy of waves/ $\text{m}^2$  and (ii) intensity of sound in joules/second/ $\text{m}^2$ . Given the velocity of sound as 332 m/s and density of air as  $1.29 \text{ kg/m}^3$ .

(i)  $159.1 \times 10^{-6} \text{ J/m}^2$

(ii)  $5.28 \times 10^{-2} \text{ J/s/m}^2$ .

22. The particle displacement due to a sound wave is given by

$$y = 2.15 \times 10^{-3} \sin \left( 100 \pi t - \frac{2\pi x}{600} \right)$$

where x is measured in cm and t in seconds. Write down the amplitude, frequency, wavelength and velocity of the wave. [ $2.15 \times 10^{-3} \text{ cm}$ , 50 vib/sec., 600 cm, 30,000 cm/sec].

23. A simple harmonic wavetrain of amplitude 2 cm and time period 0.01 second is travelling with a velocity of 10 m/s in the positive x-direction. Calculate the displacement y of the particle from the mean position, the particle velocity and particle acceleration at  $x = 150 \text{ cm}$  from the origin and at  $t = 3 \text{ second}$ . [ $0$ ,  $400 \pi \text{ cm/sec}$ ,  $0$ ]

## CHAPTER V

## INTERFERENCE OF SOUND WAVES

*Interference of sound waves-Energy distribution due to interference of sound waves- Interference of two waves of nearly same frequency and amplitude - Beats-analytical treatment of formation of beats. Superposition of two waves of nearly equal frequency and wavelength - group velocity- Interference by reflection - stationary or standing waves-formation of stationary waves - analytical treatment-Energy of a stationary wave- Solved problems-Exercises.*

## 5.1 Interference of sound waves

When two or more waves combine at a particular point, they are said to interfere. According to principle of superposition, the displacement of the particles at the points of the medium where the waves meet is given by the algebraic (vector) sum of the individual displacements of the interfering waves at those points. If the two interfering waves meet at the same phase they combine to give a large displacement which is equal to the sum of their amplitudes. But when they meet in opposite phase, the resultant displacement is given by the difference of their amplitudes. In addition, if the amplitudes of the interfering waves are equal, they cancel each other producing no disturbance at all. This phenomenon of reinforcement and cancellation is called *interference*; it applies to waves of all kinds.

Let two simple harmonic waves of the same frequency be travelling along the same, or very nearly the same path and in the same direction. If  $a$  and  $b$  be the amplitudes of the two waves and if there is a phase difference  $\phi$  between the waves, then the displacement of a particle at any instant due to each wave is given by

$$y_1 = a \sin \frac{2\pi}{\lambda} (vt - x) \quad (i)$$

$$y_2 = b \sin \frac{2\pi}{\lambda} (vt - x + \phi) \quad (ii)$$

where  $v$  is the velocity of propagation of each wave and  $\lambda$  is the wavelength.

The resultant displacement of the particle due to superposition of the two waves is

$$y = y_1 + y_2$$

$$= a \sin \frac{2\pi}{\lambda} (vt - x) + b \left[ \sin \frac{2\pi}{\lambda} (vt - x) + \phi \right]$$

$$= a \sin \frac{2\pi}{\lambda} (vt - x) + b \left[ \sin \frac{2\pi}{\lambda} (vt - x) + \cos \phi + \right.$$

$$\left. + \cos \frac{2\pi}{\lambda} (vt - x) \sin \phi \right]$$

$$= \left[ \sin \frac{2\pi}{\lambda} (vt - x) \right] (a + b \cos \phi)$$

$$+ \left[ \cos \frac{2\pi}{\lambda} (vt - x) \right] (b \sin \phi)$$

$$\text{Let } a + b \cos \phi = A \cos \theta$$

$$b \sin \phi = A \sin \theta$$

$$\therefore A = \sqrt{a^2 + b^2 + 2ab \cos \phi} \quad (\text{iii})$$

$$\text{and } \tan \theta = \frac{b \sin \phi}{a + b \cos \phi}$$

$$\text{or, } \theta = \tan^{-1} \frac{b \sin \phi}{a + b \cos \phi} \quad (\text{iv})$$

$$\therefore y = \left[ \sin \frac{2\pi}{\lambda} (vt - x) \right] [A \cos \theta]$$

$$+ \left[ \cos \frac{2\pi}{\lambda} (vt - x) \right] [A \sin \theta]$$

$$= A \sin \left[ \frac{2\pi}{\lambda} (vt - x) + \theta \right]$$

(5.1)

Thus, the resultant vibration is also simple harmonic and has the same time period (*i.e.*, frequency and wavelength). But the amplitude and phase difference are different. What is more, the amplitude is no longer constant and has different values for different values of  $\phi$ , the phase difference between the interfering waves.

$$(i) \text{ when } \phi = 0, 2\pi, 4\pi, \dots = 2n\pi$$



where  $n = 0, 1, 2, \dots$

$$A = \sqrt{a^2 + b^2 + 2ab} = (a + b)$$

The amplitude becomes maximum when  $\phi = n\pi$ . Moreover, if  $a = b$ ,  $A = 2a$ .

(ii) when  $\phi = \pi, 3\pi, 5\pi, \dots = (2n + 1)\pi$

where  $n = 0, 1, 2, \dots$

$$A = \sqrt{a^2 + b^2 - 2ab} = (a - b)$$

The amplitude becomes minimum when  $\phi = (2n + 1)\pi$  and is zero ( $A = 0$ ) when  $a = b$ .

Thus when the phase difference is zero or a multiple of  $2\pi$  (i.e., the path difference is zero or a multiple of  $\lambda$ ), the two waves reinforce each other and an intense sound is heard. The interference is known as *constructive*. When the phase difference is odd multiple of  $\pi$  (i.e., the path difference is  $\frac{\lambda}{2}$  or its odd multiple), the two waves destroy each other. The interference is known as *destructive*. When the amplitudes are different, a very feeble sound will be heard. If the amplitudes are equal, no sound is heard. They are graphically represented in (Fig. 5.1).

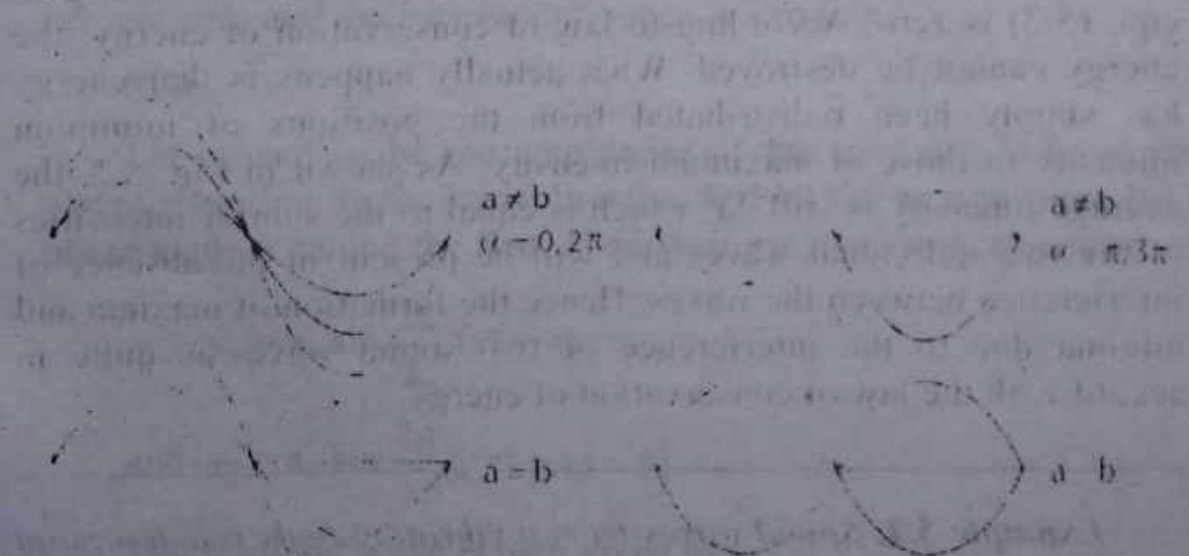


Fig. 5.1

## 5.2 Energy distribution due to interference of sound waves

The intensity of a wave is proportional to square of its amplitude. Hence the intensity of the resultant vibration due to superposition of two waves is given by [eqn. (iii), Art. 5.1]

$$I = A^2 = \left( \sqrt{a^2 + b^2 + 2ab \cos \phi} \right)^2$$

If  $a = b$  i.e., the two waves are of the same amplitude, then

$$I = A^2 = \left( \sqrt{a^2 + a^2 + 2a^2 \cos \phi} \right)^2$$

$$= \left[ \sqrt{2a^2 (1 + \cos \phi)} \right]^2$$

$$= 2a^2 (1 + \cos \phi)$$

$$= 2a^2 \cdot 2 \cos^2 \frac{\phi}{2}$$

$$= 4a^2 \cos^2 \frac{\phi}{2} \quad (5.2)$$

From eqn. (5.2), the intensity is maximum and is equal to  $4a^2$  (four times the intensity of a single wave) at the points of maximum vibration i.e., constructive interference. The intensity at the points of minima i.e., the points of destructive interference according to eqn. (5.2) is zero. According to law of conservation of energy, the energy cannot be destroyed. What actually happens is that energy has simply been redistributed from the positions of minimum intensity to those of maximum intensity. As shown in Fig. 5.2, the average intensity is still  $2a^2$  which is equal to the sum of intensities of the two individual waves and will be present in the absence of interference between the waves. Hence the formation of maxima and minima due to the interference of two sound waves is quite in accord with the law of conservation of energy.

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**Example 5.1.** Sound waves from a vibrating body reach a point by two paths. When the paths differ by 12 cm or by 36 cm, there is silence at the point. Calculate the frequency of the vibrating body if the velocity of sound in air is 330 m/sec.



**Soln.**

When silence is produced, it is clearly a case of destructive interference. For, when the path difference is 12 cm, the waves arrive at the point in opposite phase and annul each other producing silence. This path difference must be equal to odd number of half wavelengths *i.e.*, equal to  $(2k + 1) \frac{\lambda}{2}$  where  $k = 0, 1, 2, \dots$  or equal to  $\lambda/2, 3\lambda/2, 5\lambda/2, \dots$  so that the path difference between two successive points of silence must be equal to  $\lambda$ . Thus, when silence is again produced at a point when the path difference is 36 cm, this means that the difference between these two successive values of path difference must be  $\lambda$ .

$$\text{Or, } \lambda = (36 - 12) = 24 \text{ cm}$$

If  $n$  be the frequency of the wave, we have from  $v = n\lambda$

$$n = \frac{v}{\lambda} = \frac{33000}{24} = 1375 \text{ cycles per second.}$$

*Example 5.2. Find the resultant of two plane simple harmonic waves of the same period travelling in the same direction but differing in phase and amplitude. What is the amplitude of the resultant wave if those of the component waves be 3.0 and 4.0 respectively and their phase difference  $\pi/2$  radian?*

**Soln.**

Let  $a_1$  and  $a_2$  be the amplitudes of the two simple harmonic waves travelling in the same direction and let the second wave lag a phase angle  $\phi$  behind the first. Then, their equations may be written as

$$y_1 = a_1 \sin \frac{2\pi}{\lambda} (vt - x)$$

$$\text{and } y_2 = a_2 \sin \frac{2\pi}{\lambda} [(vt - x) - \phi]$$

The resultant wave is then given by

$$y = y_1 + y_2 = a_1 \sin \frac{2\pi}{\lambda} (vt - x) + a_2 \sin \frac{2\pi}{\lambda} [(vt - x) - \phi]$$



$$\begin{aligned}
 &= a_1 \sin \frac{2\pi}{\lambda} (vt - x) + a_2 \sin \frac{2\pi}{\lambda} (vt - x) \cos \phi \\
 &\quad - a_2 \cos \frac{2\pi}{\lambda} (vt - x) \sin \phi \\
 &= \sin \frac{2\pi}{\lambda} (vt - x) (a_1 + a_2 \cos \phi) \\
 &\quad - \cos \frac{2\pi}{\lambda} (vt - x) (a_2 \sin \phi)
 \end{aligned}$$

Let  $a_1 + a_2 \cos \phi = a \cos \theta$

and  $a_2 \sin \phi = a \sin \theta$ .

Then we have,

$$\begin{aligned}
 y &= a \sin \frac{2\pi}{\lambda} (vt - x) \cos \theta - a \cos \frac{2\pi}{\lambda} \sin \theta \\
 &= a \sin \left[ \frac{2\pi}{\lambda} (vt - x) - \theta \right]
 \end{aligned}$$

Thus the resultant wave is also simple harmonic having the same frequency and wavelength of the component waves. But its amplitude  $a$  is different than those of the component waves and lags in phase angle  $\theta$  behind the first wave.

$$\begin{aligned}
 a &= \sqrt{(a \cos \theta)^2 + (a \sin \theta)^2} \\
 &= \sqrt{(a_1 + a_2 \cos \phi)^2 + (a_2 \sin \phi)^2} \\
 &= \sqrt{a_1^2 + a_2^2 \cos^2 \phi + 2a_1a_2 \cos \phi + a_2^2 \sin^2 \phi} \\
 &= \sqrt{a_1^2 + a_2^2 + 2a_1a_2 \cos \phi}
 \end{aligned}$$

$$\text{And } \tan \theta = \frac{a \sin \theta}{a \cos \theta} = \frac{a_2 \sin \phi}{a_1 + a_2 \cos \phi}$$

$$\text{or, } \theta = \tan^{-1} \frac{a_2 \sin \phi}{a_1 + a_2 \cos \phi}$$

Obviously, when  $\phi = 0$ , i.e., the two component waves arrive in phase at a point,  $\cos \phi = 1$  and  $a = a_1 + a_2$ ; and when  $\phi = \pi$ , i.e., the two

component waves arrive at a point out of phase,  $\cos \phi = -1$  and  $a = a_1 - a_2$ . At all other points the amplitude lies between these two extremes.

Therefore, when  $a_1 = 3.0$  cm,  $a_2 = 4.0$  cm, and  $\phi = \frac{\pi}{2}$  rad; we have

$$\begin{aligned} a &= \sqrt{3^2 + 4^2 + 2 \cdot 3 \cdot 4 \cos \frac{\pi}{2}} \\ &= \sqrt{9 + 16 + 0} \\ &= \sqrt{25} = 5.0 \text{ cm.} \end{aligned}$$

### 5.3 Interference of two waves of nearly same frequency and amplitude : formation of beats

If two waves, travelling along the same path and in the same direction, have slightly different frequencies or wavelengths and also different amplitudes, the resulting wave motion possesses certain peculiarities. The two waves arrive in phase at some points and out of phase at others, thus producing maximum displacement (and hence maximum intensity) at the former, and minimum (but not zero) displacements (and hence minimum intensity) at the later points as they proceed along. The sound heard alternates between loud and soft. This *throbbing effect* (periodic rise and fall in intensity) in the intensity of sound is called *beats*. One maximum intensity and one succeeding minimum intensity or *vice versa* constitutes one beat. The frequency of the beats is the number of intense sound heard per second.

Let us consider a layer of air some distance away from two pure notes of nearly equal frequency, say two tuning forks of frequencies 48 and 56 Hz respectively which are being sounded together. Fig. 5.3 (i) shows the variation of the displacement  $y_1$  of the layer due to one of the notes while Fig. 5.3 (ii) shows the variation of the displacement,  $y_2$  due to the second note. Fig. 5.3 (iii) shows the result displacement,  $y$ , of the layer due to the superposition of the waves when both of them are sounded together. To understand the variation of  $y$ , suppose that the displacements  $y_1$  and  $y_2$  are in phase at some instant  $T_1$ . Since the frequency of the curve in Fig. 5.3 (i) is 48 cycles per second, the variation  $y_1$  undergoes 3 complete cycle in  $1/16$  seconds. During the same time the variation  $y_2$  undergoes



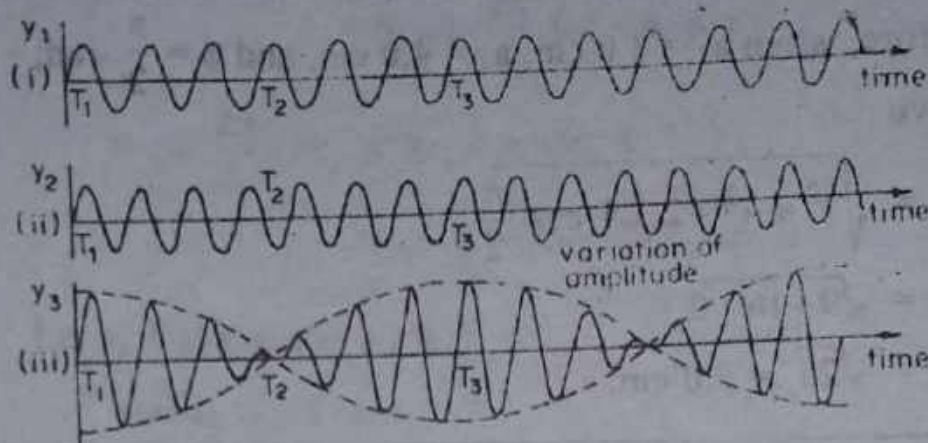


Fig. 5.3

$3\frac{1}{2}$  cycles since its frequency is 56 cycles per second. Thus  $y_1$  and  $y_2$  are  $180^\circ$  out of phase with each other at this instant and their resultant  $y$  is minimum at instant  $T_2$ . Thus  $T_1 T_2$  represents  $1/16^{\text{th}}$  of a second in Fig. 5.3 (iii). In  $1/8^{\text{th}}$  of a second from  $T_1$ ,  $y_1$  has undergone 6 complete cycles while  $y_2$  has undergone 7 complete cycles. Thus the two waves are in phase again at  $T_3$  and their resultant is again maximum at this instant.  $T_1 T_3$  represents  $1/8^{\text{th}}$  of a second. In this way it can be seen that a loud sound is heard after every  $1/8^{\text{th}}$  of a second and thus the *beat frequency* is 8 cycles per second. It will be shown below that the beat frequency is always equal to the difference of the two nearly equal frequencies.

#### 5.4 Analytical treatment of formation of beats

Let two wavetrains of frequencies  $n_1$  and  $n_2$  and of amplitudes  $a$  and  $b$  respectively start with the same phase. Then the individual displacements at a point at some instant  $t$  due to the two wavetrains are

$$y_1 = a \sin w_1 t$$

$$\text{where } w_1 = 2\pi n_1$$

$$\text{and } y_2 = b \sin w_2 t$$

$$\text{where } w_2 = 2\pi n_2$$

Then, according to the principle of superposition, the resultant displacement is given by

$$y = y_1 + y_2$$



$$\begin{aligned}
&= a \sin w_1 t + b \sin w_2 t \\
&= a \sin w_1 t + b \sin [w_1 - (w_1 - w_2)]t \\
&= a \sin w_1 t + b [\sin w_1 t \cos (w_1 - w_2) t \\
&\quad - \cos w_1 t \sin (w_1 - w_2) t] \\
&= a \sin w_1 t + b \sin w_1 t \cos (w_1 - w_2) t \\
&\quad - b \cos w_1 t \sin (w_1 - w_2) t \\
&= \sin w_1 t [a + b \cos (w_1 - w_2) t] \\
&\quad - \cos w_1 t [b \sin (w_1 - w_2) t]
\end{aligned}$$

$$\text{Let } a + b \cos (w_1 - w_2) t = A \cos \theta \quad (i)$$

$$\text{and } b \sin (w_1 - w_2) t = A \sin \theta \quad (ii)$$

$$\begin{aligned}
\therefore y &= \sin w_1 t A \cos \theta - \cos w_1 t A \sin \theta \\
&= A \sin (w_1 t - \theta) \quad (iii)
\end{aligned}$$

Eqn. (iii) gives the resultant displacement due to the superposition of the two waves. The values of the resultant amplitude  $A$  and the phase angle  $\theta$  can be obtained as follows :

From (i) and (ii)

$$A^2 \cos^2 \theta = a^2 + b^2 \cos^2 (w_1 - w_2) t + 2ab \cos (w_1 - w_2) t$$

$$A^2 \sin^2 \theta = b^2 \sin^2 (w_1 - w_2) t$$

Adding we get

$$A^2 = a^2 + b^2 + 2ab \cos (w_1 - w_2) t$$

$$\text{or, } A = \sqrt{a^2 + b^2 + 2ab \cos (w_1 - w_2) t} \quad (5.3)$$

$$\text{and } \tan \theta = \frac{A \sin \theta}{A \cos \theta} = \frac{b \sin (w_1 - w_2) t}{a + b \cos (w_1 - w_2) t}$$

$$\text{or, } \theta = \tan^{-1} \frac{b \sin (w_1 - w_2) t}{a + b \cos (w_1 - w_2) t} \quad (5.4)$$

From eqn. (5.4) it is evident that phase angle  $\theta$  changes with respect to time. Similarly eqn. (5.3) shows that the amplitude of the resultant vibration varies with time.

(a) the amplitude is maximum when

$$(w_1 - w_2) t = 2\pi (n_1 - n_2) t = 2k\pi$$

where  $k = 0, 1, 2, \dots$

The resultant amplitude is then

$$A = \sqrt{a^2 + b^2 + 2ab} = \sqrt{(a + b)^2} = (a + b)$$

Thus the resultant amplitude is maximum ( $= a + b$ )

$$\text{when } t = \frac{2k\pi}{2\pi (n_1 - n_2)} = \frac{k}{n_1 - n_2}$$

i.e., at time instants  $0, \frac{1}{n_1 - n_2}, \frac{2}{n_1 - n_2}, \dots$

As the intensity of sound is directly proportional to the square of the amplitude, the maximum intensity of sound will be heard at these instants.

(b) the amplitude is minimum when

$$(w_1 - w_2) t = 2\pi (n_1 - n_2) t = (2k + 1) \pi$$

where  $k = 0, 1, 2, 3, \dots$

The resultant amplitude is then

$$A = \sqrt{a^2 + b^2 - 2ab} = \sqrt{(a - b)^2} = (a - b)$$

Thus the resultant amplitude is minimum ( $= a - b$ )

$$\text{when } t = \frac{(2k + 1) \pi}{2\pi (n_1 - n_2)} = \frac{2k + 1}{2(n_1 - n_2)}$$

i.e., at time instants

$$\frac{1}{2(n_1 - n_2)}, \frac{3}{2(n_1 - n_2)}, \dots$$

Hence the minimum intensity of sound will be heard at these instants.



Thus the time interval between successive maxima or minima is  $\frac{1}{n_1 - n_2}$  sec. One minimum amplitude is present between two successive maxima and *vice versa*.

$$\begin{aligned}\text{Hence, the number of beats produced per second} &= \frac{1}{1/(n_1 - n_2)} \\ &= n_1 - n_2\end{aligned}$$

Thus the number of beats produced per second is equal to the difference in frequency of the two notes.

The phenomenon of beats is often used to compare an unknown frequency with a known frequency, as in the case of tuning a piano with a tuning fork. The ear can detect beats upto about 10 per second. Above this, the fluctuations in loudness are too rapid to be heard. The phenomenon of beats can be used to detect the velocity of a moving object like a car. When a beam from, say, a radar source is reflected from a moving car small frequency changes take place in the reflected beam due to Doppler effect (Art. 6.1). Due to this change in frequency in the reflected radar beam, beats will be produced when the reflected beam and the original beam from the radar source superpose. By counting the number of beats and invoking the Doppler principle, the velocity of the moving car can be determined.

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**Example 5.3.** A tuning fork A of frequency 384 Hz gives 6 beats per second when sounded with another tuning fork B. On loading B with a little wax, the number of beats per second becomes 4. What is the frequency of B?

**Soln.**

Frequency of A = 384 Hz.

Beats per second = 6

Hence frequency of B before loading is either  $384 + 6 = 390$  Hz

or,  $384 - 6 = 378$  Hz

After B is loaded, beats produced per second = 4. Then frequency of B after loading is

either  $384 + 4 = 388$  Hz

or,  $384 - 4 = 380 \text{ Hz}$ .

Since frequency decreases after loading, the frequency of B before loading is **390 Hz**.

*Example 5.4. Two tuning forks A and B when sounded together give 5 beats per second. The frequency of B is 256 Hz. When A is filed, 5 beats per second is again heard. Find the frequency of A before and after filing.*

**Soln.**

Frequency of B = 256

Beats per second = 5

$\therefore$  Frequency of A before filing is

either  $256 + 5 = 261$

or,  $256 - 5 = 251$

Since after filing number of beats per second is again 5, frequency of A after filing is

Either  $256 + 5 = 261$

or,  $256 - 5 = 251$

Let the frequency of A before filing be 261. After filing the frequency of A increases. Therefore after filing, the frequency of A cannot be either 261 or 251. Therefore, frequency of A cannot be 261.

But if the frequency of A before filing is 251 then its frequency after filing can be 261.

Hence frequency of A

**before filing = 251**

**after filing = 261**

*Example 5.5. A note produces 6 beats per second with a tuning fork of frequency 312 and 8 beats per second with a fork of frequency 314. Find the frequency of the note.*

**Soln.**

In the first case,



frequency of the tuning fork = 312

beats produced per second = 6

$\therefore$  possible frequency of the note is

either,  $312 + 6 = 318$

or,  $312 - 6 = 306$

In the second case,

frequency of the tuning fork = 314

beats produced per second = 8

$\therefore$  possible frequency of the note is

either  $314 + 8 = 322$

or,  $314 - 8 = 306$

Hence the frequency of the note is **306**.

**Example 5.6.** Determine (i) the velocity of sound in a gas in which two waves of lengths 50 cm and 50.5 cm produce 6 beats per second and (ii) the velocity of sound in water in which waves of lengths 500 cm and 512 cm produce 6 beats per second.

(i) Let  $v$  be the velocity of sound in the gas. Then the frequency  $n_1$  of the first wave

$$= \frac{v}{\lambda_1} = \frac{v}{50} \quad \because \lambda_1 = 50 \text{ cm}$$

and the frequency  $n_2$  of the second wave

$$= \frac{v}{\lambda_2} = \frac{v}{50.5} \quad \because \lambda_2 = 50.5 \text{ cm}$$

But  $n_1 - n_2 = 6$

$$\therefore \frac{v}{\lambda_1} - \frac{v}{\lambda_2} = 6$$

$$\text{or, } \frac{v}{50} - \frac{v}{50.5} = 6$$



$$\text{or, } \frac{0.5 v}{2525} = 6$$

$$\text{or, } v = 30300 \text{ cm/sec} \\ = 303 \text{ m/sec.}$$

(ii) Let  $v'$  be the velocity of sound in water.

$$\text{Then } n_1 = \frac{v'}{\lambda_1} = \frac{v'}{500}$$

$$\text{and } n_2 = \frac{v'}{\lambda_2} = \frac{v'}{512}$$

$$\text{But } n_1 - n_2 = 6$$

$$\therefore \frac{v'}{500} - \frac{v'}{512} = 6$$

$$\text{or, } \frac{12v'}{500 \times 512} = 6$$

$$\text{or, } v' = \frac{6 \times 256000}{12}$$

$$= 128000 \text{ cm/sec}$$

$$= 1280 \text{ m/sec.}$$

### 5.5 Superposition of two waves of nearly equal frequency and wavelength : group velocity

The equation of a plane progressive wave is given by

$$y = a \sin 2\pi \left( \frac{t}{T} - \frac{x}{\lambda} \right)$$

$$= a \sin (wt - kx)$$

$$\text{where } w = \frac{2\pi}{T} = 2\pi n$$

$$\text{and } k = \frac{2\pi}{\lambda}$$

$\omega$  and  $k$  are referred to as the angular frequency and angular wave number of the wave respectively.

Also, the term

$$\frac{\omega}{k} = \frac{2\pi n}{2\pi/\lambda} = n\lambda = v$$

where  $v$  is the wave velocity or phase velocity and is the velocity with which a plane progressive wavefront travels forward (see Art 4.5).

Now consider two waves of slightly different frequencies and wavelengths but having equal amplitudes moving in the same direction with slightly different velocities. Let the waves be in phase at  $t = 0$  and  $x = 0$ . Then the waves may be represented by

$$\begin{aligned} y_1 &= a \sin 2\pi \left( \frac{t}{T} - \frac{x}{\lambda} \right) \\ &= a \sin (\omega_1 t - k_1 x) \end{aligned} \quad (i)$$

$$\begin{aligned} y_2 &= a \sin 2\pi \left( \frac{t}{T'} - \frac{x}{\lambda'} \right) \\ &= a \sin (\omega_2 t - k_2 x) \end{aligned} \quad (ii)$$

Fig. 5.4 shows the two waves at time  $t = 0$ , when the waves are in phase and add constructively at  $x = 0$ . Because of the difference in wavelength, they are out of phase at other places. At some distance  $x_1$ , the waves are  $180^\circ$  out of phase and add to zero. At an equal distance beyond this point, at  $x_2$ , the waves are in phase again. The greater the difference in wavelength, the shorter the distance  $x_1$  in which they become out of phase. Fig. 5.4 can also represent the time dependence of the waves at  $x = 0$ . Initially they are in phase, but because of the difference in frequency, they are  $180^\circ$  out of phase and cancel each other at some time  $t_1$  later. Subsequently they will again be in phase and interfere constructively. The greater the difference in frequency, the sooner they will be out of phase and cancel each other.

The waves given by eqns. (i) and (ii) can be added algebraically and the resultant wave will be given by

$$y = y_1 + y_2$$



$$= a \sin (w_1 t - k_1 x) + a \sin (w_2 t - k_2 x) \quad (\text{iii})$$

Applying the trigonometric relation

$$\sin A + \sin B = 2 \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)$$

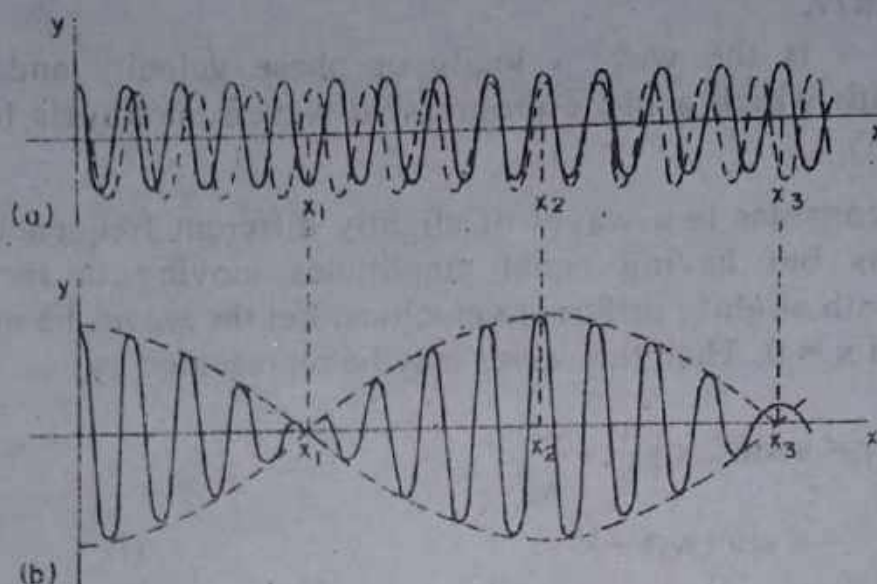


Fig. 5.4

eqn. (iii) can be written as

$$y = 2a \cos \left( \frac{w_1 - w_2}{2} \cdot t - \frac{k_1 - k_2}{2} \cdot x \right) \sin \left( \frac{w_1 + w_2}{2} \cdot t - \frac{k_1 + k_2}{2} \cdot x \right) \quad (5.5)$$

The above equation can be simplified if we use the notation  $\Delta k$  and  $\Delta w$  for the differences in wave number and frequency and  $k$  and  $w$  for their averages.

$$\Delta k = k_1 - k_2,$$

$$\Delta w = w_1 - w_2,$$

$$k = \frac{1}{2} (k_1 + k_2),$$

$$w = \frac{1}{2} (w_1 + w_2),$$

Eqn. (iv) therefore, becomes

$$y = 2a \cos \left( \frac{1}{2} \cdot \Delta w \cdot t - \frac{1}{2} \Delta k \cdot x \right) \sin (wt - k \cdot x) \quad (5.6)$$

The resultant wave is sketched in Fig. 5.4 (b). Since frequencies and wavelengths are nearly equal,  $\Delta w$  and  $\Delta k$  are small and  $w$  and  $k$  are nearly equal to the frequency and wave number of either wave. The result is a wave of about the same frequency and wavelength as the original waves but with the amplitude modulated by the factor  $\cos \left( \frac{1}{2} \Delta w \cdot t - \frac{1}{2} \Delta k \cdot x \right)$ . The velocity of the resultant wave  $v = w/k$  is nearly the same as that of the individual waves and is called the phase velocity. The envelope (dashed curve) gives the cosine wave and is called the *envelope or beat wave*. The formation of this wave is similar to the formation of beats. It consists of a group of waves, each group consisting of a number of waves, with the displacement of the wave at the centre of the group the maximum, with those of others trailing off gradually to zero on either side as shown in the figure. The envelope (or the beat wave) travels as a wave of wave number  $\frac{1}{2} \Delta k$  and angular frequency  $\frac{1}{2} \Delta w$ . The velocity with which this envelope (or the group of waves) travels can be obtained by considering the modulating factor

$$\cos \left( \frac{1}{2} \Delta w \cdot t - \frac{1}{2} \Delta k \cdot x \right)$$

$$= \cos \frac{1}{2} \Delta k \left( x - \frac{\Delta w}{\Delta k} \cdot t \right)$$

$$= \cos \frac{1}{2} \Delta k (x - ut)$$

where  $u = \frac{\Delta w}{\Delta k}$  is the velocity of the envelope and is called the *group velocity*.

$$\therefore u = \frac{\Delta w}{\Delta k}$$

But  $w = v k$  where  $v$  is the phase velocity. Hence the relation between group velocity  $u$  and phase or wave velocity  $v$  is given by

$$u = \frac{\Delta w}{\Delta k} = \frac{\Delta}{\Delta k} (vk)$$



$$\begin{aligned}
 &= v + k \frac{\Delta v}{\Delta k} \\
 &= v + k \frac{\Delta v}{\Delta \lambda} \cdot \frac{\Delta \lambda}{\Delta k} \quad \text{(vi)}
 \end{aligned}$$

Since  $k = \frac{2\pi}{\lambda}$  and hence  $\frac{\Delta \lambda}{\Delta k} = -2\pi/k^2$ .

eqn. (vi) then becomes

$$u = v - \lambda \cdot \frac{\Delta v}{\Delta \lambda} \quad (5.7)$$

The relation between the group velocity and the phase velocity depends on the medium through which the wave is transmitted. In a medium in which the phase velocity does not depend on the frequency of the wave, the group and phase velocities are equal. Such a medium is called a *dispersionless (or non-dispersive)* medium. Examples are waves on a perfectly flexible string, sound waves in air, and light waves in vacuum. On the other hand, if the phase velocity does depend on the frequency or wavelength, the group velocity and phase velocity are not equal. If  $v$  decreases with increasing wavelength the group velocity exceeds the wave velocity and is smaller than the wave velocity if the later increases with increase in wavelength. Examples are water waves, light waves in glass or water, and waves on a string which is not perfectly flexible. A medium for which the phase velocity depends on the frequency (or wavelength) is called a *dispersive* medium.

### 5.6 Interference by reflection—stationary or standing waves

A medium in which a wave is constrained to travel along a fixed linear path is known as a linear medium. As plane waves are one-dimensional, they also travel along a fixed linear path; as such they too may be regarded as waves in a linear medium. Transverse waves along a thin rod are examples of this type of wave.

If the medium be of unlimited or infinite length, the waves just continue to travel through it for an infinite time. However, if the length of the medium be limited or finite, it is referred to as a *linear bounded medium*. If a wave travelling through the medium suffer normal reflection at the boundary, we have *two identical waves - of the same*



wavelength, frequency and amplitude, travelling along the same linear path but in opposite directions. When two such waves are superposed on each other the result is a very special case of interference in which the positions of maximum and minimum displacement remain fixed throughout. The resulting waves appear to remain stationary in space, with no onward or progressive movement. For this reason, the resulting waves are referred to as the *stationary* or *standing* waves.

Now the nature of the boundary of the linear medium where the reflection takes place may be of two types : the boundary of the medium is fixed or *rigid* or the boundary is *free* (or *yielding*). Suppose a wave pulse travels down a stretched string which is fixed at one end (Fig. 5.5). Upon arrival at the end, the wave pulse exerts an upward force on the support. However the support is rigid and as such does not move. But according to Newton's third law, the support exerts an equal but oppositely directed force on the string. This force of reaction results in the generation, at the support, of a wave pulse which travels back along the string in a direction opposite to that of the incident pulse. This is expressed by saying that the incident wave pulse has been reflected at the fixed end point of the string. It may be noticed that the reflected wave pulse returns with its transverse displacement reversed. If a wavetrain is incident on the fixed end point, a reflected wavetrain is generated at that point in the same way. The displacement at any point along the string is the sum of the displacements caused by the incident and reflected waves. Since the end point is fixed, the wavetrain is not free to move and hence the displacement of the particle there will be zero at all times. Thus the two waves must always interfere destructively at that point so as to give zero displacement there. Hence the reflected wave is always  $180^\circ$  out of phase with the incident wave at a fixed boundary. This is expressed by saying that on reflection from a fixed end (boundary) a wave undergoes a phase change of  $180^\circ$ , but the wave is reflected without change of type, *i.e.*, a crest (or a condensation) is reflected as a crest (or a condensation) and a trough (or a rarefaction) is reflected as a trough (or rarefaction), but with change of sign, *i.e.*, with the direction of the displacement of the particles reversed.

Let us now consider the reflection of a pulse at a free end of a stretched string *i.e.*, at an end that is free to move transversely – the so-called *yielding wall*. This can be achieved by attaching the end to a very light ring free to slide without friction along a transverse rod,

or to a long and very much lighter string. Upon arrival at the free end,

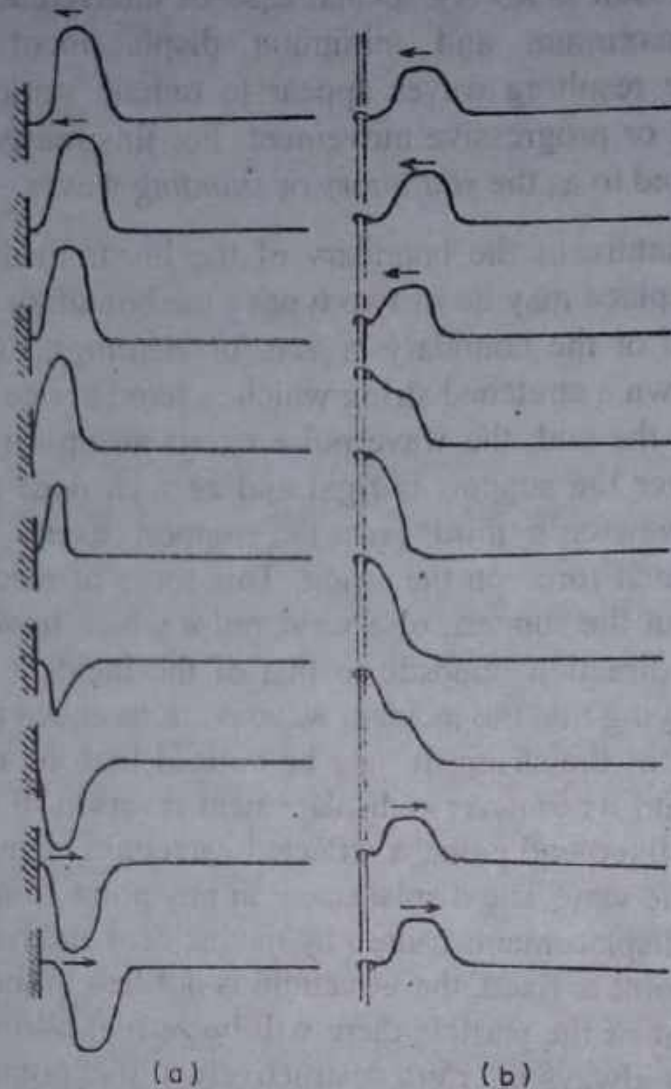


Fig. 5.5

the wave pulse exerts a force on the element of string there. This element is accelerated and its inertia carries it past the equilibrium point; it overshoots and exerts a reaction force on the string. This reaction force generates a pulse which travels back along the string in a direction opposite to that of the incident pulse. Once again we get reflection, but now at a free end. The free end or the yielding wall will obviously suffer the maximum displacement of the particles on the string. Thus the incident and the reflected wave trains must interfere constructively at that point if we are to have a maximum there. Hence the reflected wave is always in phase with the incident wave at that point (Fig. 5.5). This is expressed by saying that at a free end a wave is reflected without change of phase.



This means in other words, that the particles of the medium continue to have their displacements in the same direction as before, with no phase change so that a crest (or condensation) is reflected back as a trough (or rarefaction) and *vice versa*.

Hence, when we have a standing wave in a string, where will be zero displacement at a fixed end and maximum displacement at a free end.

### 5.6 Formation of stationary waves—analytical treatment

The formation of stationary or standing waves will be analytically discussed in this section. The case when the reflection occurs at a fixed or rigid boundary and the case when the reflection occurs at a free boundary (yielding wall) will be separately discussed.

#### Case I : reflection occurs at a fixed or rigid boundary

Let the equation of a simple harmonic wave travelling in the positive direction of the x-axis be

$$y_1 = y_m \sin \frac{2\pi}{\lambda} (vt - x)$$

where

$y_m$  = maximum displacement (amplitude)

$\lambda$  = wavelength of the wave

$v$  = velocity of the wave

If this wave is incident normally on, and reflected from, a fixed or rigid boundary, then the equation of the reflected wave will be

$$y_2 = -y_m \sin \frac{2\pi}{\lambda} (vt + x)$$

since both the direction of displacement of the particles and the direction of propagation of the wave itself, get reversed.

Both the incident and the reflected waves travel along the same linear path; hence they will be superposed on each other and the equation of the resultant stationary or standing wave is, therefore,

$$y = y_1 + y_2$$

$$= \left[ y_m \sin \frac{2\pi}{\lambda} (vt - x) \right] + \left[ -y_m \sin \frac{2\pi}{\lambda} (vt + x) \right]$$



$$\begin{aligned}
&= + y_m \left[ \sin \frac{2\pi}{\lambda} (vt - x) - \sin \frac{2\pi}{\lambda} (vt + x) \right] \\
&= y_m \left[ 2 \cos \frac{1}{2} \left( \frac{2\pi}{\lambda} vt - \frac{2\pi}{\lambda} x + \frac{2\pi}{\lambda} vt + \frac{2\pi}{\lambda} x \right) \right. \\
&\quad \left. \sin \frac{1}{2} \left( \frac{2\pi}{\lambda} vt - \frac{2\pi}{\lambda} x - \frac{2\pi}{\lambda} vt - \frac{2\pi}{\lambda} x \right) \right] \\
&= + y_m \left[ 2 \cos \frac{2\pi}{\lambda} vt \sin \left( - \frac{2\pi}{\lambda} x \right) \right] \\
&= - y_m 2 \cos 2\pi \frac{v}{\lambda} t \sin \frac{2\pi}{\lambda} x \\
&= - y_m 2 \cos \omega t \sin \frac{2\pi}{\lambda} x
\end{aligned}$$

But  $-y_m$  is the maximum displacement of the particles on the negative side. By definition, the maximum displacement on either side of the wave is the amplitude ( $a$ ) of the wave. Hence the above equation can be written as

$$y = 2a \sin \frac{2\pi}{\lambda} x \cos \omega t \quad (5.8)$$

Thus the resulting wave is also a simple harmonic wave of the same time-period and wavelength as the two constituent waves. But its amplitude has changed and is given by

$$A = 2a \sin \frac{2\pi}{\lambda} x \quad (5.9)$$

As can be seen from eqn. 5.8 that a particle at any particular point  $x$  executes simple harmonic motion as time goes on, and that all particles vibrate with the same frequency. In a travelling wave each particle vibrates with the same amplitude.

However, in the case of standing wave the amplitude is not the same for all particles but varies with the location  $x$  of the particle. The amplitude is now a function of  $x$  (eqn. 5.9).

Differentiating eqn. 5.8 with respect to time, we obtain an expression for the velocity of the particle.

$$U = \frac{dy}{dt} = \frac{4\pi av}{\lambda} \sin \frac{2\pi x}{\lambda} \sin \frac{2\pi vt}{\lambda} \quad (5.10)$$

The acceleration of the particle at any given instant of time is

$$\frac{d^2y}{dt^2} = \frac{8\pi^2 av^2}{\lambda^2} \sin \frac{2\pi x}{\lambda} \sin \frac{2\pi vt}{\lambda} \quad (5.11)$$

The strain or compression at any point of the resultant vibration is given by  $dy/dx$ .

Differentiating eqn. 5.8 with respect to  $x$

$$\frac{dy}{dx} = -\frac{4\pi a}{\lambda} \cos \frac{2\pi x}{\lambda} \cos \frac{2\pi vt}{\lambda} \quad (5.12)$$

Eqns. 5.9 to 5.12 show that the amplitude, velocity, acceleration and strain or compression vary with position and time.

### Changes with respect to position

(i) Consider the points where

$$\sin \frac{2\pi x}{\lambda} = 0 \quad \text{and} \quad \cos \frac{2\pi x}{\lambda} = \pm 1$$

From eqns. 5.8 to 5.12,

displacement  $y = 0$

amplitude  $A = 0$

velocity  $\frac{dy}{dt} = 0$

acceleration  $\frac{d^2y}{dt^2} = 0$

strain  $\frac{dy}{dx} = \mp \frac{4\pi a}{\lambda} \cos \frac{4\pi vt}{\lambda}$

Thus at these positions, displacement, amplitude, velocity and acceleration are zero but the strain is maximum. These points of permanent zero displacement are called *nodes* or *nodal points*.



Now  $\sin \frac{2\pi x}{\lambda}$  will be zero (or  $\cos \frac{2\pi x}{\lambda}$  will be  $\pm 1$ ) when

$$\frac{2\pi x}{\lambda} = m\pi,$$

where  $m$  is an integer; 0, 1, 2, 3, . . . . etc.

$$\therefore \frac{2\pi x}{\lambda} = m\pi \quad \text{or} \quad x = \frac{m\lambda}{2}$$

$$\text{or, } x = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, \dots \text{etc.}$$

Thus the nodes or nodal points are obviously equidistant and separated by  $\frac{\lambda}{2}$ . At  $x = 0$ , the position of interface is a node.

(ii) Consider the positions, where

$$\sin \frac{2\pi x}{\lambda} = \pm 1 \quad \text{and} \quad \cos \frac{2\pi x}{\lambda} = 0$$

From eqns. 5.8 to 5.12,

$$\text{displacement} \quad y = \mp 2a \cos \frac{2\pi vt}{\lambda}$$

$$\text{amplitude} \quad A = \pm 2a$$

$$\text{velocity} \quad \frac{dy}{dt} = \pm \frac{2\pi av}{\lambda} \sin \frac{2\pi vt}{\lambda}$$

$$\text{acceleration} \quad \frac{d^2y}{dt^2} = \pm \frac{8\pi^2 av^2}{\lambda^2} \cos \frac{2\pi vt}{\lambda}$$

$$\text{strain} \quad \frac{dy}{dx} = 0$$

It is, therefore, clear that at these points, the displacement will always be maximum.

Now  $\sin \frac{2\pi x}{\lambda}$  will be maximum (or  $\cos \frac{2\pi x}{\lambda}$  will be zero) when

$$\frac{2\pi x}{\lambda} = (2m + 1) \frac{\pi}{2}$$

where  $m = 0, 1, 2, 3, \dots$  etc.

$$\text{or, } x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \dots \text{ etc.}$$

These points of maximum displacements (positive or negative) are called *antinodes* or *antinodal* points. Obviously, the antinodal points are also equidistant and separated by  $\frac{\lambda}{2}$ .

It may be noted that the displacement at the antinodal points also varies simple harmonically with time. But the displacement at these points is always the maximum at any given instant relative to the displacement at all other points, where the displacement lie between the two extremes, decreasing from the maximum at an antinodal point to zero at the preceding or succeeding nodal points.

It can further be seen that no two nodes can exist without an antinode in between and *vice versa*, so that the distance between a node and a succeeding or preceding antinode is  $\frac{\lambda}{4}$ .

It is clear that energy is not transported along the string to the right or to the left, for energy cannot flow past the nodal points in the string which are permanently at rest. Hence, energy remains *standing* in the string although it alternates between vibrational kinetic energy and elastic potential energy. The motion is called a wave motion because it can be thought of as a superposition of waves travelling in opposite direction.

### Changes with respect to time

Consider the instants of time, when

$$\sin \frac{2\pi vt}{\lambda} = 0 \quad \text{and} \quad \cos \frac{2\pi vt}{\lambda} = \pm 1$$

From eqns. 5.8 to 5.12

$$\text{Displacement, } y = \mp 2a \sin \frac{2\pi x}{\lambda}$$

the displacement is maximum (positive or negative)



amplitude,  $A = 2a \sin \frac{2\pi x}{\lambda}$  (independent of time)

velocity,  $\frac{dy}{dt} = 0$

acceleration,  $\frac{d^2y}{dt^2} = \pm \frac{8\pi^2 av^2}{\lambda^2} \sin \frac{2\pi x}{\lambda}$

strain,  $\frac{dy}{dx} = \pm \frac{4\pi a}{\lambda} \cos \frac{2\pi x}{\lambda}$

These equations show that at these instants, the displacement, acceleration and strain are maximum and the velocity of the particles is zero.

Now  $\cos \frac{2\pi vt}{\lambda} = \pm 1$

when  $\frac{2\pi vt}{\lambda} = m\pi$  where  $m = 0, 1, 2, \dots$  etc.

But  $\frac{v}{\lambda} = n = \frac{1}{T}$

$\therefore t = \frac{mT}{2}$

or,  $t = 0, \frac{T}{2}, T, \frac{3T}{2}, \dots$  etc.

At these instants, although the maximum displacement of the particles are all different, each particle is at its extreme position and the velocity of the particles is zero. The pattern is, therefore, stationary at that instant. This instant is called the stationary instant.

(ii) At the instant of time, when

$\sin \frac{2\pi vt}{\lambda} = \pm 1$  and  $\cos \frac{2\pi vt}{\lambda} = 0,$

displacement,  $y = 0$

amplitude,  $A = 2a \sin \frac{2\pi x}{\lambda}$  (independent of time)

particle velocity  $\frac{dy}{dt} = \pm \frac{4\pi av}{\lambda} \sin \frac{2\pi x}{\lambda}$

$$\text{acceleration} \quad \frac{d^2y}{dt^2} = 0$$

$$\text{strain} \quad \frac{dy}{dx} = 0$$

These equations show that at these instants the displacement, acceleration and the strain are all zero.

$$\text{Now } \cos \frac{2\pi vt}{\lambda} = 0$$

$$\text{when } \frac{2\pi vt}{\lambda} = (2m + 1) \frac{\pi}{2} \quad \text{where } m = 0, 1, 2, 3, \dots \text{ etc.}$$

$$\text{or, } 2\pi \left( \frac{1}{T} \right) t = (2m + 1) \frac{\pi}{2}$$

$$\text{or, } t = (2m + 1) \frac{T}{4}$$

i.e., when  $t = \frac{T}{4}, \frac{3T}{4}, \frac{5T}{4}, \dots$  etc., i.e., half a period apart, all the particles pass through their mean positions and have their maximum velocities although these maximum velocities are different for different particles.

### *case II. reflection occurs at a free boundary*

As before, let the equation of a simple harmonic wave, of amplitude  $a$  and wavelength  $\lambda$ , travelling along the positive direction of the  $x$ -axis be

$$y_1 = a \sin \frac{2\pi}{\lambda} (vt - x)$$

Then the equation of the wave reflected at the free boundary will be

$$y_1 = a \sin \frac{2\pi}{\lambda} (vt + x)$$

since only the direction of travel of the wave is reversed and not the direction of displacement of the particles of the medium.



The equation of the resulting stationary or standing wave due to superposition of these two waves is

$$\begin{aligned}
 y &= y_1 + y_2 = a \sin \frac{2\pi}{\lambda} (vt - x) \\
 &\quad + a \sin \frac{2\pi}{\lambda} (vt + x) \\
 &= 2a \sin \frac{2\pi vt}{\lambda} \cos \frac{2\pi x}{\lambda} \\
 &= 2a \cos \frac{2\pi x}{\lambda} \sin \frac{2\pi vt}{\lambda} \\
 &= 2a \cos \frac{2\pi x}{\lambda} \sin \omega t \quad (5.13)
 \end{aligned}$$

$$\text{where } \omega = \frac{2\pi v}{\lambda} = 2\pi n$$

Thus the resulting wave is also simple harmonic having the same time period and wavelength as each of the two constituent waves. However, its amplitude has changed and is no longer constant as can be seen from the relation

$$A = 2a \cos \frac{2\pi x}{\lambda} \quad (5.14)$$

The resultant amplitude is a function of  $x$ . The particle velocity at any instant of time

$$U = \frac{dy}{dt} = \frac{4\pi av}{\lambda} \cos \frac{2\pi x}{\lambda} \cos \frac{2\pi vt}{\lambda} \quad (5.15)$$

Acceleration of the particle at any instant of time

$$\frac{d^2y}{dt^2} = -\frac{8\pi^2 av^2}{\lambda^2} \cos \frac{2\pi x}{\lambda} \sin \frac{2\pi vt}{\lambda} \quad (5.16)$$

The strain or compression at any point of the resultant vibration

$$\frac{dy}{dx} = -\frac{4\pi a}{\lambda} \sin \frac{2\pi x}{\lambda} \sin \frac{2\pi vt}{\lambda} \quad (5.17)$$

Eqns. 5.13 to 5.17 show that the amplitude, velocity, acceleration and strain or compression vary with position and time.

**Changes with respect to position**

(i) Consider the instants of time, then

$$\sin \frac{2\pi x}{\lambda} = 0 \quad \text{and} \quad \cos \frac{2\pi x}{\lambda} = \pm 1$$

Then, from eqns. 5.13 to 5.17

$$\text{displacement, } y = \pm 2a \sin \frac{2\pi vt}{\lambda}$$

$$\text{amplitude, } A = \pm 2a$$

$$\text{velocity, } \frac{dy}{dt} = \pm \frac{4\pi av}{\lambda} \cos \frac{2\pi vt}{\lambda}$$

$$\text{acceleration, } \frac{d^2y}{dt^2} = \mp \frac{8\pi^2 av^2}{\lambda^2} \sin \frac{2\pi vt}{\lambda}$$

$$\text{and strain, } \frac{dy}{dx} = 0$$

As the displacement or the amplitude is maximum, these points correspond to *antinodes*.

$$\text{Now } \sin \frac{2\pi x}{\lambda} = 0 \quad (\text{or } \cos \frac{2\pi x}{\lambda} = \pm 1)$$

$$\text{when } \frac{2\pi x}{\lambda} = m\pi \quad \text{where } m = 0, 1, 2, 3, \dots \text{ etc.}$$

$$\therefore x = \frac{m\lambda}{2}$$

$$\text{or, } x = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, \dots \text{ etc.}$$

Thus the antinodes or the antinodal points are equidistant and separated by  $\lambda/2$ . At  $x = 0$ , i.e., the position of interface is an antinode.

(ii) Consider the positions, where

$$\sin \frac{2\pi x}{\lambda} = \pm 1 \quad \text{and} \quad \cos \frac{2\pi x}{\lambda} = 0,$$

from eqns. 5.13 to 5.17

displacement,  $y = 0$

amplitude,  $A = 0$

velocity,  $\frac{dy}{dt} = 0$

acceleration,  $\frac{d^2y}{dt^2} = 0$

strain,  $\frac{dy}{dx} = \mp \frac{4\pi a}{\lambda} \sin \frac{2\pi vt}{\lambda}$

These positions are called, as we know, the *nodal points* or *nodes*.

Now  $\sin \frac{2\pi x}{\lambda} = \pm 1$  (or  $\cos \frac{2\pi x}{\lambda} = 0$ )

when  $\frac{2\pi x}{\lambda} = (2m+1) \frac{\pi}{2}$  where  $m = 0, 1, 2, 3, \dots$  etc.

or  $x = \frac{(2m+1)\lambda}{4}$

$\therefore x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \dots$  etc.

Thus the nodes are also equidistant and the distance between two successive nodes is  $\lambda/2$ .

### Changes with respect to time

Consider the instant of time, when

$\sin \frac{2\pi vt}{\lambda} = 0$  and  $\cos \frac{2\pi vt}{\lambda} = \pm 1$

From eqns. 5.13 to 5.17

displacement,  $y = 0$

amplitude,  $A = 2a \cos \frac{2\pi x}{\lambda}$  (independent of time)

velocity,  $\frac{dy}{dt} = \pm \frac{4\pi av}{\lambda} \cos \frac{2\pi x}{\lambda}$



acceleration,  $\frac{d^2y}{dt^2} = 0$

and strain,  $\frac{dy}{dx} = 0$

Now  $\sin \frac{2\pi vt}{\lambda} = 0$  (or  $\cos \frac{2\pi vt}{\lambda} = \pm 1$ )

when  $\frac{2\pi vt}{\lambda} = m\pi$  where  $m = 0, 1, 2, 3, \dots$  etc.

or,  $t = \frac{m\lambda}{2v}$

But  $\frac{v}{\lambda} = n$  (frequency)  $= \frac{1}{T}$

$\therefore t = \frac{mT}{2}$

or,  $t = 0, \frac{T}{2}, T, \frac{3T}{2}, \dots$  etc.

Thus particle displacement will be zero while its velocity will be maximum at instants of time  $t = 0, \frac{T}{2}, T, \frac{3T}{2}, \dots$ , i.e., twice in each time period.

(ii) Consider the instants of time, when

$\sin \frac{2\pi vt}{\lambda} = \pm 1$  and  $\cos \frac{2\pi vt}{\lambda} = 0$ ,

from eqns. 5.13 to 5.17

displacement,  $y = \pm 2a \cos \frac{2\pi x}{\lambda}$

amplitude,  $A = 2a \cos \frac{2\pi x}{\lambda}$  (independent of time)

velocity,  $\frac{dy}{dt} = 0$

acceleration,  $\frac{d^2y}{dt^2} = \mp \frac{4\pi a}{\lambda} \sin \frac{2\pi x}{\lambda}$

strain or compression,  $\frac{dy}{dx} = \mp \frac{4\pi a}{\lambda} \sin \frac{2\pi x}{\lambda}$

Now  $\sin \frac{2\pi vt}{\lambda} = \pm 1$  (or  $\cos \frac{2\pi vt}{\lambda} = 0$ )

when  $\frac{2\pi vt}{\lambda} = (2m+1)\frac{\pi}{2}$  where  $m = 0, 1, 2, 3, \dots$  etc.

$$\text{or } t = \frac{(2m+1)\lambda}{4v}$$

$$\text{But } \frac{v}{\lambda} = n = \frac{1}{T}$$

$$\therefore t = \frac{(2m+1)T}{4}$$

$$\text{or, } t = \frac{T}{4}, \frac{3T}{4}, \frac{5T}{4}, \dots \dots \dots \text{etc.}$$

At these instants, all particles attain their maximum displacements (although different for different particles), suffer maximum strain and have maximum acceleration at all positions. But the velocity of the particles at these instants, will be zero. At these instants, each particle is at its extreme position and the pattern becomes stationary at that instant.

### 5.7 Energy of a stationary wave

When a longitudinal wave propagates through a fluid, the bulk modulus (K) of the fluid is given by

$$K = -\frac{p}{dy/dx}$$

where  $p$  is the excess pressure (volume stress) and  $\frac{dy}{dx}$  is the volume strain.

$$\therefore p = -K \cdot \frac{dy}{dx} \dots \dots \dots (i)$$

In the case of a stationary wave formed by reflection at a free boundary, the strain is given by

$$\frac{dy}{dx} = -\frac{4\pi a}{\lambda} \sin \frac{2\pi x}{\lambda} \sin \frac{2\pi vt}{\lambda}$$

$$\text{Also } v = \sqrt{\frac{K}{\rho}} ; \quad \text{or, } v^2 = \sqrt{\frac{K}{\rho}} ; \quad \text{or, } K = v^2 \rho .$$

where  $\rho$  is the density of the fluid.

Substituting the values of  $\frac{dy}{dx}$  and  $K$  in eqn. (1).

$$p = v^2 \rho \cdot \frac{4\pi a}{\lambda} \sin \frac{2\pi x}{\lambda} \sin \frac{2\pi vt}{\lambda} \dots\dots\dots (ii)$$

$$\text{when } \sin \frac{2\pi x}{\lambda} = 1 \text{ and } \sin \frac{2\pi vt}{\lambda} = 1,$$

the excess pressure is maximum.

$$p = p_0 = v^2 \rho \cdot \frac{4\pi a}{\lambda} \dots\dots\dots (iii)$$

From eqns. (ii) and (iii)

$$p = p_0 \sin \frac{2\pi x}{\lambda} \sin \frac{2\pi vt}{\lambda} \dots\dots\dots (iv)$$

$$\text{Putting } p_0 \sin \frac{2\pi x}{\lambda} = p_x$$

eqn. (iv) becomes

$$p = p_x \sin \frac{2\pi vt}{\lambda} \dots\dots\dots (v)$$

The particle velocity at a point is given by

$$U = \frac{dy}{dt} = \frac{4\pi av}{\lambda} \cos \frac{2\pi x}{\lambda} \cos \frac{2\pi vt}{\lambda} \dots\dots (vi)$$

$$\text{Putting } \frac{4\pi av}{\lambda} \cos \frac{2\pi x}{\lambda} = U_x$$

$$\text{we have } U = U_x \cos \frac{2\pi vt}{\lambda} \dots\dots\dots (vii)$$



Now, the energy transferred per unit area in a small interval of time  $dt$  is equal to the work done. Or

$$dI = p \cdot U \cdot dt$$

So, the energy transferred during the whole time period  $T$  is given by

$$\begin{aligned} I &= \int_0^T p \cdot U \cdot dt \\ &= \int_0^T p_x \sin \frac{2\pi vt}{\lambda} \cdot U_x \cos \frac{2\pi vt}{\lambda} \cdot dt \end{aligned}$$

Then, the rate of energy transfer or the average energy transferred per second, say

$$\begin{aligned} I_{av} &= \frac{\int_0^T p \cdot U \cdot dt}{T} \\ &= \frac{p_x U_x}{T} \int_0^T \sin \frac{2\pi vt}{\lambda} \cos \frac{2\pi vt}{\lambda} dt \\ &= \frac{p_x U_x}{T} \int_0^T \sin \frac{4\pi vt}{\lambda} \cdot dt \end{aligned}$$

$$\text{But } \int_0^T \sin \frac{4\pi vt}{\lambda} dt = 0$$

So, the rate of energy transfer = 0.

Thus, there is no transference of energy across any section of the medium in the case of a stationary or standing wave.

### 5.9 Distinction between progressive and stationary waves

Progressive wave	Stationary wave
(i) The vibration characteristics of each particle in the path of a progressive wave is the same and is handed over from particle to particle so that there is an	(i) The vibration characteristics of each particle of the medium is its own which it does not pass on to the others. Thus there is no onward propagation of the wave through the medium – it

onward propagation of the wave through the medium.	remains confined within the space where it is produced.
(ii) The maximum displacement of all particles of the medium is same which they attain one after another.	(ii) The maximum displacement of the particles are not same but they attain their respective maximum displacements simultaneously. The displacement decreases progressively from its maximum value at an antinode to zero at the adjoining node.
(iii) All particles pass through their mean positions with the same maximum velocity but one after another.	(iii) All particles pass through their mean positions simultaneously but with different maximum velocities.
(iv) No particle of the medium is permanently at rest.	(iv) Certain particles of the medium (the displacement nodes) are permanently at rest.
(v) Every region in the path of the wave becomes successively a region of compression, normal pressure and rarefaction. Thus all particles of the medium undergo the same changes of pressure but one after the other.	(v) Condensation and rarefaction do not move along as in the progressive wave. They simply appear and disappear again to be succeeded by the opposite conditions in the same place. The change of pressure is maximum at the nodal points and zero at the antinodal points but occur simultaneously at all points.
(vi) A regular transfer of energy takes place across every section of the medium.	(vi) There is no transference of energy across any section of the medium.

**Example 5.6.** A plane progressive harmonic wave is travelling with a velocity of 340 m/sec in a fluid medium of density 0.0015 gm/sec. If the amplitude of the wave be  $10^{-4}$  cm and its frequency 300 cps, obtain the values of (i) pressure amplitude, (ii) the energy density and (iii) energy current for it.

**Soln.**

(i) In the case of a plane harmonic wave



$$y = a \sin \frac{2\pi}{\lambda} (vt - x)$$

$$\text{and } p = -K \frac{dy}{dx} \text{ (see Art. 5.7)}$$

$$= K \frac{2\pi}{\lambda} a \cos \frac{2\pi}{\lambda} (vt - x)$$

Obviously  $p$  will be maximum when  $\cos \frac{2\pi}{\lambda} (vt - x)$  is maximum i.e., is equal to 1.

So the maximum value of  $p$  or the pressure amplitude  $= K \frac{2\pi}{\lambda} a$ .

Now  $v = \sqrt{\frac{K}{\rho}}$  ; or,  $K = \rho v^2$  where  $\rho$  is the density of the medium.

Hence, pressure amplitude  $= \rho v^2 \cdot \frac{2\pi}{\lambda} a$ .

Again, from  $v = n\lambda$ , we have  $\frac{1}{\lambda} = \frac{n}{v}$ .

$$\begin{aligned} \therefore \text{pressure amplitude} &= 2\pi \rho v^2 a \cdot \frac{n}{v} \\ &= 2\pi \rho v a n \end{aligned}$$

Here  $\rho = 0.0015 \text{ gm/cc}$ ,  $v = 340 \text{ m/sec} = 340 \times 10^2 \text{ cm/sec}$ ,  $a = 10^{-4} \text{ cm}$  and  $n = 300 \text{ cps}$ .

$$\begin{aligned} \therefore \text{pressure amplitude} &= 2\pi \times 0.0015 \times 340 \times 10^2 \times 10^{-4} \times 300 \\ &= 9.6084 \text{ dyne/cm}^2. \end{aligned}$$

$$\begin{aligned} \text{(ii) Energy density, } E &= 2\pi^2 n^2 a^2 \rho \text{ (Art. 4.9)} \\ &= 2\pi^2 \times (300)^2 \times (10^{-4})^2 \times 0.0015 \\ &= 2.665 \times 10^{-5} \text{ erg/cc.} \end{aligned}$$

(iii) Energy current (or energy flux),

$$\begin{aligned} I &= E \cdot v \\ &= 2.665 \times 10^{-5} \times 340 \times 10^2 \\ &= 0.9059 \text{ erg/cm}^2 - \text{sec.} \end{aligned}$$



**Example 5.7 (a).** A sound wave in air, having an amplitude of 0.005 cm and frequency 700 cps. travelling along the direction of positive x-axis with a velocity of 350 m/sec suffers reflection at a free boundary. Obtain the values of (i) the displacement amplitude and (ii) the pressure amplitude in the resulting stationary wave at a point  $x = 50$  cm. (Density of air = 1.293 gm/litre).

(b) Also obtain the distance between (i) two successive displacement nodes and antinodes, (ii) two successive pressure nodes and antinodes, (iii) a displacement node and an adjacent pressure node.

**Soln.**

The equation of a stationary wave formed due to reflection at a free boundary is given by

$$y = 2a \cos \frac{2\pi x}{\lambda} \sin \frac{2\pi vt}{\lambda}$$

where the displacement amplitude

$$= 2a \cos \frac{2\pi x}{\lambda}$$

$a = 0.005$  cm,  $x = 50$  cm and

$$\lambda = \frac{v}{n} = \frac{350 \times 100}{700} = 50 \text{ cm}$$

$\therefore$  displacement amplitude

$$= 2 \times 0.005 \cos 2\pi \times \frac{50}{50} = 0.01 \text{ cm.}$$

Since the cosine term is equal to 1, the displacement amplitude is maximum and the point is, therefore, a displacement antinode.

(ii) excess pressure,

$$p = \rho v^2 \frac{4\pi a}{\lambda} \sin \frac{2\pi x}{\lambda} \sin \frac{2\pi vt}{\lambda}$$

$\therefore$  pressure amplitude

$$= \rho v^2 \frac{4\pi a}{\lambda} \sin \frac{4\pi x}{\lambda}$$

$$= \frac{0.001293 \times (35000)^2 \times 4\pi \times 0.005}{50} \times \sin \frac{2\pi \times 50}{50} = 0$$

Thus the point is a pressure node which is also obvious since the point is a displacement antinode.

(b) Since the distance between two successive displacement nodes or antinodes is  $\lambda/2$ , the distance between a displacement node and an adjacent displacement antinode is  $\lambda/4$  and also because a displacement node coincides with a pressure antinode and *vice versa*.

distance between two successive displacement nodes = distance between two successive displacement antinodes

$$= \lambda/2 = 50/2 = 25.0 \text{ cm.}$$

Therefore, the distance between two successive pressure nodes (or antinodes) is also  $= \lambda/2 = 50/2 = 25.0 \text{ cm.}$

The distance between a displacement node (or pressure antinode) and an adjacent pressure node (or displacement antinode)  $= \lambda/4 = 50/2 = 12.5 \text{ cm.}$

### EXERCISES

- [1] What is interference of sound waves? Distinguish between constructive and destructive interference.
- [2] Analytically discuss the interference of sound and obtain the condition for maximum and minimum intensity.
- [3] Show that in interference of sound waves, energy is not destroyed – it is simply redistributed from the points of minimum intensity to those of maximum intensity.
- [4] What are beats? How are they produced? Analytically discuss the formation of beats and show that the number of beats per second is equal to the difference in frequency of the two notes.
- [5] Distinguish between phase velocity and group velocity of a train of waves and establish a relationship between the two. Show that in a non-dispersive medium they are same.
- [6] Show that the group velocity is given by the relation

$$U = v - \lambda \frac{dv}{d\lambda}$$



- [7] Discuss analytically the formation of stationary waves in a linear bounded medium. Why are they referred to as stationary or standing waves? How do they differ from ordinary progressive waves?
- [8] Discuss analytically the formation of stationary waves due to reflection
- (i) at a rigid boundary
  - and (ii) at a free boundary

Explain how the characteristics change (i) with position and (ii) with time.

- [9] What are nodes and antinodes? Analytically discuss the formation of nodes and antinodes and hence show that two nodes or two antinodes are separated from each other by a distance  $\lambda/2$  apart. Discuss the property of the medium at the positions of node and antinode.
- [10] What is meant by the principle of superposition of waves? Use it to explain the formation of stationary waves in a gaseous medium. Show that in the case of stationary wave, no energy is transferred across any section of the medium.
- [11] Two tuning forks A and B give 5 beats per second. The frequency of A = 512. When B is filed, 5 beats per second are again produced. Find the frequency of B before and after filing.

[before filing 507; after filing 517]

- [12] When two tuning forks are sounded together, 20 beats are produced in 8 seconds. After loading one of the tuning forks with a little wax, they produce 32 beats in 8 seconds. If the unloaded fork had a frequency of 512 Hz, calculate the frequency of the other. [509.5]
- [13] A note produces 4 beats per second with a tuning fork of frequency 512 and 6 beats per second with a tuning fork of frequency 514. Find the frequency of the note. [508].
- [14] A sound wave is represented by the equation  $5 \sin 0.3142 (500t - x)$  where  $t$  is in sec and  $x$  in cm. Write the equation of the wave which, on superposition with it, would produce a standing wave.

[(i)  $-5 \sin 0.3142 (500t + x)$ ; (ii)  $5 \sin 0.3142 (500t + x)$ ]

- [15] A progressive wavetrain in air, of amplitude 0.005 cm, frequency 1000 cps and wavelength, 80 cm, and travelling along the positive direction of  $x$ , is reflected normally at a rigid boundary. Obtain the values of (i) displacement amplitude, (ii) particle velocity amplitude, and (iii) amplitude of pressure variation at  $x = 50$  cm. [(i) 0.01 cm, (ii) 62.84 cm/sec, (iii) 0]



## CHAPTER VI

### VELOCITY OF SOUND AND VIBRATION OF STRINGS

*Velocity of longitudinal waves in a gaseous medium-Velocity of sound in liquids-Velocity of sound waves in solids-Velocity of transverse waves along a stretched string-Frequency of a vibrating string-Laws of transverse vibrations of a stretched string-Sonometer-Verification of the laws of transverse vibrations of strings-Melde's experiment-Verification of the laws of transverse vibrations of a string by Melde's experiment-Determination of the frequency of a tuning fork by Melde's experiment-Solved problems-Exercises.*

#### 6.1 *Velocity of longitudinal waves (or sound waves) in a gaseous medium.*

In a gaseous or a fluid medium, only a longitudinal wave motion is possible, with its particles executing a simple harmonic motion along the direction of propagation of the wave. During the wave motion the particles crowd together at some region of the medium while they get further apart at some other region *i.e.*, the wave travels in the form of condensations and rarefactions. Thus there is a continuous variation of pressure all along the direction of propagation of the wave.

To obtain an expression for the velocity of longitudinal wave in a gaseous medium, let us imagine the wave to be travelling from left to right along the axis of  $x$  through a uniform cylindrical tube of area of cross-section  $\alpha$  and its axis coinciding with the axis of  $x$ .

Let AB and CD be two layers of the medium (or two plane sections of the tube perpendicular to its axis) whose positions before the passage of the wave are at distances  $x$  and  $x + \delta x$  respectively from some arbitrary origin O (Fig 6.1). The two layers, therefore, lie  $\delta x$  apart and the volume of the cylindrical slice of the gas (or the fluid) enclosed in between them is  $\delta x \cdot \alpha$  where  $\alpha$  is the area of cross-section of each of the layer (*i.e.*, the tube).

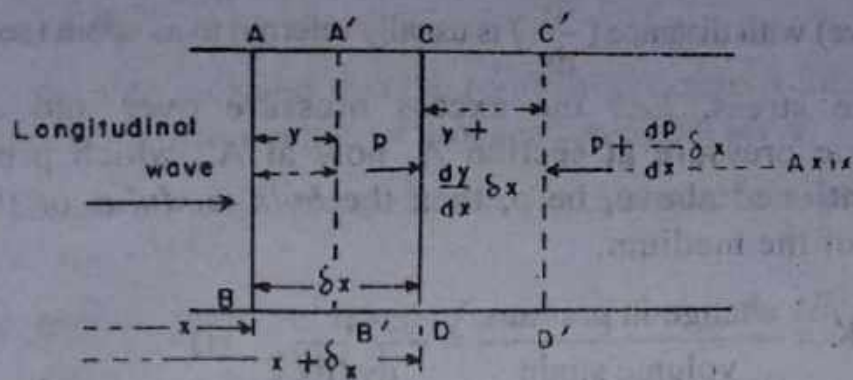


Fig. 6.1

When the pulses of compressions or rarefactions pass through the medium, let A'B' and C'D' be the new positions of the layers. In moving to these new positions let the displacement of AB be  $y$  i.e.

$AA' = BB' = y$ . The displacement of CD will, then, be  $y + \left(\frac{dy}{dx}\right) \delta x$ ,

where  $\frac{dy}{dx}$  is the rate of change of displacement with distance.

$$\text{or, } CC' = DD' = y + \frac{dy}{dx} \cdot \delta x.$$

The two layers thus get displaced further apart and the increase in the distance of separation between them  $= \left(y + \frac{dy}{dx} \cdot \delta x\right) - y = \frac{dy}{dx} \cdot \delta x$ .

Therefore, the increase in the volume of the cylindrical slice of the gas in between the two layers = increase in distance (length)  $\times$  cross-section

$$\text{of the layer} = \frac{dy}{dx} \cdot \delta x \cdot \alpha.$$

Hence, the volume strain (or simply strain) produced in the medium

$$= \frac{\text{increase (or change) in volume}}{\text{original volume}}$$

$$= \frac{\left(\frac{dy}{dx}\right) \cdot \delta x \cdot \alpha}{\delta x \cdot \alpha}$$

$$= \frac{dy}{dx}$$



Thus the rate of change of displacement of a longitudinal wave motion (sound wave) with distance ( $\frac{dy}{dx}$ ) is usually referred to as strain (see Art. 4.5).

If the stress, *i.e.*, the excess pressure over and above the atmospheric pressure at section A, now at A', which produces the strain mentioned above, be  $p$ , then the *bulk modulus* or the *volume elasticity* of the medium.

$$K = \frac{\text{change in pressure}}{\text{volume strain}} = -\frac{p}{dy/dx} \quad (i)$$

The minus sign indicates that an increase in pressure is accompanied by a decrease in volume and *vice versa*.

The excess pressure  $p$  at A now at A' is given by

$$p = K \left( -\frac{dy}{dx} \right) \quad (ii)$$

If  $\frac{dy}{dx}$  is negative,  $p$  is positive, *i.e.*, it is a region of compression. If

$\frac{dy}{dx}$  is positive,  $p$  is negative, *i.e.*, it is a region of rarefaction. Thus  $\frac{dy}{dx}$  is a measure of the change of pressure and hence, of the density of the medium.

If the pressure gradient *i.e.*, the rate of change of pressure with distance along the x-axis be  $dp/dx$ , then the excess pressure at section CD now at C' D' is given by

$$\begin{aligned} & p + \frac{dp}{dx} \cdot \delta x \\ &= p + \frac{d}{dx} \left( -K \frac{dy}{dx} \right) \cdot \delta x \\ &= p - K \frac{d^2 y}{dx^2} \cdot \delta x \end{aligned}$$

Hence the resultant pressure on the slice of the gas enclosed between A'B' and C'D'.

$$= p - K \frac{d^2 y}{dx^2} \cdot \delta x - p$$



$$= -K \frac{d^2 y}{dx^2} \cdot \delta x$$

The minus sign indicates that the resultant pressure is directed from B' to A', opposite to the direction of propagation of the wave.

Now, *pressure = force per unit area* on the slice of the gas enclosed between A'B' and C'D'.

Hence, *force = pressure  $\times$  area of cross section of the slice*

$$= (-K \cdot \frac{d^2 y}{dx^2} \cdot \delta x) \times \alpha \quad (v)$$

Now, force is also equal to mass  $\times$  acceleration. If  $-\frac{d^2 y}{dt^2}$  is the acceleration of the slice of the gas enclosed between the two sections A'B' and C'D' in the direction C'D' to A'B', then the force acting on it

$$= \text{mass} \times (-\frac{d^2 y}{dt^2}).$$

If  $\rho$  is the density of the gas (or the fluid), then the mass of the gas enclosed

$$= \alpha \cdot \delta x \cdot \rho$$

Hence

$$\text{Force, } F = \alpha \cdot \delta x \cdot \rho \cdot (-\frac{d^2 y}{dt^2}) \quad (vi)$$

Equating the two values of force, we have

$$\alpha \cdot \delta x \cdot \rho \cdot (-\frac{d^2 y}{dt^2}) = (-K \cdot \frac{d^2 y}{dx^2} \cdot \delta x) \cdot \alpha$$

$$\text{or, } \frac{d^2 y}{dt^2} = \frac{K}{\rho} \cdot \frac{d^2 y}{dx^2} \quad (6.1)$$

Eqn. (6.1) is of the same form as the standard differential equation of wave motion (eqn. 4.15). Hence the velocity of the longitudinal wave through the gas is given by

$$v^2 = \frac{K}{\rho}; \quad \text{or,} \quad v = \sqrt{\frac{K}{\rho}} \quad (6.2)$$

i.e., the velocity of the wave

$$= \sqrt{\frac{\text{volume elasticity of the gas}}{\text{density of the gas}}}$$

Eqn.(6.2) is true only in the case of *plane waves* whose original amplitude is very small – a requirement fulfilled by sound waves of ordinary intensity.

### Newton's formula for velocity of sound

Newton assumed that the passage of the train of longitudinal wave through air is so slow that the heat generated in the region of compression is dissipated in the adjacent colder region of rarefaction, thereby equalizing the temperature. Thus the temperature of the gas (air) remains unaffected. In other words, the sound wave passes through a gas under isothermal condition.  $K$  in eqn.(6.2) is, therefore, the isothermal elasticity of the gas. Now the isothermal elasticity

$$K = - \frac{P}{dV/V} = P$$

where  $P$  is the pressure of the gas.

Newton's formula for the velocity of a sound wave (longitudinal wave) in a gaseous medium, therefore, becomes

$$v = \sqrt{\frac{P}{\rho}} \quad (6.3)$$

Taking air at N.T.P.,

$$P = 76 \times 13.6 \times 981 \text{ dyne/cm}^2$$

$$\rho = 0.001293 \text{ gm/cm}^3$$

Therefore,

$$v = \sqrt{\frac{76 \times 13.6 \times 981}{0.001293}}$$



= 280 metres per second.

Actually, the velocity of sound in air at N.T.P. as measured by Newton himself is found to be 332 meters per second.

Newton could not offer any explanation for this large discrepancy between his experimentally measured value and the theoretically calculated value. Strangely enough, it was 140 years before Laplace pointed out the error in Newton's formula in 1816.

### Laplace's correction

The discrepancy between experimentally determined and theoretically calculated values was explained by Laplace after nearly a century and a half. He pointed out that the condensations and rarefactions in sound wave in a gas succeed each other much too rapidly. This coupled with the fact that a gas (or air) is a poor conductor of heat does not make for any equalization of temperature due to passage of heat from the hotter regions of condensation to the colder ones of rarefaction. Thus, although the total quantity of heat in the medium (gas or air) remains constant, there are temperature variations throughout the medium. A sound wave, therefore, passes through a gas (or air) under adiabatic, and not isothermal, condition.

Now *adiabatic elasticity*

=  $\gamma \times$  isothermal elasticity =  $\gamma \times P$  where  $\gamma$  is the ratio between the specific heat at constant pressure of the gas to its specific heat at constant volume.

Thus, Newton's formula, as corrected by Laplace becomes

$$v = \sqrt{\frac{\gamma \cdot P}{\rho}} \quad (6.4)$$

where  $\gamma = 1.4$  for diatomic gases (oxygen, hydrogen, etc.).

For air, which is a mixture of gases,  $\gamma = 1.41$

$$\therefore v = \frac{1.41 \times 76 \times 13.6 \times 981}{0.001293}$$

$$= \sqrt{1.41} \times 280$$

$$= 331.6 \text{ meters per second at N.T.P.}$$



The calculated value is in excellent agreement with the measured value of 331.5 m/sec.

### Effect of pressure and temperature

Let  $v_0$  be the velocity of sound at  $0^\circ\text{C}$  and pressure  $P_0$  and density  $\rho_0$ . Then

$$v_0 = \sqrt{\frac{\gamma \cdot P_0}{\rho_0}}$$

When pressure is  $P_1$ , let the corresponding values of temperature and density be  $T_1$   $^\circ\text{C}$  and  $\rho_1$ . If the velocity of sound is now  $v_1$ , then

$$v_1 = \sqrt{\frac{\gamma \cdot P_1}{\rho_1}}$$

Therefore,

$$\begin{aligned} \frac{v_1}{v_0} &= \sqrt{\frac{\gamma P_1}{\rho_1}} \times \frac{\rho_0}{\gamma P_0} \\ &= \sqrt{\frac{P_1}{P_0}} \times \frac{\rho_0}{\rho_1} \end{aligned}$$

If the quantity of the gas (or air) considered be 1 gm-molecule, then we have  $PV = RT$  where  $V$  is the volume of a gm-molecule of the gas,  $T$  its absolute temperature and  $R$ , the gas constant.

If  $M$  be the molecular weight of the gas, then  $\rho = \frac{M}{V}$ . Then

$$\begin{aligned} \frac{v_1}{v_0} &= \sqrt{\frac{P_1}{P_0}} \cdot \frac{\rho_0}{\rho_1} = \sqrt{\frac{P_1}{P_0}} \cdot \frac{M}{V_0} \times \frac{V_1}{M} \\ &= \sqrt{\frac{P_1 V_1}{P_0 V_0}} \end{aligned}$$

But from the gas law, we have

$$P_1 V_1 = RT_1 \text{ and } P_0 V_0 = RT_0$$

$$\begin{aligned}
 \text{or, } \frac{P_1 V_1}{P_0 V_0} &= \frac{RT_1}{RT_0} = \frac{T_1}{T_0} \\
 \therefore \frac{v_1}{v_0} &= \sqrt{\frac{P_1 V_1}{P_0 V_0}} = \sqrt{\frac{T_1}{T_0}} = \sqrt{\frac{273+t}{273}} \\
 &= \left(1 + \frac{t}{273}\right)^{1/2} \quad (6.5)
 \end{aligned}$$

where  $t$  is the temperature in degree Celsius.

Thus, the velocity of sound in a gas does not depend upon the pressure to which the gas is subjected; but it depends upon the temperature and increases with the rise of temperature as given by eqn. (6.5)

### Effect of moisture

Let  $v_d$  and  $v_m$  be the velocities of sound in dry and moist air respectively. If  $\rho_d$  and  $\rho_m$  denote the density of dry air and moist air and  $P$  is the pressure of air, then we have

$$v_d = \sqrt{\frac{\gamma P}{\rho_d}} \quad \text{and} \quad v_m = \sqrt{\frac{\gamma P}{\rho_m}}$$

where  $\gamma$  is the ratio of the two specific heats of air.

Now since moist air is lighter than dry air.

$$\rho_m < \rho_d$$

$$\text{or, } \frac{1}{\rho_m} > \frac{1}{\rho_d}$$

$$\text{or, } \frac{\gamma P}{\rho_m} > \frac{\gamma P}{\rho_d}$$

$$\text{or, } \sqrt{\frac{\gamma P}{\rho_m}} > \sqrt{\frac{\gamma P}{\rho_d}}$$

$$\text{or, } v_m > v_d$$

Thus the velocity of sound in moist air is greater than the velocity of sound in dry air.

## 6.2 Velocity of sound in liquids

The formula for the velocity of propagation of sound, derived in the previous article, applies to the cases of fluids, *i.e.*, it will hold good both for a gas as well as a liquid. If  $v$  is the velocity of sound in a liquid then  $v$  is given by the relation

$$v = \sqrt{\frac{K}{\rho}}$$

where  $K$  is the bulk modulus and  $\rho$  is the density of the liquid. The bulk modulus referred to in the formula is the adiabatic bulk modulus and not the isothermal bulk modulus. However, for water,  $\gamma$  is nearly equal to unity so that the adiabatic bulk modulus of water is very nearly equal to its isothermal elasticity. Thus the velocity of sound in water is given by the square root of the ratio of the isothermal bulk modulus of water to its density.

## 6.3 Velocity of sound waves in isotropic solids

Newton's formula for the velocity of sound (longitudinal waves) in a fluid may be equally applied to the case of propagation of sound through a solid. However, the solid should be isotropic and should be in the form of a thin rod or wire (*i.e.*, one whose diameter is very much smaller than its length). Obviously, the elasticity involved here will be the linear elasticity or Young's modulus  $Y$ , so that the velocity of a longitudinal wave travelling along a solid rod is given by  $v = \sqrt{Y/\rho}$ . This relation may, however, also be deduced directly in the same manner as used in Art. 6.1.

Let  $AB$  and  $CD$  be two plane sections of a solid rod of uniform cross-section and let their distances measured along the  $x$ -axis with reference to some arbitrary origin  $O$  be  $x$  and  $x + \delta x$  respectively (Fig. 6.2).

Let a longitudinal wave propagating along the  $x$ -axis be incident on these sections and displace them to their new positions  $A'B'$  and  $C'D'$ . If, after the passage of the wave, the displacement of the section  $AB$  be  $y$  *i.e.*,  $AA' = BB' = y$ , then the displacement of the section  $CD$  will be  $CC' = DD' = (y + \frac{dy}{dx} \delta x)$ , where  $dy/dx$  is the rate of displacement with distance.



Now the length of the slice AC (=BD) in the undisplaced position is  $\delta x$  and that in the displaced position is  $\delta x + \frac{dy}{dx} \cdot \delta x$ . Then, obviously, increase in length of element  $\delta x$  of the rod

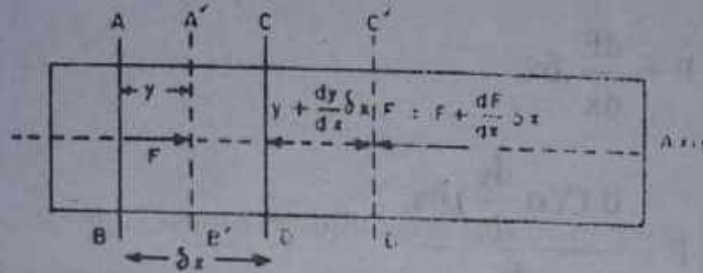


Fig. 6.2

$$= \delta x + \frac{dy}{dx} \delta x - \delta x = \frac{dy}{dx} \cdot \delta x$$

Therefore, the *longitudinal (or tensile) strain* set up in the rod

$$= \frac{\text{increase in length}}{\text{original length}}$$

$$= \frac{(dy/dx) \cdot \delta x}{\delta x} = \frac{dy}{dx}$$

This increase in length of the element gives rise to restoring forces in the material of the rod, tending to bring it back to its original length. If  $F$  is the force acting at section A now at A', then the longitudinal (or tensile) stress =  $F/\alpha$ , where  $\alpha$  is the area of cross-section of the rod.

Therefore, the Young's modulus for the material of the rod.

$$Y = \frac{\text{tensile stress}}{\text{tensile strain}}$$

$$= \frac{F/\alpha}{dy/dx}$$

$$\text{or, } F = Y \cdot \alpha \frac{dy}{dx}$$

in the direction A' to C'

If  $\frac{dF}{dx}$  is the rate of change of force with distance, then the force acting at section C now at C', in the direction C' to A' is

$$F' = F + \frac{dF}{dx} \cdot \delta x.$$

$$\begin{aligned} \text{or, } F' &= F + \frac{d(Y\alpha \frac{dy}{dx}) \cdot \delta x}{dx} \\ &= Y \cdot \alpha \cdot \frac{dy}{dx} + Y \cdot \alpha \cdot \frac{d^2y}{dx^2} \cdot \delta x \end{aligned}$$

Hence, the resultant force acting on the element  $\delta x$  of the rod,

$$\begin{aligned} &= F' - F \\ &= Y \cdot \alpha \cdot \frac{dy}{dx} + Y \cdot \alpha \cdot \frac{d^2y}{dx^2} \cdot \delta x - Y \cdot \alpha \cdot \frac{dy}{dx} \\ &= Y \cdot \alpha \cdot \frac{d^2y}{dx^2} \cdot \delta x \end{aligned}$$

in the direction C' to A'.

If  $\rho$  denotes the density of the solid, then the mass of the slice =  $\alpha \cdot \delta x \cdot \rho$ . And if  $\frac{d^2y}{dt^2}$  is the instantaneous acceleration produced in the element  $\delta x$  by the resultant force, in the direction C' to A', we have

Force acting on the element  $\delta x$

$$= \text{mass of element} \times \frac{d^2y}{dt^2}$$

$$\text{or, } Y \cdot \alpha \cdot \frac{d^2y}{dx^2} \cdot \delta x = \alpha \cdot \delta x \cdot \rho \cdot \frac{d^2y}{dt^2}$$

$$\text{or, } \frac{d^2y}{dt^2} = \frac{Y}{\rho} \cdot \frac{d^2y}{dx^2} \quad (6.6)$$

The above equation is of the same form as the standard differential equation of wave motion. Comparing these equations, we have

$$v^2 = \frac{Y}{\rho}$$

$$\text{or, } v = \sqrt{\frac{Y}{\rho}}$$

or, the velocity of sound (longitudinal) wave in a rod is  $v = \sqrt{Y/\rho}$ ,

**Example 6.1.** A sound wave of frequency 1000 and amplitude  $2.5 \times 10^{-5}$  cm is travelling through air at N. T. P. (a) Obtain the values of its velocity and wavelength. (b) Write down the equation of the wave. (c) Obtain the values of particle velocity amplitude and amplitude of pressure variation. (Density of air at N. T. P is 1.293 gm/litre and  $\gamma$  for air = 1.4)

**Soln.**

(a) The velocity of the wave is given by

$$v = \sqrt{\frac{\gamma P}{\rho}}$$

Now  $P = 76 \times 13.6 \times 981$  dynes/cm<sup>2</sup>

$\rho = 1.293$  gm / litre

$= 1.293 \times 10^{-3}$  gm / c.c

$$\therefore v = \sqrt{\frac{1.4 \times 76 \times 13.6 \times 981}{1.293 \times 10^{-3}}}$$

$= 33130$  cm/sec.

And from  $v = n\lambda$  we have

$$\text{wavelength, } \lambda = \frac{v}{n} = \frac{33130}{1000} = 33.13 \text{ cm.}$$

(b) The most general form of the equation of a plane progressive wave is given by



$$y = a \sin \frac{2\pi}{\lambda} (vt - x)$$

Substituting the values of  $a$ ,  $\lambda$  and  $v$ , we have

$$y = 2.5 \times 10^{-5} \sin \frac{2\pi}{33.13} (33130t - x)$$

(c) Particle velocity

$$U = \frac{dy}{dt} = \frac{2\pi\pi}{\lambda} a \cos \frac{2\pi}{\lambda} (vt - x)$$

Therefore, the particle velocity amplitude or the maximum value of the particle velocity

$$= \frac{2\pi v}{\lambda} \cdot a$$

$$= 2\pi na$$

$$= 2 \times 3.14 \times 1000 \times 2.5 \times 10^{-5}$$

$$= 0.157 \text{ cm/sec.}$$

The pressure variation is given by

$$p = -K \frac{dy}{dx} \quad (\text{Art. 5.1})$$

$$= -K \cdot \frac{2\pi}{\lambda} a \cos \frac{2\pi}{\lambda} (vt - x)$$

Hence the maximum value (or amplitude) of pressure variation

$$= K \cdot \frac{2\pi}{\lambda} \cdot a \quad (\text{ignoring the - sign})$$

$$= v^2 \cdot \rho \cdot \frac{2\pi}{\lambda} \cdot a \quad (\because v = \sqrt{\frac{K}{\rho}})$$

$$= 2\pi v \rho n a$$

$$= 2\pi \times 33130 \times 1.293 \times 10^{-3} \times 1000 \times 2.5 \times 10^{-5}$$

$$= 6.726 \text{ dynes/cm}^2.$$

**Example 6.2.** Prove that the velocity of sound in air increases by 61 cm/sec for each degree Celsius rise in temperature. The velocity of sound in air at 0°C is 332 m/sec.

**Soln.**

Let  $v_0$  and  $v_t$  be the velocities of sound in air at 0°C and  $t$ °C respectively. Then

$$\frac{v_t}{v_0} = \sqrt{\frac{T}{T_0}} \text{ where } T \text{ and } T_0 \text{ are in degree absolute.}$$

$$= \sqrt{\frac{273 + t}{273 + 0}}$$

$$= \left(1 + \frac{t}{273}\right)^{1/2}$$

$$= 1 + \frac{1}{2} \times \frac{t}{273}$$

$$\text{or, } v_t = v_0 \left(1 + \frac{t}{546}\right)$$

$$= 332 \left(1 + \frac{t}{546}\right)$$

$$= 332 + 0.61 = 332.61 \text{ m/sec. } [t = 1^\circ\text{C}]$$

$$\therefore v_t - v_0 = .61 \text{ m/sec} = 61 \text{ cm/sec.}$$

**Example 6.3** The velocity of sound in air at N.T.P is 33200 cm per second. Find the velocity at a temperature of 50°C and a pressure of 70 cms. of mercury.

**Soln.**

Let  $v_2$  and  $v_1$  be the velocity of sound in air at 50°C and 0°C, respectively. Since the velocity of sound is independent of pressure, we have

$$\frac{v_2}{v_1} = \sqrt{\frac{T_2}{T_1}} = \sqrt{\frac{273 + 50}{273 + 0}}$$

$$\begin{aligned}
 \text{or, } v_2 &= v_1 \times \sqrt{\frac{323}{273}} \\
 &= 33200 \times 1.0877 \\
 &= 36120 \text{ cm/sec.}
 \end{aligned}$$

**Example 6.4.** Find the temperature at which sound travels in hydrogen with the same velocity as in oxygen at  $1000^\circ\text{C}$ . Assume  $\gamma$  to be the same for the two gases. Density of oxygen is 16 times that of hydrogen.

**Soln.**

Let the required temperature of hydrogen be  $T_2$  K.

Here  $T_1\text{K} = 273 + 1000$

$= 1273$  K is the temperature of oxygen.

If  $\rho_1$  and  $\rho_2$  be the densities of oxygen and hydrogen, then we have

$$\frac{T_1}{\rho_1} = \frac{T_2}{\rho_2}; \quad \text{or, } T_2 = T_1 \times \frac{\rho_2}{\rho_1}$$

$$\text{Here } \frac{\rho_2}{\rho_1} = \frac{1}{16}$$

$$\begin{aligned}
 \therefore T_2 &= 1273 \times \frac{1}{16} = 79.56 \text{ K} \\
 &= -193.44^\circ\text{C.}
 \end{aligned}$$

**Example 6.5.** If the velocity of sound in hydrogen at  $0^\circ\text{C}$  is 1280 m/sec, what will be the velocity of sound (at the same temperature) in a mixture of two parts by volume of hydrogen to one of oxygen? Density of oxygen is 16 times that of hydrogen.

**Soln.**

Let  $\rho_H \rightarrow$  density of hydrogen

$\rho_M \rightarrow$  density of mixture of hydrogen and oxygen.

Then, if  $v_H$  and  $v_M$  be the velocities of sound in hydrogen and the mixture respectively, we have



$$v_M = \sqrt{\frac{\gamma P}{\rho_M}} \quad \text{and} \quad v_H = \sqrt{\frac{\gamma P}{\rho_H}}$$

$$\text{Therefore, } \frac{v_M}{v_H} = \sqrt{\frac{\rho_H}{\rho_M}}$$

Now the mixture consists of two parts by volume of hydrogen and one part by volume of oxygen. So that, if each part by volume be  $V$ , then

mass of two parts by volume of hydrogen =  $2V \times 1$  and mass of one part by volume of oxygen =  $V \times 16$

$\therefore$  mass of a volume  $3V$  of the mixture =  $(2V + 16V) = 18V$  mass units.

So the density of the mixture,

$$\rho_M = \frac{\text{mass}}{\text{volume}} = \frac{18V}{3V} = 6 \text{ mass units/c.c.}$$

$$\text{Hence } \frac{v_M}{v_H} = \sqrt{\frac{1}{6}}; \quad \text{or, } v_M = v_H \sqrt{\frac{1}{6}}$$

$$= 1280 \times \sqrt{\frac{1}{6}} = 523 \text{ m/sec}$$

or, the velocity of sound in the given mixture = **523 m/sec.**

**Example 6.6.** The planet Jupiter has an atmosphere of a mixture of ammonia and methane at a temperature of  $-130^\circ\text{C}$ . If  $\gamma$  for the mixture is 1.3 and molecular weight of the mixture is 16.5, calculate the velocity of sound on this planet, (Gas constant  $R = 8.3$  joules per degree per gm. molecule).

**Soln.**

The molecular weight of the mixture of ammonia and methane, i.e., the mass of 1 gm. molecule of the mixture = 16.5. Therefore, if  $\rho$  is the density of the mixture, then the

$$\text{volume of 1 gm-molecule of it, i.e., } V = \frac{16.5}{\rho}$$

From the standard gas equation we have,

$$PV = RT.$$

Here  $R = 8.3$  joules per degree per gm. molecule

$$= 8.3 \times 10^7 \text{ ergs per degree per gm. molecule}$$

$$T = -130^\circ\text{C} = (-130 + 273)$$

$$= 143^\circ\text{K.}$$

$$\text{and } V = \frac{16.5}{\rho}$$

Substituting these values, we have

$$P \cdot \frac{16.5}{\rho} = 8.3 \times 10^7 \times 143$$

$$\text{or, } \frac{P}{\rho} = \frac{8.3 \times 10^7 \times 143}{16.5} = 71.9 \times 10^7$$

If  $v$  be the velocity of sound in the mixture *i.e.*, on the planet Jupiter, then

$$\begin{aligned} v &= \sqrt{\frac{\gamma P}{\rho}} = \sqrt{1.3 \times 71.9 \times 10^7} \\ &= 30.59 \times 10^3 \text{ cm/sec} \\ &= 305.9 \text{ m/sec.} \end{aligned}$$

**Example 6.7.** Prove that  $v = c \sqrt{\frac{\gamma}{3}}$  where  $v$  is the velocity of sound in a gas,  $c$ , the root mean square velocity of the molecules of the gas and  $\gamma$  is the ratio of its specific heats.

**Soln.**

Let  $P$  be the pressure and  $\rho$  the density of the gas. Then the velocity of sound,  $v$ , in the gas is

$$v = \sqrt{\frac{\gamma P}{\rho}}.$$

Now, in accordance with the kinetic theory of gases, we have (Art. ....),

$$p = \frac{1}{3} \rho c^2; \quad \text{or, } \frac{p}{\rho} = \frac{1}{3} c^2.$$

where  $c$  is the root mean square velocity of the molecules of the gas.

$$\therefore v = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\frac{\gamma c^2}{3}} = c \sqrt{\frac{\gamma}{3}}.$$

**Example 6.8.** Prove that at any point in the course of a plane progressive wave, the ratio of the excess pressure to the velocity of the air particles at that place is constant, provided the temperature and pressure do not alter.

**Soln.**

We know (Arts. 4.7 and 4.8),

$$\text{particle velocity, } U = \frac{dy}{dt} = -v \cdot \frac{dy}{dx}$$

$$\text{and excess pressure, } p = -K \cdot \frac{dy}{dx}$$

$$= -\gamma \cdot P \frac{dy}{dx}$$

since  $K$  = adiabatic elasticity of the gas

$$= \gamma \cdot P$$

where  $P$  is the atmospheric pressure,

$$\therefore \frac{p}{U} = \frac{-\gamma P \cdot \frac{dy}{dx}}{-v \cdot \frac{dy}{dx}} = \frac{\gamma P}{v}$$

Since  $\gamma$  is a constant for air and conditions of pressure and temperature remain unaltered,  $P$  and  $v$  too are constants. Therefore

$$\frac{p}{U} = \frac{\text{excess pressure}}{\text{particle velocity}} = \text{constant.}$$



**Example 6.9.** Young's modulus for steel is  $210 \times 10^{10}$  dynes/cm<sup>2</sup>. If the density of steel is 7.8 gm/c.c., find the velocity of propagation of sound through a steel bar.

**Soln.**

The velocity of sound,  $v$ , in solid is given by  $v = \sqrt{\frac{Y}{\rho}}$

where  $Y$  is the Young's modulus for the solid and  $\rho$  its density. Therefore, the velocity of sound through the solid bar

$$\begin{aligned} v &= \sqrt{\frac{210 \times 10^{10}}{7.8}} \\ &= 5.189 \times 10^5 \text{ cm/sec} \\ &= 5189 \text{ m/sec.} \end{aligned}$$

**Example 6.10.** The lowest tone emitted by a cylindrical metal rod has a frequency 1000 cps and wavelength 300 cm. If the density of the metal be 7.5 gm/c.c., obtain the value of Young's modulus for it.

**Soln.**

The velocity of the wave emitted by the rod,

$$v = n\lambda = 1000 \times 300 = 3 \times 10^5 \text{ cm/sec.}$$

$$\text{Now } v = \sqrt{\frac{Y}{\rho}}$$

where  $Y$  is the Young's modulus for the material of the rod and  $\rho$  its density.

$$\therefore v^2 = \frac{Y}{\rho}$$

$$\text{or, } Y = v^2 \cdot \rho.$$

Substituting the values of  $v$  and  $\rho$ , we have Young's modulus for the material of the rod

$$\begin{aligned} &= 7.5 \times (3 \times 10^5)^2 \\ &= 7.5 \times 9 \times 10^{10} \end{aligned}$$

$$= 67.5 \times 10^{10}$$

$$= 6.75 \times 10^{11} \text{ dynes/cm}^2.$$

**Example 6.11.** (a) The speed of sound in a certain metal is  $V$  cm/sec. One end of a pipe of that metal of length  $l$  cm is struck a blow. A listener at the other end hears two sounds, one from the wave that has travelled along the pipe and the other from the wave that has travelled through air. If  $v$  is the speed of the sound in air, what time interval  $t$  elapses between the two sounds? (b) Suppose  $t = 1.4$  sec, Young's modulus for the metal  $= 7.2 \times 10^{11}$  dynes/cm<sup>2</sup>, density of the metal  $= 8$  gm/c.c., atmospheric pressure  $10^6$  dynes/cm<sup>2</sup>, density of air  $= 0.00129$  gm/c.c. and  $\gamma$  for air  $= 1.4$ . Find the length  $l$ .

(a) velocity of sound along the pipe,

$$V = \sqrt{\frac{Y}{\rho}}$$

where  $Y$  is the Young's modulus of the metal and  $\rho$  its density.

and velocity of sound through air,

$$v = \sqrt{\frac{\gamma P}{\rho'}}$$

where  $P$  is the atmospheric pressure,  $\rho'$ , the density of air and  $\gamma = \frac{C_p}{C_v}$ .

Time taken by the sound travelling along the pipe to cover the length  $l$  of the pipe  $= l/V$  and time taken by sound travelling through air to cover the same distance  $l$  in air  $= l/v$ .

$$t = \frac{l}{v} - \frac{l}{V} = l \left( \frac{1}{v} - \frac{1}{V} \right)$$

$$= l \left( \frac{V - v}{Vv} \right) \text{ sec.}$$

(b) Now  $t = 1.4$ , therefore, we have

$$t = 1.4 = l \cdot \frac{V - v}{vV}$$

$$\text{or, } l = \frac{1.4 \times v \times V}{V - v}$$

$$\text{Now } V = \sqrt{\frac{Y}{\rho}} = \sqrt{\frac{7.2 \times 10^{11}}{8}}$$

$$= 3 \times 10^5 \text{ cm/sec.}$$

$$\text{and } v = \sqrt{\frac{\gamma P'}{\rho'}} = \sqrt{\frac{1.4 \times 10^6}{0.00129}}$$

$$= 3.294 \times 10^4 \text{ cm/sec.}$$

$$\therefore l = \frac{1.4 \times 3.294 \times 10^4 \times 3 \times 10^5}{3 \times 10^5 - 3.294 \times 10^4}$$

$$= \frac{13.8348 \times 10^9}{267060} = 0.0000518 \times 10^9$$

$$= 51800 \text{ cm.}$$

$$= 518 \text{ m.}$$

#### 6.4 Velocity of transverse waves along a stretched string

Let AB represent the undisturbed position of a string of uniform cross-section having a constant mass per unit length. The string is taken to be perfectly flexible, its stiffness being negligible. It is supposed not to be subjected to appreciable change in length while in vibration.

Let the string be stretched between the fixed points A and B parallel to the  $x$ -axis and let  $T$  be the tension on the string. If by some means the string is displaced perpendicular to its length by a small amount and then released, it will execute transverse vibrations setting up transverse waves in the string. These vibrations are simple harmonic in nature. The force tending to bring any element of the string back to the equilibrium position is the component of tension acting at right angles to the element.

Let ACDB denote the displaced position of the string at any instant. Thus the undisturbed position of the string is along the  $x$ -



axis while its displacement is along the Y-axis. Because the string is supposed to be perfectly flexible, the tension  $T$  is the same at all points on it both in its displaced and undisplaced positions. But since it acts tangentially to the string at the point considered, its inclination to the axis of  $x$  (i.e., the normal position of the string) is different at different points. Let us consider an infinitesimally small element of length  $CD = \delta x$  which may be supposed to be straight (Fig. 6.3). The tangents drawn at  $C$  and  $D$  make angles  $\theta$  and  $\theta - d\theta$  with the  $x$ -axis.

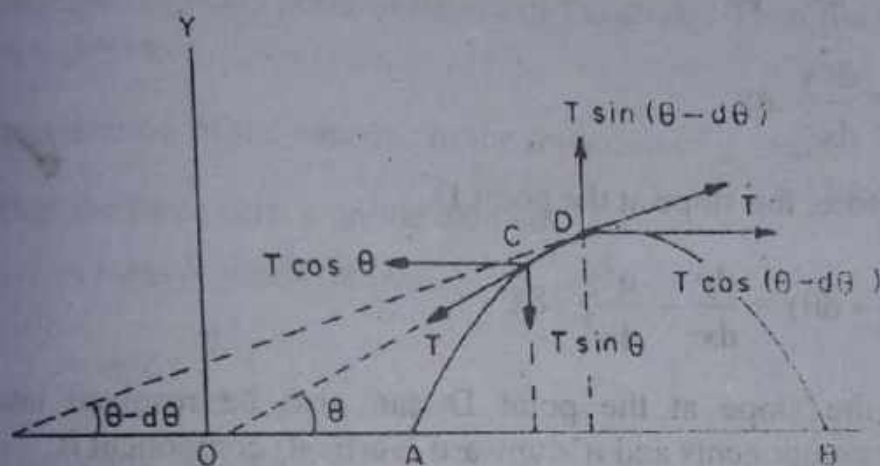


Fig. 6.3

The tension at  $C$  and  $D$  can be resolved into two rectangular components. The element of length  $\delta x$  of the string and its displacement being infinitesimally small, the horizontal and vertical components of  $T$  at  $C$  and  $D$  may be taken to act along the same horizontal and vertical lines respectively, though of course in opposite directions to each other. Then we have resultant horizontal force on element  $\delta x$  of the string  $= T \cos (\theta - d\theta) - T \cos \theta = 0$

since  $d\theta$  is much too small.

Now the downward component of tension at  $C$  is  $T \sin \theta$ . As  $\theta$  is small,  $\sin \theta = \tan \theta$

$\therefore$  the downward component of  $T$  at  $C = T \tan \theta$

But  $\tan \theta = \frac{dy}{dx}$  at  $C$  (i.e., slope at  $C$ ).

So the downward component at  $C$

$$= T \cdot \frac{dy}{dx} \quad (i)$$

Let  $\frac{d}{dx} \left( \frac{dy}{dx} \right)$  be the rate of change of slope with respect to the length of the element. Then the change in slope for a distance  $\delta x$

$$= \frac{d}{dx} \left( \frac{dy}{dx} \right) \delta x$$

$$= \frac{d^2 y}{dx^2} \cdot \delta x$$

Therefore, the slope at the point D.

$$\tan (\theta - d\theta) = \frac{dy}{dx} - \frac{d^2 y}{dx^2} \cdot \delta x$$

Now the slope at the point D can also be resolved into two rectangular components and it's upward (vertical) component is

$$T \sin (\theta - d\theta)$$

Again, as  $\theta$  is small,

$$\sin (\theta - \delta\theta) = \tan (\theta - \delta\theta)$$

$\therefore$  the upward component of tension acting at D

$$= T \tan (\theta - \delta\theta)$$

$$= T \left[ \frac{dy}{dx} - \frac{d^2 y}{dx^2} \cdot \delta x \right] \quad (ii)$$

From eqns. (i) and (ii),

The resultant downward tension acting on the element  $\delta x$

$$= T \frac{dy}{dx} - T \left[ \frac{dy}{dx} - \frac{d^2 y}{dx^2} \cdot \delta x \right]$$

$$= T \cdot \frac{d^2 y}{dx^2} \cdot \delta x$$

Therefore, the change in the component of tension along the axis, i.e., the resultant downward tension is the moving force on the element  $\delta x$  of the string and this causes it to vibrate transversely. Thus the force responsible for the transverse vibration of the string.

$$F = T \cdot \frac{d^2 y}{dx^2} \cdot \delta x \quad \text{(iii)}$$

Let  $m$  be the mass per unit length of the string. Then the mass of the element  $CD = m \cdot \delta x$ .

$$\text{Acceleration of the element in the direction of } y \text{-axis} = \frac{d^2 y}{dt^2}$$

Then the force acting on the element

$$= \text{mass} \times \text{acceleration}$$

$$= m \cdot \delta x \cdot \frac{d^2 y}{dt^2}$$

From eqns. (iii) and (iv)

$$m \cdot \delta x \cdot \frac{d^2 y}{dt^2} = T \cdot \frac{d^2 y}{dx^2} \cdot \delta x$$

$$\text{or, } \frac{d^2 y}{dt^2} = \frac{T}{m} \cdot \frac{d^2 y}{dx^2}$$

This is similar to the differential equation of the wave motion,

$$\frac{d^2 y}{dt^2} = c^2 \frac{d^2 y}{dx^2} \quad \text{(vi)}$$

where  $c$  is the velocity of propagation of the wave

Combining eqns. (v) and (vi), we have

$$c^2 = \frac{T}{m}$$

$$\text{or, } c = \sqrt{\frac{T}{m}} \quad \text{(6.8)}$$



Thus the velocity of the transverse wave along a stretched string is given by the square root of the ratio of tension applied to the string to the mass per unit length of the string.

### 6.5 Frequency of a vibrating string

Suppose we have a string, stretched between two fixed points A and B parallel to the  $x$ -axis. If the string is now displaced perpendicular to its length by a small amount and then released, it will execute transverse vibrations setting up transverse waves in the string. On being reflected at the fixed ends, these waves will form stationary waves.

As derived in the previous article, the velocity  $c$  of the propagation of the transverse waves along the string is given by

$$c = \sqrt{\frac{T}{m}}$$

where  $T$  is the tension on the string and  $m$  is the mass per unit length of the string. But  $c = n\lambda$  where  $n$  is the frequency of vibration (also referred to as pitch) of the wave and  $\lambda$  its wavelength.

Hence

$$n = \frac{c}{\lambda} = \frac{1}{\lambda} \sqrt{\frac{T}{m}} \quad (6.9)$$

Now the string may be set into different modes of vibration, *i.e.*, stationary waves of different wavelengths and frequencies may be produced. In every case, the two fixed points always remain at rest (*i.e.*, they remain nodal points or displacement nodes).

Thus if the string vibrates as a whole in one single segment (or loop), as shown in Fig. 6.4(i), it will give its lowest frequency. If  $l$  = length of the string, then  $\lambda = 2l$ . Hence

$$n = \frac{1}{2l} \sqrt{\frac{T}{m}} \quad (6.10)$$

and the string is said to vibrate in its fundamental note (*i.e.*, lowest frequency).

If the string vibrates with two segments or loops, then  $\lambda = l$  and

$$n = \frac{1}{l} \sqrt{\frac{T}{m}}$$

In general if the number of segments or loops between the two fixed points be  $p$  where  $p = 1, 2, 3, \dots$  then the length of each segment is  $l/p$  and each segment corresponds to  $\lambda/2$ .

$$\therefore \frac{l}{p} = \frac{\lambda}{2}; \quad \text{or, } \lambda = \frac{2l}{p}$$

Therefore,

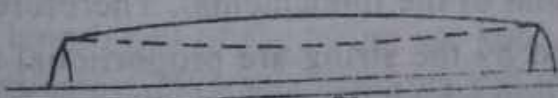
$$c = n\lambda = \frac{n(2l)}{p}$$


Substituting this value of  $c$  in eqn. (6.8) we have

$$\begin{aligned} \frac{n(2l)}{p} &= \sqrt{\frac{T}{m}} \\ \text{or, } n &= \frac{p}{2l} \sqrt{\frac{T}{m}} \end{aligned} \quad (6.11)$$

Thus if the string vibrates in one segment

$$n = n_0 = \frac{1}{2l} \sqrt{\frac{T}{m}}$$

(i)   
 $n_0 = \frac{1}{2l} \sqrt{\frac{T}{m}}$

(ii)   
 $n = \frac{1}{l} \sqrt{\frac{T}{m}} = 2n_0$

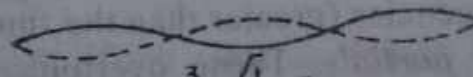

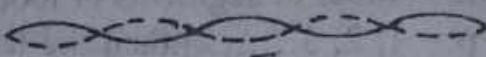
(iii)   
 $n = \frac{3}{2l} \sqrt{\frac{T}{m}} = 3n_0$

Fig. 6.4

(iv) 

$$n = \frac{2}{l} \sqrt{\frac{T}{m}} = 4n_0$$

(v) 

$$n = \frac{5}{2l} \sqrt{\frac{T}{m}} = 5n_0$$

Fig. 6.4

The next mode of vibration in order of complexity is when the string vibrates with two segments or loops shown in Fig. 5.4 (ii). Here  $\lambda = l$  and

$$n = \frac{1}{l} \sqrt{\frac{T}{m}} = 2n_0$$

where  $n_0$  is the fundamental note.

The note then emitted is the *first overtone* or *first harmonic*.

In a similar manner, the frequency of vibration associated with the mode of vibration of Fig. 6.4(iii) i.e., when the string vibrates with three segments, is

$$n = \frac{3}{2l} \sqrt{\frac{T}{m}} = 3n_0$$

The note then emitted is the *second overtone* or *second harmonic*.

In general, if the string vibrates with  $p$  segments or loops between the two fixed points, then the frequency of vibration of the string is  $p$  times that of the fundamental. Therefore, the frequencies of the notes emitted by the string are proportional to the number of segments or loops into which the string is thrown into vibration, or

$$n_1 : n_2 : n_3 : \dots : n_p : : 1 : 2 : 3 : \dots : p.$$

As already pointed out, the sound of the lowest pitch for which the frequency is minimum is called the *fundamental tone*. The sounds of other frequencies (greater than the fundamental) are called *overtones* or *upper partials*. Those overtones which are integral multiples of the fundamental are called *harmonic overtones* or simply *harmonics* and those which are non-integral multiples of the fundamental are called *inharmonic overtones* or simply *overtones*.



The harmonics whose frequencies are *three* times, *four* times, ..... that of the fundamental are called *second*, *third*, ..... harmonics. The special name octave is reserved for the first overtone having twice the frequency of the fundamental.

A sound of only one frequency is called a *pure tone* or simply a *tone* and a sound in which several frequencies are present is called a *note*. In fact, a note consists of several tones.

### 6.6 Laws of transverse vibrations of a stretched string

The transverse vibrations set up in a string of a given length  $l$  and of uniform mass per unit length  $m$ , stretched between two fixed points with a given tension  $T$ , are found to obey certain laws which are embedded in the formula given by

$$\begin{aligned} n &= \frac{1}{2l} \sqrt{\frac{T}{m}} \\ &= \frac{1}{2lr} \sqrt{\frac{T}{\pi\rho}} \quad [\because m = \pi r^2 \rho] \end{aligned} \quad (5.12)$$

where  $n$  = frequency of vibration of the string

$r$  = radius of the string

$\rho$  = density of the material of the string.

These laws are :

(i) *law of length* : If the tension ( $T$ ) applied to the string and the mass per unit length ( $m$ ) of the string are kept constant, then the frequency of vibrations is inversely proportional to the length of the vibrating segment of the string. Or,

$$n \propto \frac{1}{l} \text{ when } T \text{ and } m \text{ are constants}$$

or,  $n.l = \text{constant}$ .

(ii) *law of tension* : If the vibrating length ( $l$ ) of the string and the mass per unit length ( $m$ ) of the string are kept constant, then the frequency of vibration varies directly as the square root of the tension applied. Or,

$n \propto \sqrt{T}$  when  $l$  and  $m$  are constants

or,  $n / \sqrt{T} = \text{constant}$ .

(iii) *law of mass* : If the vibrating length ( $l$ ) and the tension applied ( $T$ ) are kept constant then the frequency of vibration varies inversely as the square root of the mass per unit length. Or,

$n \propto \frac{1}{\sqrt{m}}$  when  $l$  and  $T$  are constants

or,  $n \cdot \sqrt{m} = \text{constant}$ .

(iv) *law of density* : If the vibrating length ( $l$ ), the tension applied ( $T$ ) and the radius of the string ( $r$ ) are kept constant, then the frequency of vibration varies inversely as the square root of the density of the material of the string. Or,

$n \propto \frac{1}{\sqrt{\rho}}$  when  $l$ ,  $T$  and  $r$  are constant.

(v) *law of radius* : If the vibrating length ( $l$ ), the tension applied ( $T$ ) and the density of the material of the string ( $\rho$ ) are kept constant, then the frequency of vibration varies inversely as the radius of the string. Or,

$n \propto \frac{1}{r}$  when  $l$ ,  $T$  and  $\rho$  are constants.

## 6.7 Sonometer

The laws of transverse vibrations of fixed string, can be verified with the help of a *sonometer* or a *monochord*, designed for this purpose.

The sonometer consists of a rectangular hollow wooden board  $Q$  with a fine uniform horizontal steel wire attached to  $A$  at one end (Fig. 6.5). The other end of the wire passes over a grooved wheel  $H$  and is kept taut by a mass  $M$  placed over a hanger. Slotted loads can be put upon the hanger and thereby the amount of stretching can be adjusted. Wooden bridges  $B$ ,  $C$  can be placed beneath the wire so that a definite length of the wire (vibrating segment) is obtained, and the length of the wire can be varied by moving one of the bridges.



The length of the wire between  $B$ ,  $C$  can be read from a fixed horizontal scale  $D$ , graduated in millimetres, on the box below the wire.

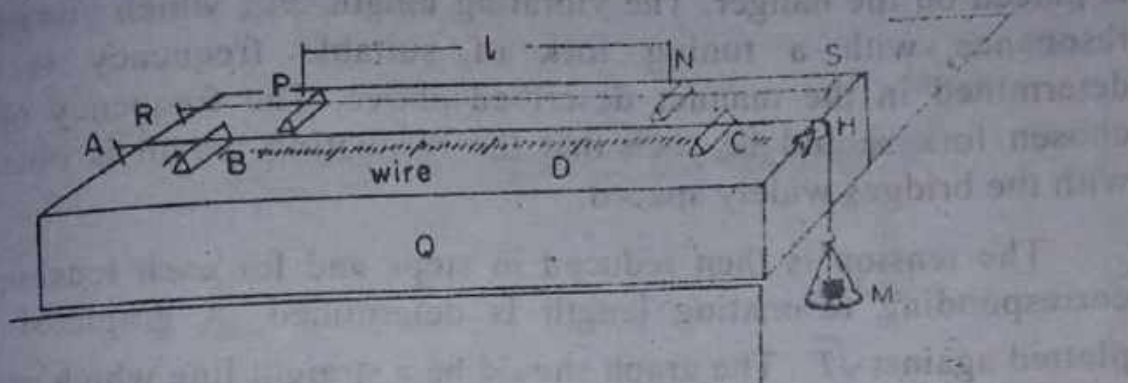


Fig. 6.5

In some apparatus there are two pieces of wires, one of which is stretched by adjustable loads and the other is stretched parallel to this wire and is fixed between two pegs. The fixed wire is called the *comparison* wire and is used to verify the laws of transverse vibrations of string by the method of beats.

### 6.8 Verification of the laws of transverse vibrations of strings

(i) *law of length* : To verify the law of length, the mass  $M$  placed on the hanger is kept constant so that the tension  $T$ , in the wire  $AH$  is constant.

A small piece of paper in the form of an inverted V, called the *rider*, is placed on the wire mid-way between the bridges. A tuning fork of known frequency is then set into vibration and placed vertically near the string with its shank on the board of the sonometer. The vibrations of the fork are communicated to the string, first through the board and then through the bridges. The vibrating length  $BC$  is slowly adjusted by moving the bridges until the rider is thrown over upon placing the tuning fork on the board. When this happens the string vibrates in *resonance* with the tuning fork. The length  $BC$  ( $l$ ) is noted.

Different tuning forks of known frequencies ( $n$ ) are taken, and the corresponding lengths ( $l$ ) are observed. A graph of  $n$  against  $l$  or  $1/l$  is then plotted. Within the limits of experimental error, the graph



is found to be a rectangular hyperbola in the first case ( $n \propto 1/l$ ) or a straight line in the second case ( $n \propto 1/l$ ), which proves the law.

(ii) *law of tension* : To verify the law of tension, maximum load is placed on the hanger. The vibrating length,  $BC$ , which vibrates in resonance, with a tuning fork of suitable frequency is then determined in the manner described above. The frequency of the chosen fork should be such that the resonating length is obtained with the bridges widely spaced.

The tension is then reduced in steps and for each tension the corresponding resonating length is determined. A graph of  $l$  is plotted against  $\sqrt{T}$ . The graph should be a straight line which proves that the frequency is proportional to the square root of tension.

(iii) *law of mass* : A number of wires of different material or of the same material but different diameters, therefore having different masses per unit length, are selected. A tuning fork of suitable frequency is taken. The wires are then placed in turn on the sonometer and subjected to the same tension. For each wire the resonating length for the given fork is determined in the manner described above. A graph of  $1/l$  against  $\sqrt{m}$  is then plotted which should yield a straight line which proves that the frequency is inversely proportional to the square root of the mass per unit length.

## 6.9 Melde's experiment

In Melde's experiment, one end  $B$  of the string, which may be a piece of stout thread or thin cord, is attached to one prong of a tuning fork  $F$  which is screwed vertically on a wooden box  $W$  (Fig. 6.6) at the edge of a table. The other end  $A$  of the string passes over a small pulley and is attached to a scale pan  $S$  on which loads can be placed to stretch the string. When the fork is excited, the string is set in to vibration. The superposition of the direct and reflected waves will form stationary waves, in which the extreme fixed ends of the string will always be nodes with one or more antinodes in between depending on the length of the string or the tension to which the string is subjected. Hence loops are formed which are visible in the string. By adjusting the tension or the length, the frequency  $n$  of the fork may be made to be equal to the frequency  $n'$  of the fundamental or any one of the higher tones of the string. When this happens a resonance is said to have occurred between

the fork and the particular mode of vibration of the string. If the mode is assumed to be fundamental *i.e.*, the string vibrates in one segment then  $\lambda = 2l$  where  $l$  is the length of the string. The frequency of the fork will then be given by the relation

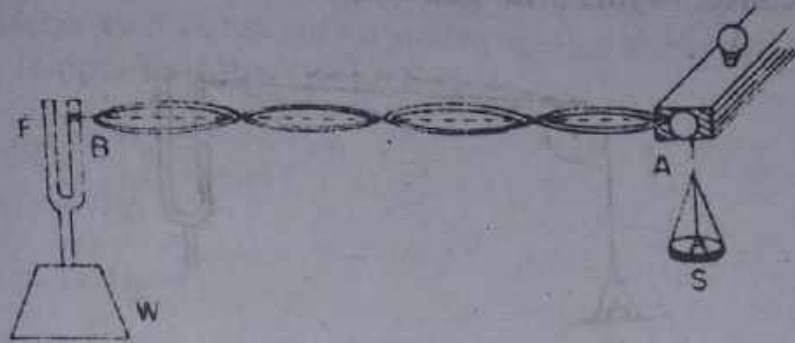


Fig. 6.6

$$n = \frac{1}{\lambda} \sqrt{\frac{T}{m}}$$

$$\therefore = \frac{1}{2l} \sqrt{\frac{T}{m}}$$

where  $m$  is the mass per unit length of the vibrating string, and  $T$  is the tension applied to the string.

Now the motion of the prongs of the fork, which sets the string in resonant vibration may be in two different directions (i) in a direction perpendicular to the length of the string *i.e.*, transverse position (or mode) (Fig. 6.7) and (ii) in a direction along the length of the string *i.e.*, longitudinal position (or mode) (Fig. 6.8). In the

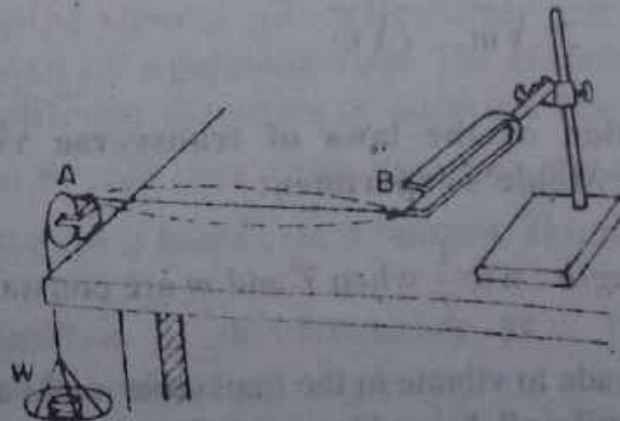


Fig. 6.7

transverse arrangement, the string moves in the same way as the fork and hence the frequency  $n$  of the fork is the same as the frequency  $n'$  of the string. But in the longitudinal arrangement the frequency of the fork is twice as great as the frequency of the string. This is because, in this arrangement,



Fig. 6.8

the vibration is produced by the alternating pulls upon the end of the string by the prong of the fork. Each movement of the prong away from the pulley pulls the string tight *i.e.*, the string is stretched; but the string is sufficiently loose when the prong comes nearer to the pulley. The string, therefore, makes half a vibration during the time the fork completes one full vibration. Thus in this arrangement, the frequency of the string is half that of the fork.

Therefore, for transverse position, the frequency  $n$  of the fork is

$$n = n' = \frac{1}{\lambda} \sqrt{\frac{T}{m}} = \frac{1}{2l} \sqrt{\frac{T}{m}} \quad (6.13)$$

and for longitudinal position

$$n = 2n' = \frac{2}{2l} \sqrt{\frac{T}{m}} = \frac{1}{l} \sqrt{\frac{T}{m}} \quad (6.14)$$

### 6.10 Verification of the laws of transverse vibrations of a string by Melde's experiment

(i) *law of length* :  $n \propto \frac{1}{l}$  when  $T$  and  $m$  are constants.

The fork is made to vibrate in the transverse mode and loads on the pan are adjusted until well defined loops are formed in the string. Let  $l_1$



be the length between two consecutive nodes and  $n_1$  the frequency of the string and hence the fork. The fork is then turned through  $90^\circ$ , so that the prongs now vibrate along the length of the string i.e., in the longitudinal mode. For the same load on the pan, let  $l_2$  be the distance between two consecutive nodes in the string of which the frequency  $n_2$  is now half that of the fork.

Then if it can be shown that

$$n_1/n_2 = l_2/l_1 = 2$$

the law is verified.

(ii) *law of tension* :  $n \propto \sqrt{T}$  when  $l$  and  $m$  are constants.

Hence for a constant frequency fork,  $\sqrt{T} \propto l$ . So the law of tension will be verified if it can be shown that  $l/\sqrt{T} = \text{a constant}$ .

In either transverse or longitudinal arrangement, the length of the string between two consecutive nodes for different loads on the pan are determined. Let  $l_1, l_2, l_3$ , etc. be the lengths for loads  $w_1, w_2, w_3$ , etc. on the pan. It was found that  $l/\sqrt{T}$ , where  $T = wg$ , is a constant. This proves the law.

(iii) *law of mass* :  $n \propto \frac{1}{\sqrt{m}}$  when  $l$  and  $T$  are constants.

This means that for a constant frequency fork,  $\sqrt{m} \propto \frac{1}{l}$  so that if it can be shown that  $l \cdot \sqrt{m}$  is a constant, the law will be verified.

When the string vibrates with well defined loops, the length of a loop  $l$  is measured for a particular load. The experiment is repeated with strings of different diameters or materials but with same load and  $l$  is measured in each case. Using the value of  $l$  and  $\sqrt{m}$  for each string, the product  $l \cdot \sqrt{m}$  is found to be a constant. This proves the law.

### 6.11 Determination of the frequency of a tuning fork by Melde's experiment

As explained in Art. 6.9, a tuning fork may be set up so that its vibration is along the length of the string (longitudinal mode of

vibration) or in a direction perpendicular to the length of the string (transverse mode of vibration). The tuning fork is set up for, say, the longitudinal mode of vibration and some small load is placed on the scale pan. The fork is then set into vibration by a smart light blow by a rubber padded hammer. This sets the string into vibration. As stationary waves are formed due to reflection from the other fixed end, several nodes and loops appear. The loops will possibly be ill-defined. The mass on the scale pan is increased or decreased, until the loops are well defined, *i.e.*, for a given number of loops between the two ends, the amplitude of the loops are maximum, the nodal points are fixed in position and the loops are of equal length. When this happens, resonance is said to have occurred between the fork and the particular mode of vibration of the string. The distance between the ends of a definite number  $N$  of loops is located by placing pointed stands at two extreme nodes and the length  $L$  measured which gives  $l = L/N$ , the length of a single loop between two consecutive nodes. Since the frequency of vibration of the string,  $n'$ , in the longitudinal mode is half the frequency of vibration,  $n$ , of the tuning fork, we have

$$n' = \frac{n}{2} = \frac{1}{2l} \sqrt{\frac{T}{m}}$$

$$\text{or, } n = 2n'.$$

The tuning fork is then turned through  $90^\circ$  about the fixing screw and again set into vibration. This time the fork vibrates in the transverse mode, *i.e.*, in a direction at right angles to the length of the string. The string is set into vibration and a number of, possibly ill-defined, loops appear. The loops are made well defined and the length of a loop measured in the manner described above. Since, in transverse mode, the frequency of the fork is equal to the frequency of vibration of the string, we have

$$n = n' = \frac{1}{2l} \sqrt{\frac{T}{m}}$$

---

**Example 6.12.** Calculate the frequency of the fundamental note of a string, 1 metre long and weighing 2 gms when stretched by a weight of 400 kg.

**Soln.**

The expression for frequency of the fundamental note is given by

$$n = \frac{1}{2l} \sqrt{\frac{T}{m}}$$

here,  $l = 1 \text{ metre} = 100 \text{ cm.}$

Total mass = 2 gm.

$$\text{Hence } m = \frac{2}{100} = .02 \text{ gm/cm.}$$

$$\begin{aligned} T &= (400 \times 10^3) \text{ gm} \times 980 \text{ cm/sec}^2 \\ &= 400 \times 980 \times 10^3 \text{ dynes/cm}^2. \end{aligned}$$

$$\begin{aligned} \therefore n &= \frac{1}{2 \times 100} \sqrt{\frac{400 \times 980 \times 10^3}{0.02}} \\ &= \frac{1}{2 \times 100} \times 14 \times 10^4 \\ &= 7 \times 10^2 = 700 \text{ Hz.} \end{aligned}$$

**Example 6.13.** A fork vibrates along the length of a string that is attached to one prong and is stretched by a load of 41.8 grams. If the length of the string be 20 cms, and its mass per unit vibrates in 4 segments, calculate the frequency of the fork.

**Soln.**

The relevant expression is

$$n = \frac{p}{2l} \sqrt{\frac{T}{m}} \quad (\text{eqn. 6.11})$$

here  $l = 20 \text{ cm}$

$p = \text{no. of segments} = 4$

$T = 41.8 \times 980 \text{ dynes/cm}^2$

$m = 0.025 \text{ gm.}$



$$\therefore n = \frac{4}{2 \times 20} \sqrt{\frac{41.8 \times 980}{0.025}}$$

$$= 256 \text{ vibrations per second.}$$

**Example 6.14.** Wires of equal length of brass and steel are stretched on a sonometer and adjusted to emit the same fundamental note. If the tensions in the two cases are 5 and 3 kilogram-weights respectively and the diameter of the steel wire is 0.8 mm, find that of the brass wire, the densities for brass and steel being 8.4 and 7.8 respectively.

**Soln.**

The note emitted by steel and brass wires are given by

$$n = \frac{1}{2l} \sqrt{\frac{T}{m}} \quad \text{and} \quad n' = \frac{1}{2l'} \sqrt{\frac{T'}{m'}}$$

where the primed quantities refer to brass wire

Here  $n = n'$  and  $l = l'$ , so that we have

$$\sqrt{\frac{T}{m}} = \sqrt{\frac{T'}{m'}}; \quad \text{or, } \frac{T}{m} = \frac{T'}{m'}$$

$$\text{or, } \frac{m'}{m} = \frac{T'}{T} = \frac{5}{3} \quad (T' = 5 \text{ kg - wt., and } T = 3 \text{ kg - wt.})$$

Now mass per unit length,  $m = \pi r^2 \times \rho$  where  $\rho$  is the density of the material of the wire, and  $r$  is the radius of the wire.

$$\therefore \frac{\pi r^2 \times 8.4}{\pi (0.4)^2 \times 7.8} = \frac{5}{3}$$

$r$  is the radius of the brass wire

0.4 mm is the radius of the steel wire.

$$\text{or, } r^2 = \frac{5 \times 0.16 \times 7.8}{3 \times 8.4}$$

$$= \frac{6.24}{25.2} = 0.247619$$

$$\therefore r = 0.4976 \text{ mm.}$$

$$\text{Hence diameter} = 2 \times r = 2 \times 0.4976$$

$$= 0.9952 \text{ mm} = 0.09952 \text{ cm.}$$

**Example 6.15.** A wire gives out a fundamental note of 256 cycles per second when it is under a tension of 10 kg-wt.

(i) Under what tension will the string emit a frequency of 512 cycles per second?

(ii) How would you make the wire emit a note of 768 cycles per second keeping the tension at 10 kg-wt?

**Soln.**

(i) From  $n = \frac{1}{2l} \sqrt{\frac{T}{m}}$  we have

$$\frac{n_2}{n_1} = \sqrt{\frac{T_2}{T_1}} \text{ since } l \text{ and } m \text{ are same in both cases,}$$

$$\text{Here } n_1 = 256, n_2 = 512,$$

$$T = 10 \text{ kg - wt.}, T_2 = ?$$

$$\therefore \frac{512}{256} = \sqrt{\frac{T_2}{10}}; \text{ or, } T_2 = 4 \times 10$$

$$= 40 \text{ kg - wt.}$$

(ii) In the second case, since  $T$  and  $m$  are same in both cases, we have

$$\frac{n_2}{n_1} = \frac{l_1}{l_2}$$

$$\text{Here } n_1 = 256 \text{ and } n_2 = 768$$

$$\therefore \frac{768}{256} = \frac{l_1}{l_2}; \text{ or, } l_2 = \frac{l_1}{3}$$

Thus, by shortening the length to one-third of the length in the first case, the wire can be made to emit a note of frequency 768.

**Example 6.16.** In an experiment it was found that the string vibrated in 5 loops when 10 gms were placed in the scale pan. What mass must be placed in the pan to make the string vibrate in 7 loops?

**Soln.**

Since  $l$  and  $m$  are constants, the frequency,  $n \propto \sqrt{T}$ .

Let  $n_1$  and  $n_2$  be the frequencies when the string vibrates in 5 loops and 7 loops respectively and let  $T_1$  and  $T_2$  be the corresponding tensions. Then

$$n_1 \propto \sqrt{T_1} \text{ and } n_2 \propto \sqrt{T_2}$$

$$\text{or, } \frac{n_2}{n_1} = \sqrt{\frac{T_2}{T_1}}$$

Now, when the string vibrates in 5 loops,  $n_1 = 5n_0$  where  $n_0$  is the frequency when the string vibrates in the fundamental mode. Similarly,  $n_2 = 7n_0$ . (see Art. 6.5)

Therefore,

$$\frac{n_2}{n_1} = \frac{7n_0}{5n_0} = \sqrt{\frac{T_2}{T_1}}$$

$$\text{or, } \sqrt{\frac{T_2}{T_1}} = \frac{7}{5}$$

$$\text{or, } \frac{T_2}{T_1} = \frac{49}{25} ; \text{ or, } T_2 = \frac{49}{25} \times 10 \text{ (} T_1 = 10 \text{ gms)}$$

$$= 1.96 \times 10$$

$$= 19.6 \text{ gms.}$$

**Example 6.17.** In a transverse arrangement of Melde's experiment, the string vibrated in 3 loops when the tension is 200 gms. Calculate the tension required to make the string vibrate in 2 loops in the longitudinal arrangement.



**Soln.**

For transverse arrangement,

$$n = \frac{3}{2l} \sqrt{\frac{T}{m}} = \frac{3}{2l} \sqrt{\frac{200 \times g}{m}} \quad (i)$$

for the longitudinal arrangement,

$$n' = \frac{n}{2} = \frac{2}{2l} \sqrt{\frac{T}{m}}$$

$$\text{or, } n = \frac{4}{2l} \sqrt{\frac{T}{m}} \quad (ii)$$

From (i) and (ii)

$$4 \times \sqrt{T} = 3 \times \sqrt{200 \times g}$$

$$\text{or, } T = Mg = \frac{9}{16} \times 200 \times g$$

$$\text{or, } M = \frac{9}{16} \times 200$$

$$= 112.5 \text{ gm.}$$

Hence tension required = 112.5 gm-wt.

**Example 6.18.** A tuning fork of frequency 160 is sounded along with a sonometer wire of length 25 cm stretched to a tension of 1.25 kg-wt. Calculate the number of beats produced per second. (mass per unit length = 0.025 gm).

**Soln.**

The frequency of vibration of the sonometer wire is

$$n = \frac{1}{2l} \sqrt{\frac{T}{m}}$$

here  $l = 25 \text{ cm}$ ,  $m = 0.025 \text{ gm}$

and  $T = 1.25 \times 1000 \times 980 \text{ dynes}$

$$\begin{aligned} \therefore n &= \frac{1}{2 \times 25} \sqrt{\frac{1.25 \times 1000 \times 980}{0.025}} \\ &= \frac{7000}{50} = 140. \end{aligned}$$

Frequency of the tuning fork,  $N = 160$ .

Therefore, the number of beats produced per second

$$= N - n = 160 - 140 = 20.$$

**Example 6.19.** Two tuning forks when sounded together give 5 beats per second. One is in unison with a length of 128 cms of a sonometer wire and the other with 130 cms of the same wire, the tension remaining unaltered. What are the frequencies of the forks?

**Soln.**

We have

$$n = \frac{1}{2l} \sqrt{\frac{T}{m}}$$

In the first case,

$$n_1 = \frac{1}{2l_1} \sqrt{\frac{T}{m}} = \frac{1}{2 \times 128} \sqrt{\frac{T}{m}} \quad (i)$$

In the second case,

$$n_2 = \frac{1}{2l_2} \sqrt{\frac{T}{m}} = \frac{1}{2 \times 130} \sqrt{\frac{T}{m}} \quad (ii)$$

It is evident from (i) and (ii) that  $n_1 > n_2$ .

Dividing (i) by (ii) we get

$$\frac{n_1}{n_2} = \frac{130}{128}$$

$$\text{or, } \frac{n_1 - n_2}{n_2} = \frac{2}{128}$$

But  $n_1 - n_2 = \text{number of beats produced per second} = 5$

$$\therefore \frac{5}{n_2} = \frac{2}{128}; \text{ or, } n_2 = 320.$$

Hence  $n_1 - 320 = 5$ .

or,  $n_1 = 325$

The frequencies of the forks are **325** and **320**.

**Example 6.20.** A fine wire 500 cms long is fixed at both ends under tension and its fundamental frequency is 50 cycles per second. Find the distance from its centre where a bridge is to be placed, so that the two segments of the wire vibrating transversely would produce 4 beats per second.

**Soln.**

Since tension  $T$  and mass per unit length  $m$  are constants,  $n \times l = a$  constant.

Let the bridge be placed at a distance  $x$  cm from the centre so that the wire is divided into segments of lengths  $(250 + x)$  and  $(250 - x)$  respectively. If  $n_1$  and  $n_2$  are the corresponding frequencies of vibrations of the segments, then

$$(250 + x) n_1 = (250 - x) n_2.$$

$$\text{But } n \times l = 50 \times 500 = 25000$$

$$\therefore n_1 = \frac{25000}{250 + x} \text{ and } n_2 = \frac{25000}{250 - x}$$

It is evident that  $n_2 > n_1$

$$\therefore \frac{25000}{250 - x} - \frac{25000}{250 + x} = n_2 - n_1 = 4$$

$$\text{or, } \frac{250 + x - 250 \times x}{(250)^2 - x^2} - \frac{4}{25000} = \frac{1}{6250}$$

$$\text{or, } \frac{2x}{(250)^2 - x^2} = \frac{1}{6250}$$



$$\text{or, } 12500x = 62500 - x^2$$

$$\text{or, } x^2 + 12500x + (6250)^2 - 39062500 = 62500$$

$$\text{or, } (x + 6250)^2 = 39125000 = (6254.998)^2$$

$$\text{or, } x + 6250 = 6254.998$$

$$\text{or, } x = 6254.998 - 6250$$

$$= 4.998 \text{ cm.}$$

**Example 6.21.** A load of 20 kg is suspended by a steel wire. Its frequency when rubbed by a resined cloth is found to be 20 times its frequency when plucked. Find the area of cross-section of the wire. (Young's modulus for steel =  $19.6 \times 10^{11}$  dynes/cm<sup>2</sup>;  $g = 980$  cm/sec<sup>2</sup>).

**Soln.**

For longitudinal vibration,

$$n_l = \frac{1}{2l} \sqrt{\frac{Y}{\rho}}$$

and for transverse vibrations,

$$n_t = \frac{1}{2l} \sqrt{\frac{T}{m}} = \frac{1}{2l} \sqrt{\frac{T}{l \times a \times \rho}}$$

where  $l \times a \times \rho = m$ , mass per unit length,  $a$  being the required area of across-section.

$$\text{But } n_l = 20n_t$$

$$\therefore 20 \times \frac{1}{2l} \sqrt{\frac{Y}{\rho}} = \frac{1}{2l} \sqrt{\frac{T}{a \times \rho}}$$

$$\text{or, } \frac{400 \times T}{a} = Y$$

$$\text{or, } a = \frac{400 \times T}{Y}$$

$$= \frac{400 \times 20 \times 1000 \times 980}{19.6 \times 10^{11}}$$

$$= 0.004 \text{ sq. cm.}$$

$$= 0.004 \times 100 \text{ sq. mm.}$$

$$= 0.4 \text{ sq. mm.}$$

**Example 6.22.** Two similar sonometer wires of the same material, under the same tension, produce 2 beats per second. The length of one wire is 50 cm and that of the other is 50.1 cm. Calculate the frequencies of the two wires.

**Soln.**

As the tensions and the materials of the wires are the same

$$n_1 l_1 = n_2 l_2$$

$$\text{or, } \frac{n_1}{n_2} = \frac{l_2}{l_1}$$

$$\text{Here } n_1 - n_2 = 2$$

(i)

$$l_1 = 50 \text{ cm}$$

$$l_2 = 50.1 \text{ cm}$$

$$\therefore \frac{n_1}{n_2} = \frac{50.1}{50}$$

$$\text{or, } n_1 = \left( \frac{50.1}{50} \right) n_2 \quad \text{(ii)}$$

Substituting this value of  $n_1$  in eqn. (i),

$$\left( \frac{50.1}{50} \right) n_2 - n_2 = 2$$

$$\text{or, } n_2 = 1000 \text{ Hz.}$$

$$\text{and } n_1 = n_2 + 2$$

$$= 1000 + 2$$

$$= 1002 \text{ Hz.}$$

**Example 6.23.** In Melde's experiment when a string is stretched by a piece of glass it vibrates with 7 loops. When the glass piece is completely immersed in water the string vibrates in 9 loops. What is the specific gravity of glass?

**Soln.**

$$T_1 p_1^2 = T_2 p_2^2$$

$$\frac{T_1}{T_2} = \frac{p_2^2}{p_1^2} = \frac{(9)^2}{(7)^2} = \frac{81}{49}$$

$$\text{or, } \frac{T_2}{T_1} = \frac{49}{81}$$

$$\text{or, } 1 - \frac{T_2}{T_1} = 1 - \frac{49}{81} = \frac{32}{81}$$

$$\text{or, } \frac{T_1 - T_2}{T_1} = \frac{32}{81}$$

$$\text{or, } \frac{T_1}{T_1 - T_2} = \frac{81}{32} = 2.531$$

Therefore, the sp. gr. of glass = 2.531

**Example 6.24.** A stretched wire is observed to vibrate with a frequency 30 cycles per second in the fundamental mode when the supports are 0.6m apart. The string has a mass of 0.05kg/m.

(i) Find the velocity of propagation of a transverse wave in the string (ii) compute the tension of the string.

**Soln.**

(i) Here  $n = 30$

$$l = 0.6 \text{ m, } \therefore \lambda = 2 \times l = 2 \times 0.6 = 1.2 \text{ m}$$

$$\text{Hence, } v = n\lambda = 30 \times 1.2 = 36 \text{ m/sec.}$$

$$(ii) \quad v = \sqrt{\frac{T}{m}}$$

$$\text{or, } v^2 = \frac{T}{m}$$

$$\begin{aligned} \text{or, } T &= v^2 \cdot m = (36)^2 \times 0.05 \\ &= 64.8 \text{ newton.} \end{aligned}$$



**Example 6.25.** A wire 50 cm long and of mass  $6.5 \times 10^{-3}$  kg is stretched so that it makes 80 vibrations per second. Find the stretching force in kg - wt.

**Soln.**

$$n = \frac{1}{2l} \sqrt{\frac{T}{m}}$$

$$n = 80, l = 50 \text{ cm} = 0.5 \text{ m}$$

$$m = \frac{6.5 \times 10^{-3}}{0.5} = 13 \times 10^{-3} \text{ kg/m.}$$

$$\therefore 80 = \frac{1}{2 \times 0.5} \sqrt{\frac{T}{13 \times 10^{-3}}}$$

$$\text{or, } T = 83.2 \text{ newton.}$$

$$\text{Now } 1 \text{ kg - wt} = 9.81 \text{ newton.}$$

$$\begin{aligned} \therefore 83.2 \text{ newton} &= \frac{83.2}{9.81} \text{ kg - wt} \\ &= 8.48 \text{ kg - wt.} \end{aligned}$$

**Example 6.26.** Two wires of the same material and length but of diameters in the ratio 1: 1.44 are kept stretched by weights of 8 kg and 11.52 kg respectively. If the first wire has a fundamental frequency of vibration equal to 240, find that of the second.

**Soln.**

$$n = \frac{1}{2lr} \sqrt{\frac{T}{\pi\rho}}$$

$$= \frac{1}{l.d} \sqrt{\frac{T}{\pi\rho}}$$

$$2r = d = \text{diameter of the wire.}$$

Since the length and material of the two wires are the same,

$$n \propto \frac{\sqrt{T}}{d}$$

$$\therefore \frac{n_1}{n_2} = \sqrt{\frac{T_1}{T_2}} \times \frac{d_2}{d_1}$$

Here  $n_1 = 240$ ,  $T_1 = 8$  kg,  $T_2 = 11.52$  kg,

$$\frac{d_1}{d_2} = \frac{1}{1.44}$$

$$\therefore \frac{240}{n_2} = \sqrt{\frac{8}{11.52}} \times \frac{1.44}{1}$$

$\therefore n_2 = 200$  vibrations per second.

**Example 6.27.** Find the ratio between the fundamental frequencies of transverse and longitudinal vibrations of a steel wire 1mm diameter, mounted on a sonometer and stretched by a force of 10 kg. Young's modulus for steel is 20 million newton per  $\text{cm}^2$ .

**Soln.**

When the steel wire vibrates in transverse mode, the fundamental frequency of vibration,

$$n_t = \frac{1}{2l} \sqrt{\frac{T}{m}} = \frac{1}{2l} \sqrt{\frac{T}{\pi r^2 \rho}}$$

where  $r$  is the radius of the wire and  $\rho$  its density.

When the wire vibrates in longitudinal mode, the fundamental frequency,

$$n_l = \frac{1}{2l} \sqrt{\frac{Y}{\rho}}$$

$$\therefore \frac{n_l}{n_t} = \left( \frac{1}{2l} \sqrt{\frac{Y}{\rho}} \right) \div \left( \frac{1}{2l} \sqrt{\frac{T}{\pi r^2 \rho}} \right)$$

$$= r \sqrt{\frac{\pi Y}{T}}$$

Here,  $r = 0.5 \text{ mm} = 0.5 \times 10^{-3} \text{ m} = 5 \times 10^{-4} \text{ m}$

$$T = 10 \text{ kg} - \text{wt} = 10 \times 9.81 = 98.1 \text{ newton}$$

$$Y = 20 \times 10^6 = 2 \times 10^7 \text{ N/cm}^2$$

$$= 2 \times 10^{11} \text{ N/m}^2$$

$$\therefore \frac{n_l}{n_t} = 5 \times 10^{-4} \sqrt{\frac{\pi \times 2 \times 10^{11}}{98.1}} = 40$$

**Example 6.28.** A tuning fork sounds 6 beats when sounded together with 95 cm and 100 cm length of sonometer wire. Calculate the frequency of the tuning fork.

**Soln.**

It is obvious that the frequency  $n$  of the fork lies in between the frequencies of the vibrating strings of length 95 cm and 100 cm.

$$\text{Now } n_1 \propto \frac{1}{95} \text{ and } n_2 \propto \frac{1}{100}$$

$$\therefore \frac{n_1}{n_2} = \frac{100}{95} = \frac{20}{19} \quad (\text{i})$$

$$\text{Also } n_1 - n = 6$$

$$\text{and } n - n_2 = 6$$

$$\therefore n_1 - n_2 = 12 \quad (\text{ii})$$

solving for  $n_1$  from (i) and (ii) we have

$$n_1 = 240 \text{ and } n_2 = 228$$

Obviously,  $n = 234 \text{ vib/sec.}$



## EXERCISES

- [1] Obtain an expression for the velocity of sound in a gas. Show how it is affected by temperature, pressure and humidity.
- [2] Derive from elementary principles an expression for the velocity of sound in air (Newton's formula). Explain Laplace's correction.
- [3] Obtain an expression for the velocity of sound in a gas discussing in detail Newton's formula and Laplace's correction. What is the effect of temperature variation on the velocity of sound in a gas?
- [4] Derive the formula for the velocity of a plane progressive wave in a fluid.
- [5] Obtain an expression for the velocity of longitudinal waves in a gas. Discuss Laplace's correction.
- [6] Find a general expression for the velocity of sound in a gas and discuss the formula due to Newton and Laplace.
- [7] If the velocity of sound in air be given by  $v = \sqrt{\frac{E}{\rho}}$ , show that  $E$  is equal to the atmospheric pressure if the process involved be regarded as isothermal. What other view can be taken of this process? Which of the two is correct and why?
- [8] Show how the velocity of sound in a gas is affected by (a) temperature (b) pressure and (c) humidity.
- [9] Show that the velocity of sound in isotropic solids is given by  

$$v = \sqrt{Y/\rho}.$$
- [10] Derive an expression for the velocity of longitudinal waves in a metallic rod.
- [11] Deduce an expression for the velocity of transverse waves in a string. Explain how the pitch of a string vibrating transversely depends on the tension, length and the density.
- [12] Obtain the expression for the velocity of transverse waves in a string.
- [13] Prove that the frequency of vibrations of a stretched string is equal to  $(1/2l) \times \sqrt{T/m}$ . Explain the symbols used.
- [14] Deduce an expression for the velocity of a transverse wave in a sonometer wire.

- [15] State and explain the laws of transverse vibrations of a stretched string.
- [16] State the laws of transverse vibrations of stretched strings and describe experiments to verify the law concerning (i) length (ii) tension and (iii) linear density.
- [17] Describe Melde's experiment and explain how the laws of vibration of strings can be verified with this experiment.
- [18] Describe a sonometer. Explain how the laws of transverse vibrations of a stretched string can be verified with it. Explain how the diameters of two wires can be compared using sonometer.
- [19] Derive an expression for the velocity of transverse waves in a string under tension and hence calculate the expression for the fundamental frequency of vibration.
- [20] Describe Melde's experiment for determining the frequency of a tuning fork. Show that the frequency of vibration of a string in the longitudinal mode is half that in the transverse mode of vibration.
- [21] Calculate the velocity of sound in air at *N.T.P.* and deduce the change in velocity per degree centigrade rise in temperature if  $\alpha = 0.00367$ . The ratio of the two specific heats of air = 1.41. [332 m/sec, 0.61 m/s].
- [22] Determine the temperature at which the velocity of sound in air becomes 1.414 times its value at  $0^\circ\text{C}$ . [230°C].
- [23] The velocity of sound in air at  $14^\circ\text{C}$  is 340 m/sec. What will it be when the pressure of the gas is doubled and its temperature is raised to  $157.5^\circ\text{C}$ ? [416.14 metres per second].
- [24] Prove that the velocity of sound in hydrogen is four times the velocity of sound in oxygen.
- [25] A sound is emitted by a source placed at one end of an iron tube one kilometre long, and two sounds are heard at the other end at an interval of 2.8 secs. If the velocity of sound in air in the condition of the experiment is 330 metres per second, find that for iron. [4342 metres per second].
- [26] In the faintest sound that can be heard at 1000 cps, the pressure amplitude is  $2.0 \times 10^{-5} \text{ N/m}^2$ . Find the corresponding displacement amplitude. Atmospheric pressure =  $10^5 \text{ N/m}^2$ , velocity of sound in air = 331 m/sec and  $\rho$  for air =  $1.22 \text{ kg/m}^3$ . [7.883  $\times 10^{-12}$  metre].



- [27] Show that the velocity of sound in a gas, for which  $\gamma = 1.41$  is  $0.68 c$ , where  $c$  is the root mean square velocity of the molecules.
- [28] If the frequency of the longitudinal waves produced in a solid rod be 1000 cps, the density of the material of the rod,  $9 \text{ gm/cc}$ , the value of Young's modulus for it,  $9 \times 10^{12} \text{ dynes/cm}^2$ , calculate the wavelength of the waves. [1000 cm].
- [29] The velocity of a longitudinal wave in a wire of area of cross-section  $0.004 \text{ sq. cm.}$  is twice the velocity of a transverse wave along it. Determine the ratio between the Young's modulus for the material of the wire and the tension to which the wire is subjected. [ $Y:T=1000:1$ ].
- [30] If in the above problem the velocities of the longitudinal and transverse waves along the wire are the same, then show that the stress in the wire is equal to the value of  $Y$  for its material.
- [31] Find the fundamental frequency of transverse vibration of steel wire, 1 metre long and 1mm diameter hanging vertically from a rigid support with a load of 20 kg attached to its lower end. Density of steel =  $7.9 \text{ gms/cc}$  [85.5 cps].
- [32] A steel piano wire 50 cms long of mass 5 kg is stretched with a tension of  $400 \times 10^5 \text{ dynes}$ . Determine the frequency of its fundamental mode of vibration. [200 Hz].
- [33] A wire 100 cms long and fixed at both ends vibrates transversely with a fundamental frequency of 50 cycles per second. A bridge is placed under the wire and the two segments of the wire vibrating simultaneously, produce 5 beats per second. Find the distance of the bridge from the centre. [1.25 cm].
- [34] A string when stretched by a weight of 4 kg gives a note of frequency 256. What weight will produce an octave of this note? [16 kg.wt.].
- [35] A flexible string of length 0.99 metre and mass one gram is stretched by a tension of  $T$  newtons. The string vibrates in three segments with a frequency of 500 Hz. Calculate the tension. [ $T = 110 \text{ newtons}$ ].
- [36] A flexible string of length 0.88 m and mass 1 gram is stretched by a force of 55 newtons. If the string vibrates in 5 segments, calculate its frequency of vibration. [625 Hz].
- [37] Sonometer wire is stretched by means of a piece of metal hanging from its other end. With the effective length of the wire *i.e.*, the distance between the two bridges) equal to 100 cm, it emits its fundamental note in unison with that of a given tuning fork. When the suspended piece of metal is immersed in water, the length of the wire



has to be shortened by 7 cm in order that the fundamental note emitted by the wire may again be in unison with that of the fork. Obtain the value of the density of the metal piece. [7.4 gm/c.c.].

[38] One end of a string, 100 cm long and of mass 5 gm, is attached to a prong of an electrically driven tuning fork, with weights suspended from the other so that it is quite horizontal and under tension. The fork vibrates in a direction perpendicular to the length of the string which is thrown into vibration in 4 loops, emitting a note of frequency 60. Calculate the tension in the string.

[39] What would happen if the fork be turned so as to vibrate parallel to the length of the string? [ $4.5 \times 10^5$  dynes; frequency of the note emitted = 30].

[40] Calculate the velocity of sound in dry hydrogen at 363K assuming the density of hydrogen at NTP =  $0.089 \text{ kg/m}^3$  and  $\gamma = 1.41$  [1461 m/s].

[41] A metal rod 1.5 m long is fixed at the centre. When it vibrates longitudinally, the frequency is found to be 1200. Calculate the Young's modulus of the material of the rod. Its density =  $8 \times 10^3 \text{ kg/m}^3$ . [ $10.4 \times 10^{10} \text{ N/m}^2$ ].

[42] The adiabatic bulk modulus of a gas at a pressure of 76 cm of mercury is  $1.317 \times 10^6 \text{ dynes/cm}^2$ ; calculate the value of  $\gamma$ . If the density of the gas is  $1.98 \times 10^{-3} \text{ g/cm}^3$ , calculate the wavelength of sound of frequency 1000 vibrations/s. [1.3; 25.8 cm].

[43] A string weighing 10 gm and measuring 100 cm is stretched tightly. Calculate the tension in kg that is required for producing a note of frequency 140 vib/sec when the string vibrates so as to emit (i) fundamental tone and (ii) first overtone. [(i) 80 kg (ii) 20 kg].

[44] Two wires A and B of the same material and having equal lengths are in unison when stretched by forces of 3.5 and 10.5 kg weight respectively. What is the ratio of their diameters? If the force on A is increased to 5 kg, what is the new ratio of the frequencies? [ $\sqrt{3} : 1$  ;  $10 : \sqrt{70}$ ].

[45] Calculate the frequency of the fundamental note of a string 1 metre long weighing  $2 \times 10^{-3} \text{ kg}$  when stretched by a weight of 0.4 kg [70 vib/s].

[46] Two identical wires, when stretched with the same tension, have a fundamental frequency of 400 vibrations/second. By what fractional amount must the tension in one wire be increased in order that 4 beats/second occur when both wires vibrate simultaneously? [2%].

## CHAPTER VII

## DOPPLER EFFECT

*Doppler effect-Calculation of apparent frequency-General expression for Doppler effect-Application of Doppler effect-Doppler effect in light-Solved problems-Exercises.*

## 7.1 Doppler effect

Everyone is familiar with the fact that the whistle of a train or a jet airplane appears to increase in pitch (note of the sound emitted) as it approaches a stationary observer; as the moving object passes the observer, the pitch changes and becomes lower. Same effect is produced when an observer approaches a stationary source emitting sound. Stated in more general term, there will be an alteration in the pitch of the sound emitted whenever the source emitting it or the observer receiving it is in relative motion with each other.

This apparent alteration or shift in the frequency of sound wave due to the relative motion between the source and the observer was first predicted in 1845 by Doppler, an Austrian physicist, and is known as *Doppler effect*.

Fig. 7.1 illustrates how the change of wavelengths, and hence frequency, occurs when a source of sound is moving towards a stationary observer. Let the position of a moving source at a certain instant be at 4. At four successive seconds before this instant the source had been at the positions 3, 2, 1, 0 respectively. If  $v$  is the velocity of sound, the wavefront of the sound wave emitted by the source when it was in position 3 reaches the surface A of a sphere of radius  $v$  and centre 3 by the time the source just reaches 4. In the same manner, the wavefront of the wave emitted when the source was in position 2 reaches the surface B of a sphere of radius  $2v$  and centre 2 by the time the source reaches 4. The wavefront C corresponds to the source when it was in the position 1, and the wavefront D to the source when it was in the position 0. Thus if the observer is on the right of S, he receives wavefronts which are relatively more crowded together than if S were stationary; the frequency of S thus appears to increase. When the observer is on the left of S, in which case the source is moving away from him, the



wavefronts are farther apart than if S were stationary and hence the observer receives correspondingly fewer waves per second. The apparent frequency of S thus appear to be lowered.

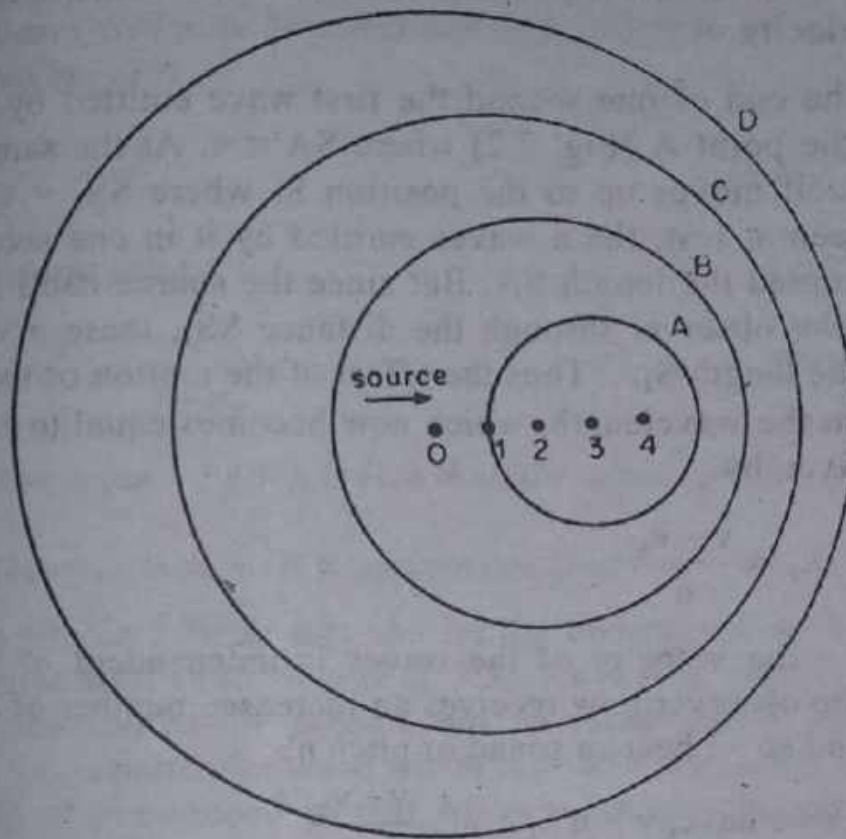


Fig. 7.1

## 7.2 Calculation of apparent frequency

While calculating the apparent frequency, the cases that will be considered are (i) when both the source and the observer are at rest, (ii) when the source only is in motion (iii) when the observer only is in motion, (iv) when the source and the observer are both in motion and (v) that due to the presence of wind in all these cases.

### Case I : Source and observer are both at rest

Let a source which is at rest send out  $n$  waves per second. These waves travel through a distance  $v$ , the velocity of sound waves in air in one second. The observer will hear a sound of frequency (or pitch)  $n$ .



*Case II : Source is in motion and observer at rest*

Let  $S$  be position of the source and  $A$  that of the observer at rest. The source emits  $n$  waves per second which travel towards the observer with a velocity  $v$  and let the source follow these waves with a velocity  $v_s$ .

At the end of one second the first wave emitted by the source reaches the point  $A$  (Fig. 7.2) where  $SA = v$ . At the same time the source itself moves up to the position  $S_1$  where  $SS_1 = v_s$ . Had the source been at rest, the  $n$  waves emitted by it in one second would have occupied the length  $SA$ . But since the source itself has moved towards the observer through the distance  $SS_1$ , these  $n$  waves now occupy the length  $S_1A$ . Thus the effect of the motion of the source is to shorten the wavelength, which now becomes equal to  $\lambda_1$  (say).  $\lambda_1$  is then given by

$$\lambda_1 = \frac{v - v_s}{n}$$

Since the velocity of the waves is independent of the wavelength, the observer now receives an increased number of waves per second and so he hears a sound of pitch  $n'$ .

$$\text{Thus we have, } v = n'\lambda_1 = n' \cdot \frac{v - v_s}{n}$$

$$\text{Therefore, } n' = \frac{nv}{v - v_s} = n\left(\frac{v}{v - v_s}\right) \quad (7.1)$$

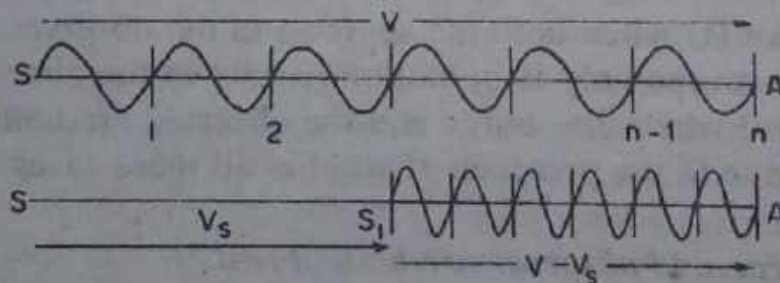


Fig. 7.2

From eqn. (7.1) it is obvious that the apparent frequency or pitch  $n'$  is greater than the actual frequency or pitch  $n$ .

**Note :** If the source moves away from the observer i.e., moves in the opposite direction to that in the above case,  $v_s$  becomes negative.  $n$  waves will now be contained in a length  $(v + v_s)$ ; so the apparent wavelength is

$$\lambda_1 = \frac{v + v_s}{n}$$

The apparent pitch

$$n' = \frac{v}{\lambda'} = \left( \frac{v}{v + v_s} \right) \cdot n. \quad (7.2)$$

Hence the apparent pitch is less than the actual pitch.

**Case III : Observer is in motion and source is at rest**

Let the source  $S$  be at rest and let the observer  $A$  move away from the source with velocity  $v_0$  (Fig. 7.3). As before, let  $n$  waves be emitted by the source per second and let these waves reach the observer  $A$ . Let a particular wave which is now at  $A$  reach a position  $B$  at the end of one second so that  $AB = v$ . During the same time interval (one second) the observer has moved up to the position  $C$ , so that  $AC = v_0$ .

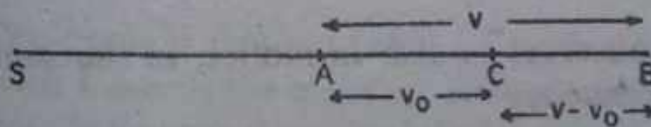


Fig. 7.3

In this case, the wavelength is not affected by the motion of the observer; the observer only receives a fewer number of waves—those waves lying in between  $C$  and  $B$ , i.e., in the length  $v - v_0$  and hears a sound of pitch  $n'$ .

Since  $n$  waves occupy a length  $v$ , so for a length  $v - v_0$  we will have  $n(v - v_0)/v$  waves

$$\begin{aligned}\text{Thus } n' &= \frac{n(v - v_o)}{v} = \frac{(v - v_o)}{v} n \\ &= n \{1 - (v_o/v)\}\end{aligned}\quad (7.3)$$

Thus  $n'$  is less than  $n$  and the pitch is lowered.

**Note :** If the observer moves towards the source i.e., in a direction opposite to that in the above case,  $v_o$  becomes negative so that the apparent pitch becomes

$$n' = \left(\frac{v + v_o}{v}\right)n = n\left[1 + \frac{v_o}{v}\right] \quad (7.4)$$

The observer now receives more number of waves in one second and the apparent pitch is higher than the actual pitch.

**Case IV : Source and observer both in motion.**

Equations (7.1) and (7.3) can be combined to give an expression for the pitch of the sound when both the source and the observer are in motion in the same direction.

When the source is in motion and the observer is at rest, we have, for the apparent frequency

$$n_1 = \frac{nv}{v - v_s} \quad (7.5)$$

From eqn. (7.1)

Again, if the observer is also in motion in the same direction then the apparent frequency

$$n' = \left(\frac{v - v_o}{v}\right) n_1 \quad (7.6)$$

From eqn. (7.3)

Substituting the value of  $n_1$  as given by eqn. (7.5) in eqn. (7.6), we get,

$$\begin{aligned}n' &= \left(\frac{v - v_o}{v}\right) n_1 \\ &= \left(\frac{v - v_o}{v}\right) \left(\frac{nv}{v - v_s}\right)\end{aligned}$$



$$= \left( \frac{v - v_o}{v - v_s} \right) n \quad (7.7)$$

$n'$  will be greater or less than  $n$  depending on whether  $v_s$  is greater or less than  $v_o$ .

### Special cases :

(a) *When the source and observer move towards each other :*

In eqn. (7.7), taking  $v_o$  to be negative, we have

$$n' = \left[ \frac{v - (-v_o)}{v - v_s} \right] n = \left( \frac{v + v_o}{v - v_s} \right) n \quad (7.8)$$

(b) *When the source and the observer move away from each other*

Taking  $v_s$  to be negative in eqn. (7.7), we have

$$n' = \left[ \frac{v - v_o}{v - (-v_s)} \right] n = \left[ \frac{v - v_o}{v + v_s} \right] \cdot n \quad (7.9)$$

(c) *Source moving away from the observer and the observer moving towards the source*

Taking both  $v_o$  and  $v_s$  negative in eqn. (7.7), we have

$$n' = \left[ \frac{v - (-v_o)}{v - (-v_s)} \right] n = \left[ \frac{v + v_o}{v + v_s} \right] \cdot n \quad (7.10)$$

**Note :** The general relation (eqn. 7.7) refers to the case when the source moves towards the observer and the observer moves away from the source. When the direction of any one of the two changes, the signs of  $v_s$  and  $v_o$  have to be changed. While solving numerical problems, the general relation [eqn. (7.7)], with appropriate signs of  $v_s$  and  $v_o$ , should be used.

### 7.3 General expression for Doppler effect : the source, the observer and the wind all moving in the same direction

Let a source be situated at S and give out  $n$  waves per second. Let an observer, located at D, receive  $n$  waves per second. He, therefore, hears a sound of pitch  $n$  (Fig. 7.4). Let  $v$  be the velocity of sound waves, i.e., the distance traveled by the waves in one second

be represented by SB. If  $w$ , the velocity of the wind in the same direction as the waves be denoted by BC, then the distance travelled by the sound waves in one second will be SC.

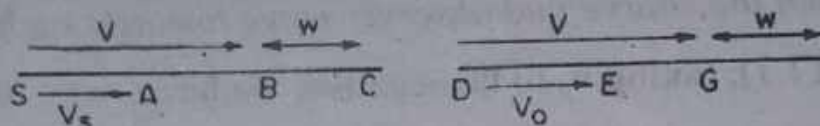


Fig. 7.4

Let the velocity  $v_s$  of the source emitting sound waves be represented in magnitude and direction by SA. Now if the source is considered to be at rest, the waves starting from the source would pass the observer at D and would travel through a distance DG in one second. If the wind blows with a velocity  $w$  in the same direction as the waves, the distance travelled by the waves in one second is  $DG + GF$  or  $DF$  and is the same as SC.

Let us now consider a case when the source and the observer are moving in the same direction *i.e.*, towards the right with velocities  $v_s$  and  $v_o$  respectively. At any instant of motion, let  $n$  be the number of waves emitted by the source per second and let  $n'$  be the number of waves received by the observer per second. Let S be the position of the source at the beginning of the time when the first wave starts from it.

Now A is the position of the source just after one second and at that instant the  $(n+1)$ th wave is about to start from it and by this time the first wave has reached the point C. Therefore, in the first second, there are  $n$  waves in the length AC.

Let the first wave reach the observer at an instant when he is at D. After one second this wave will reach F. During this interval the observer has moved to E to receive the last wave of the second. Therefore, there are  $n'$  waves in the length EF in one second.

Thus we have



$$\frac{n'}{EF} = \frac{n}{AC}$$

$$\text{or, } \frac{n'}{n} = \frac{EF}{AC} = \frac{DF - DE}{SC - SA} = \frac{v + w - v_0}{v + w - v_s}$$

$$\text{or, } n' = \frac{v + w - v_0}{v + w - v_s} \cdot n \quad (7.11)$$

= pitch (or frequency) of the sound as perceived by the observer.

Proper signs of  $w$ ,  $v_s$  and  $v_0$  are to be considered when the motion of the wind, the source and the observer be not in the direction from the source to the observer, *i.e.*, towards the right.

#### 7.4 Doppler effect at low speeds

Doppler effect when the source is in motion (eqns. 7.1 and 7.2) and when the observer is in motion (eqns. 7.3 and 7.4) are different, even though the source and the observer may be moving at the same speed. However, if the speeds are low enough compared to the speed of sound (*i.e.*,  $v_s \ll v$  and  $v_0 \ll v$ ), the frequency changes produced by these two motions are essentially the same.

By using binomial theorem, it can be shown that the equation for change in frequency when both the source and the observer are moving (eqns. 7.6 – 7.10) can be written as

$$n' = n \left( 1 \pm \frac{u}{v} \right)$$

where  $u = v_s \pm v_0$  is the relative speed of the source with respect to the detector. If the source and the observer are moving towards each other, the plus sign is chosen and a greater frequency is anticipated. On the other hand, if the source and the observer move away from each other, the minus sign is chosen, and a decrease in frequency is anticipated.

#### 7.5 Doppler effect at supersonic speeds

If a source moves toward a stationary observer at a speed equal to the speed of sound (*i.e.*,  $v_s = v$ ), eqn. 7.1 predicts that the frequency



heard ( $n'$ ) will be infinitely great. This means that the source moves so fast that it keeps pace with its own spherical wavefronts as suggested by Fig. 7.5. What happens if the source moves with supersonic speed *i.e.*, a speed greater than the speed of sound.

Eqn. 7.11 no longer holds for such supersonic speeds. Fig. 7.6 depicts the spherical wavefront that originated at various positions of the source. The radius of any wavefront in the figure is given by  $vt$  where  $v$  is the speed of sound and  $t$  is the time that has elapsed since the source emitted that wavefront. As can be seen from the figure, all the wavefronts bunch along a V-shaped envelope. In three dimensions, this envelope is a cone, called the *Mach cone*. Because the bunching of the wavefronts causes an abrupt rise and fall of air pressure as the surface passes through any point, a *shock wave* is said to exist along the surface of this cone. The half-angle  $\theta$  of the cone (Fig. 7.6) is called the *Mach cone angle* and is given by

$$\sin \theta = \frac{vt}{v_s t} = \frac{u}{v_s}$$

The ratio  $v_s/v$  is called the *Mach number*. When we say that a particular plane has flown at Mach 1.8, it means that the speed of the plane was 1.8 times the speed of sound in air through which the plane was flying. The shock wave generated by a supersonic plane or a projectile produces a burst of sound called *sonic boom*.

## 7.6 Application of Doppler effect

The phenomenon of Doppler effect can be conveniently used to track an earth satellite. Let the earth satellite emit a radio signal of constant frequency  $n$ . As the satellite moves from position 1 to position 2 (Fig. 7.6), the component velocity of the satellite towards the earth station A decreases. The apparent frequency  $n'$  of the signal as received on the earth, therefore, decreases as the satellite moves from position 1 to position 2. Also this velocity component points away from the earth as the satellite moves from position 2 to position 3. If the signal received from the satellite is combined with a signal of constant frequency at the earth station, a beat frequency of audible note can be heard, which decreases as the satellite passes over head.

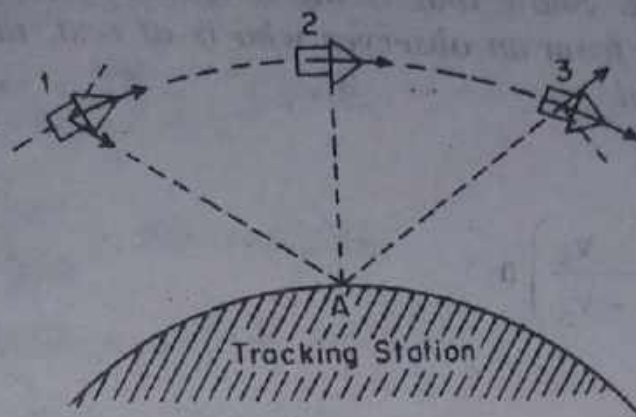


Fig. 7.5

The phenomenon of Doppler effect is used by the traffic police to detect the speed of automobiles. The technique is to emit an electro-magnetic wave of constant frequency by a source attached to the police van. This wave is reflected by the moving car, which in turn acts as a moving source. This results in a Doppler shift in the frequency of the reflected wave. From the shift in frequency, the speed of the car can be measured.

**Example 7.1.** A train approaches a stationary observer at a speed of 75 kilometers per hour sounding a whistle of frequency 1000. What will be the apparent frequency of the whistle to the observer? (velocity of sound = 330 m/sec)

**Soln.**

$$n' = \left( \frac{V - v_0}{V - v_s} \right) n$$

Here  $n = 1000$ ,  $V = 330 \text{ m. sec}^{-1}$

$$v_0 = 0$$

$$v_s = 75 \text{ km/hr} = 20.833 \text{ m. sec}^{-1}$$

$$\therefore n' = \left( \frac{330}{330 - 20.833} \right) 1000$$

$$= \frac{330}{309.167} \times 1000 = 1067.$$

**Example 7.2.** Show that if the source moves away with the velocity of sound from an observer who is at rest, the frequency of vibration is halved.

**Soln.**

$$n' = \left( \frac{V - v_0}{V - v_s} \right) n$$

here  $v_0 = 0$

and  $v_s = -V$ .

$$\begin{aligned} n' &= \frac{V}{V + V} \cdot n \\ &= \frac{V}{2V} \cdot n = \frac{1}{2} n. \end{aligned}$$

**Example 7.3.** A car sounding a horn producing a note of 500 Hz, approaches and then passes a stationary observer at a steady speed of 20 m. sec<sup>-1</sup>. What will be the frequencies apparent to the observer when the car is (i) approaching and (ii) receding? What is the interval between these two notes? (velocity of sound = 340 m. s<sup>-1</sup>).

**Soln.**

$$(i) \quad n' = \left( \frac{V - v_0}{V - v_s} \right) n$$

here  $V = 340 \text{ m.s}^{-1}$

$v_0 = 0, v_s = 20 \text{ m.s}^{-1}$

$n = 500 \text{ Hz.}$

$$\begin{aligned} \therefore n' &= \frac{V}{V - v_s} \cdot n \\ &= \left( \frac{340}{340 - 20} \right) \times 500 \\ &= \frac{340}{320} \times 500 = 531 \text{ Hz.} \end{aligned}$$



(ii) here  $v_s = -20 \text{ ms}^{-1}$  and  $v_o = 0$ .

$$\therefore n' = \left[ \frac{340}{340 - (-20)} \right] \times 500$$

$$= \frac{340}{360} \times 500 = 472 \text{ Hz.}$$

Interval between the notes

= ratio of the two frequencies

$$= \frac{531}{472} = 1.125$$

**Example 7.4.** Calculate the velocity at which a source of frequency 10000 per second should approach the observer at rest in order to produce Doppler's shift of 200 per second. (velocity of sound = 340 m per sec.)

**Soln.**

$$n' = \left( \frac{V - v_o}{V - v_s} \right) n$$

Here  $V = 340 \text{ m/sec}$

$v_o = 0$ ,  $n = 10,000$

$v_s = ?$

Since the source approaches the observer,  $v_s$  is positive and, therefore,  $n' > n$ .

$$\therefore n' = n + 200 = 10200$$

$$\therefore 10200 = \left( \frac{340}{340 - v_s} \right) \times 10000$$

$$\text{or, } 340 - v_s = \frac{340}{10200} \times 10000$$

$$= 333.33$$

$$\therefore v_s = 340 - 333.33$$

$$= 6.66 \text{ m per sec.}$$

**Example 7.5.** Calculate the velocity at which an observer should approach a source of frequency one thousand per second at rest in order to produce a Doppler shift of 40 per second. (velocity of sound in air = 340 m/sec).

**Soln.**

$$n' = \left( \frac{V - v_0}{V - v_s} \right) n$$

Here  $n = 1000$ ,

$v_s = 0$ ,  $V = 340$  m/sec.

Observer moves towards the source i.e.,  $v_0$  is negative. Therefore  $n' > n$ ;

or,  $n' = 1000 + 40 = 1040$ .

$$\begin{aligned} \therefore n' &= \left[ \frac{V - (-v_0)}{V} \right] \times n \\ &= \left( \frac{V + v_0}{V} \right) n \end{aligned}$$

$$\begin{aligned} \text{or, } 1040 &= \left( \frac{340 + v_0}{340} \right) \times 1000 \\ &= \left( 1 + \frac{v_0}{340} \right) \times 1000 \\ &= 1000 + \frac{1000 v_0}{340} \end{aligned}$$

$$\text{or, } 1040 - 1000 = \frac{1000 v_0}{340}$$

$$\text{or, } \frac{1000 v_0}{340} = 40$$

$$\therefore v_0 = \frac{13600}{1000} = 13.6 \text{ m/sec.}$$

**Example 7.6.** Two aeroplanes pass each other in opposite directions and one of them is blowing a whistle of frequency 540 Hz.

Calculate the frequencies of the notes heard in the other aeroplane (i) before and (ii) after they have passed each other. Velocity of either of the aeroplanes is  $540 \text{ km hr}^{-1}$  and the velocity of sound =  $350 \text{ ms}^{-1}$ .

**Soln.**

$$V = 350 \text{ m/s.}$$

$$v_s = v_0 = \pm 540 \text{ km/hr} = 150 \text{ m/sec.}$$

$$n = 540 \text{ Hz.}$$

$$\text{and } n' = \left( \frac{V - v_0}{V - v_s} \right) \times n$$

$$(i) \quad v_0 = -ve \text{ and } v_s = +ve.$$

$$\therefore n' = \left( \frac{V + v_0}{V - v_s} \right) \times n$$

$$= \left( \frac{350 + 150}{350 - 150} \right) \times 540$$

$$= \frac{500}{200} \times 540$$

$$= 1350 \text{ Hz.}$$

$$(ii) \quad v_0 = +ve \text{ and } v_s = -ve$$

$$\therefore n' = \left( \frac{V - v_0}{V + v_s} \right) \times n$$

$$= \left( \frac{350 - 150}{350 + 150} \right) \times 540$$

$$= \frac{200}{500} \times 540 = 216 \text{ Hz.}$$

**Example 7.7.** Two aeroplanes A and B are approaching each other and their velocities are  $108 \text{ km/hr}$  and  $144 \text{ km/hr}$  respectively. The frequency of a note emitted by A as heard by the passengers in B is  $1170 \text{ Hz}$ . Calculate the frequency of the note as heard by the passengers in A. (velocity of sound =  $350 \text{ m/sec}$ ).



**Soln.**

Here  $V = 350$  m/sec.

$$v_0 = 144 \text{ km/hr} = 40 \text{ m/sec}$$

$$v_s = 108 \text{ km/hr} = 30 \text{ m/sec}$$

$$n' = 1170 \text{ Hz.}$$

$$n = ?$$

The source and the observer are approaching each other. So  $v_s$  is positive and  $v_0$  is negative.

Hence, from  $n' = \left( \frac{V - v_0}{V - v_s} \right) \times n$  we have

$$\begin{aligned} n' &= \left( \frac{V + v_0}{V - v_s} \right) \times n \\ &= \left( \frac{350 + 40}{350 - 30} \right) \times n \end{aligned}$$

$$\text{or, } 1170 = \frac{390}{320} \times n$$

$$\text{or, } n = \frac{320 \times 1170}{390}$$

$$= 960 \text{ Hz.}$$

**Example 7.8.** The apparent frequency of the whistle of an engine changes in the ratio 6 : 5 as the engine passes an stationary observer. If the velocity of sound is 352 m/sec, calculate the velocity of the engine.

**Soln.**

Let the frequencies as heard by the observer when the train is moving towards and away from him be  $n_1$  and  $n_2$  respectively. Then

$$\frac{n_1}{n_2} = \frac{6}{5}$$

$$\text{Now } n_1 = \left( \frac{V - v_0}{V - v_s} \right) \times n$$

$$= \left( \frac{V}{V - v_s} \right) \times n \quad (\because v_0 = 0 \text{ and } v_s = +ve)$$

$$\text{and } n_2 = \left( \frac{V - v_0}{V + v_s} \right) \times n$$

$$= \left( \frac{V}{V + v_s} \right) \times n \quad (\because v_0 = 0 \text{ and } v_s = -ve)$$

$$\therefore \frac{n_1}{n_2} = \left( \frac{V}{V - v_s} \right) \times \left( \frac{V + v_s}{V} \right)$$

$$= \frac{V + v_s}{V - v_s}$$

$$\text{or, } \frac{6}{5} = \frac{352 + v_s}{352 - v_s}$$

$$\text{or, } v_s = 13.8 \text{ m.s}^{-1}$$

The speed of the engine is 13.8 m/sec.

**Example 7.9.** A car emitting sound of frequency 200 Hz is moving away from a stationary observer and towards a rigid flat wall. The velocity of the car is 5 m/sec. Calculate the number of beats heard per second by the observer. Velocity of sound in air = 350 m/sec.

**Soln.**

$$n' = \left( \frac{V - v_0}{V - v_s} \right) n$$

$$n = 200 \text{ Hz.}$$

$$v_0 = 0, v_s = 5 \text{ m/sec.}$$

$$V = 350 \text{ m/sec.}$$

As the source moves away from the observer,  $v_s$  is negative.

$$\begin{aligned}\therefore n_1 &= \left( \frac{350}{350 - (-5)} \right) \times 200 \\ &= \frac{350}{355} \times 200 = 197.2 \text{ Hz.}\end{aligned}$$

The observer also hears a sound of apparent frequency  $n_2$  due to reflection of sound waves by the wall. This means the source is now moving towards the stationary observer. Or,  $v_s$  is positive.

$$\begin{aligned}\therefore n_2 &= \left( \frac{V}{V - v_s} \right) \times n \\ &= \frac{350}{350 - 5} \times 200 \\ &= \frac{350}{345} \times 200 = 202.9 \text{ Hz.}\end{aligned}$$

Therefore, the number of beats produced per second.

$$\begin{aligned}&= n_2 - n_1 = 202.9 - 197.2 \\ &= 5.7.\end{aligned}$$

**Example 7.10.** A motor car fitted with two sounding horns which differ in frequency by 320 vibrations per second is speeding at the rate of 36 km/hr towards a stationary observer. Calculate the difference in the frequencies of the notes heard by him. Velocity of sound in air = 330 m/sec.

**Soln.**

Let  $n_1$  and  $n_2$  be the frequencies.

For  $n_1$ , the apparent frequency is

$$n'_1 = \frac{V}{V - v_s} \times n_1 \quad (i)$$

and for  $n_2$ , the apparent frequency is

$$n'_2 = \frac{V}{V - v_s} \times n_2 \quad (ii)$$



Subtracting (ii) from (i), we have

$$n'_1 - n'_2 = \frac{V}{V - v_s} (n_1 - n_2)$$

Here  $V = 330$  m/sec.

$$v_s = 36 \text{ km/hr} = 10 \text{ m/sec}$$

$$n_1 - n_2 = 320$$

$$\therefore n'_1 - n'_2 = \frac{330}{330 - 10} \times 320$$

$$= \frac{330}{320} \times 320 = 330 \text{ vibrations per second.}$$

**Example 7.11.** At each of the two stations A and B, a siren is sounding with a constant frequency of 250 cycles per second. A cyclist from A proceeds straight towards B with a velocity of 12 km/hr and hears 5 beats per second. Calculate the velocity of sound.

**Soln.**

(i) In the first case the cyclist is moving away from the source A. So  $v_0$  is positive. We, therefore, have

$$n_1 = \left( \frac{V - v_0}{V} \right) n$$

$$v_0 = 12 \text{ km/hr} = \frac{10}{3} \text{ m/sec.}$$

$$n = 250$$

$$\therefore n_1 = \left( \frac{V - \frac{10}{3}}{V} \right) \times 250$$

(ii) In the second case the cyclist is moving towards the source B. So  $v_0$  is negative. Therefore, we have,

$$n_2 = \left( \frac{V - (-v_0)}{V} \right) n$$

$$\begin{aligned}
 &= \left( \frac{V + v_0}{V} \right) n \\
 &= \left( \frac{V + \frac{10}{3}}{V} \right) \times 250 \\
 \therefore n_2 - n_1 &= \frac{250}{V} \times \frac{20}{3} \\
 \text{or, } 5 &= \frac{250 \times 20}{3 \times V} \\
 \text{or, } V &= \frac{250 \times 20}{3 \times 5} \\
 &= 333.33 \text{ m/sec.}
 \end{aligned}$$

### 7.5 Doppler effect in light

There is a change in the frequency of light radiation when the source or the observer move with respect to one another. This phenomenon is known as Doppler effect and is similar to the apparent change in the pitch of sound when either the source or the observer is in motion with respect to one another.

There is, however, one difference between the Doppler effects in sound and in light. In deriving the Doppler effect in sound, it was found that the change in frequency for a given velocity  $v$  depends on whether it is the source or the observer that is moving with that speed. Such a distinction is possible for sound because there is a medium (the air) relative to which the motion takes place, and so it is not surprising that the motion of the source or the observer relative to the still air can be distinguished. Thus Doppler effect in sound is *asymmetric*, the apparent frequency when the source is moving towards a stationary observer is different than that when the observer is moving towards a stationary source. In the first case we have

$$n' = \left( \frac{V}{V - v_s} \right) \cdot n$$

and in the second case

$$n' = \left( \frac{V + v_0}{V} \right) n$$

Here  $n$  and  $n'$  are the actual and apparent frequencies of sound respectively,  $V$  is the velocity of sound in air,  $v_0$  and  $v_s$  are the respective velocities of the observer and the source.

But it is not so in case of light. Doppler effect in light is *symmetric*. According to theory of relativity no material medium is necessary for the propagation of light or electro-magnetic waves. The apparent frequency is the same when either the source moves towards a stationary observer or the observer moves towards a stationary source.

(i) Suppose the observer is stationary and the source is moving towards the observer with a velocity  $v$ . The observer would then receive  $v/\lambda$  waves in addition to the number  $v = c/\lambda$  that would reach him if the source were at rest. Then the apparent frequency

$$v' = \frac{c}{\lambda} + \frac{v}{\lambda} = v + \frac{vv}{c} = v \left( 1 + \frac{v}{c} \right) \quad (7.12)$$

Here  $c$  is the velocity of light and  $\lambda$  its wavelength.

(ii) When the source is stationary and the observer is moving towards it with a velocity  $v$ , then the apparent frequency

$$v' = \frac{c}{\lambda} + \frac{v}{\lambda} = v + \frac{vv}{c} = v \left( 1 + \frac{v}{c} \right) \quad (7.13)$$

Eqns. 7.12 and 7.13 are therefore similar.

(iii) When the source moves away from the stationary observer or the observer moves away from the stationary source with a velocity  $v$ , then

$$v' = \frac{c}{\lambda} - \frac{v}{\lambda} = v - \frac{vv}{c} = v \left( 1 - \frac{v}{c} \right) \quad (7.14)$$

(iv) When the source and the observer move towards each other and each is moving with a velocity  $v$ , then the apparent frequency

$$v' = \left[ v \left( 1 + \frac{v}{c} \right) \right] \left( 1 + \frac{v}{c} \right)$$



$$= v \left(1 + \frac{v}{c}\right)^2 \quad (7.15)$$

(v) When the source and the observer move away from each other with the same speed  $v$ , then the apparent frequency

$$\begin{aligned} v' &= \left[v \left(1 - \frac{v}{c}\right)\right] \left(1 - \frac{v}{c}\right) \\ &= v \left(1 - \frac{v}{c}\right)^2 \end{aligned} \quad (7.16)$$

The phenomenon of Doppler effect in light is of great importance in astronomy. In astronomy, it is the wavelength rather than the frequency that is more readily measured. Thus replacing  $n'$  by  $c/\lambda'$  and  $n$  by  $c/\lambda$  we find from eqns. 7.12 and 7.13

$$\lambda' = \lambda \left(1 \pm \frac{v}{c}\right)^{-1} \approx \lambda \left(1 \mp \frac{v}{c}\right)$$

This can be written as

$$\frac{\lambda' - \lambda}{\lambda} = \mp \frac{v}{c}$$

$$\text{or, } v = \frac{\Delta\lambda}{\lambda} \cdot c \quad (\text{for light waves } v \leq c) \quad (7.17)$$

in which  $\Delta\lambda$  is the magnitude of the Doppler wavelength shift. When the wavelength decreases, it is called a *blue shift* because the blue portion of the visible spectrum has the shortest wavelength. On the other hand if the wavelength increases, the shift is called a *red shift* as the red portion of the visible spectrum has the longest wavelength. Thus, if it is a blue shift, the frequency necessarily increases and – according to convention of signs – this means that the distance between the source and the observer is decreasing. If the wavelength increases (red shift), the distance between the source and the observer is increasing. The phenomenon has been used to estimate the speed of distant stars and planets. It has been used to measure the speed of rotation of the sun. By employing the phenomenon of Doppler effect it has been possible to establish that some of the stars that appear to be single are actually double stars that revolve about each other. A number of this double-star (spectroscopic binaries) systems have been found. By Doppler effect it has been possible to establish that the important spectral lines of

some distant nebulae which are moving with a velocity greater than  $20 \times 10^3$  km/sec appear to shift towards the red end of the spectrum by  $200 \text{ \AA}$ . This red shift gives the idea that the universe is expanding. By combining kinetic interpretation of temperature with Doppler effect it is possible to measure temperature of the order of millions of degrees Celsius as found in very hot gases or plasma which are used in thermonuclear fusion experiments.

**Example 7.12.** A spectroscopic examination of light from a certain star shows that the apparent wavelength of a certain spectral line is  $5001 \text{ \AA}$ , whereas the observed wavelength of the same line produced by a terrestrial source is  $5000 \text{ \AA}$ . In what direction and at what speed do these figures suggest that the star is moving relative to the earth? (velocity of light =  $3 \times 10^{10} \text{ cm/sec}$ ).

**Soln.**

$$n' = \left( \frac{V - v_0}{V - v_s} \right) n$$

$$\text{or, } \frac{n'}{n} = \frac{V - v_0}{V - v_s}$$

$$\text{Now } n' = \frac{v}{\lambda'} \text{ and } n = \frac{v}{\lambda}$$

$$\text{or, } \frac{n'}{n} = \frac{\lambda}{\lambda'} = \frac{V - v_0}{V - v_s} = \frac{V}{V - v_s} \quad (\because v_0 = 0)$$

$$\text{or, } \frac{\lambda'}{\lambda} = \frac{V - v_s}{V} = 1 - \frac{v_s}{V}$$

$$\text{or, } 1 - \frac{\lambda'}{\lambda} = \frac{v_s}{V}$$

$$\text{or, } 1 - \frac{5001}{5000} = \frac{v_s}{V}$$

$$\text{or, } -\frac{1}{5000} = \frac{v_s}{3 \times 10^{10}}$$

$$\therefore v_s = -\frac{3 \times 10^{10}}{5000} = 6 \times 10^6 \text{ cm/sec.}$$

Since  $v_s$  is negative, the star is moving away from the earth with a velocity of  $6 \times 10^6$  cm/sec.

**Example 7.13.** The wavelength of the most intense spectral line emitted by oxygen atoms in a distant galaxy was measured to be 513 nm. This wavelength is 12 nm greater than the wavelength that would be measured for the same line emitted by a laboratory source. What is the speed of the galaxy with respect to the earth? Is it approaching or receding? (Velocity of light =  $3 \times 10^8$  m/sec).

Soln. From eqn. 7.17, we have

$$\begin{aligned} v &= \frac{\Delta\lambda}{\lambda} \cdot c \\ &= \frac{(12 \text{ nm}) (3 \times 10^8 \text{ m/sec})}{513 \text{ nm}} \\ &= 7.0 \times 10^6 \text{ m/sec.} \\ &= 7000 \text{ km/sec.} \end{aligned}$$

The observed Doppler shift is toward the longer wavelength. Thus the observed frequency is smaller because of the motion of the galaxy. According to convention of signs, the galaxy must be receding from the earth.

## EXERCISES

- [1] Explain Doppler's principle in sound. Obtain an expression for the apparent frequency of a note when the source and the listener are (i) moving towards each other and (ii) moving away from each other.
- [2] Explain Doppler's effect. A source produces a note of frequency  $n$  and is moving towards a stationary observer with a uniform speed  $a$ . Show that the apparent pitch is

$$n' = n \left[ \frac{v}{v - a} \right]$$

where  $v$  is the velocity of sound in air.



- [3] Explain clearly Doppler's effect in sound. Show that the change in frequency due to this effect is greater when the source approaches the observer than when the observer approaches the source with the same speed.
- [4] Explain Doppler's effect. Obtain an expression for the frequency of a note heard by an observer, when both the source and the observer are in motion towards each other.
- [5] Explain Doppler's effect. Find an expression for the ratio of the apparent frequency to the real frequency of the notes emitted where there is relative movement between the source and the listener.
- [6] What is Doppler's effect in sound? Obtain expressions for shift in frequency separately for the cases of observer and source in motion. Are these expressions the same? Explain.
- [7] What is Doppler's effect? Deduce an expression for the apparent frequency due to Doppler effect when the source, the observer and the medium are moving.
- [8] Explain Doppler's principle. Deduce the formula connecting the pitch of the note heard with velocities of the wind, source and the observer.
- [9] Discuss Doppler's effect in light. Does the shift depend individually on the state of motion of the observer or that of the source? If not, why?
- [10] An observer on a railway platform observed a train passing through the station at a speed  $a$ . Show that the frequency of the whistle changes by
- $$n v \left[ \frac{2a}{v^2 - a^2} \right].$$
- [11] A person is standing on a railway platform. An engine while approaching the platform blows a whistle of pitch 600 Hz. If the speed of the engine is 72 km/hr, calculate the apparent pitch of the whistle as heard by the person. (velocity of sound = 350 m/sec.) [700 Hz].
- [12] A railway engine moving away from a person standing on a platform with a speed of 90 km/hr blows a whistle of pitch 730 Hz. If the velocity of sound in air is 340 m/s, calculate the apparent pitch of the whistle as heard by the person. [680 Hz].
- [13] A person is standing near a railway track and a train moving with a speed of 72 km/hr is approaching him. The apparent pitch of the whistle as heard by the person is 680 Hz. Calculate the actual frequency of the whistle. Velocity of sound in air = 340 m/sec. 640 Hz].

- [14] Two aeroplanes A and B are approaching each other with a speed of 360 km/hr. The frequency of the whistle emitted by A is 1000 Hz. Calculate the apparent pitch of the whistle as heard by the passengers of aeroplane B. Velocity of sound in air = 350 m/sec. [1350 Hz].
- [15] Two aeroplanes A and B are moving away from one another with a speed of 540 km/hr. The frequency of the whistle emitted by A is 1000 Hz. Calculate the apparent pitch of the whistle as heard by the passengers on B. Velocity of sound in air = 350 m/sec. [400 Hz].
- [16] A locomotive engine moves towards you with a ground speed of 18.8 km per hour blowing a whistle which has a frequency of 1000 per second. At the same time you are moving towards the engine in a car with an equal speed. What will be the apparent frequency of the whistle to you? Velocity of sound in air = 333 metres per second. [1031 per second].
- [17] An observer on the railway platform observed that as a train passed through the station at 90 km/hr, the frequency of the whistle appeared to drop by 400 Hz. Find the frequency of the whistle. Velocity of sound in air = 350 m/sec. [2785.7 Hz].
- [18] A motor car sounding a horn at a frequency of 100 Hz moves away from a stationary observer towards a rigid flat wall with a velocity of 36 km/hr. How many beats per second will be heard by the observer? The velocity of sound in air = 350 m/sec. [5.67].
- [19] Two trains travelling in opposite directions at 100 km/hr each, cross each other while one of them is whistling. If the frequency of the note is 800 Hz, find the apparent pitch as heard by an observer in the other train :
- (i) before the trains cross each other
  - (ii) after the trains have crossed each other. Velocity of sound in air = 340 m/s.
- [20] An observer on a railway platform noticed that when a train passed through the station, at a speed of 72 km/hr, the frequency of the whistle appeared to drop by 500 Hz. Calculate the actual frequency of the note given by the whistle. Velocity of sound in air = 340 m/sec. [4235.3 Hz].



## CHAPTER VIII

## INTENSITY OF SOUND AND ACOUSTICS

*Intensity of sound-Measurement of intensity of sound - Bel-Sound pressure level - Phon-Limits of audibility-Acoustic intensity-Acoustic measurements-Architectural acoustics - Determination of absorption coefficient-Requisites for good acoustics-Solved problems-Exercises.*

## 8.1 Intensity of sound

The intensity of sound is defined as the rate of transfer of energy per unit area, the area being perpendicular to the direction of propagation of sound. The determination of the intensity of sound is of great importance in practical acoustics.

The intensity of sound or amount of energy transfer per unit area per second is (Art. 4.9)

$$I = 2\pi^2 \rho n^2 a^2 v \quad (8.1)$$

where  $\rho$  = density of the medium

$v$  = velocity of sound

$n$  = frequency of the sound wave

and  $a$  = amplitude of the sound wave

$$\text{Now } v = \sqrt{\frac{E}{\rho}}$$

$$\text{and } E = -\frac{p}{dV/V}$$

where  $dV$  is the change in volume,  $V$  is the original volume and  $p$  is the excess pressure.

$$\therefore v = \sqrt{\frac{-p}{\left(\frac{dV}{V}\right) \rho}}$$

$$\text{Now } \frac{dV}{V} = \frac{dy}{dx} \quad (\text{Art. 5.1})$$



Therefore, on simplification

$$p = -v^2 \rho \cdot \frac{dy}{dx}$$

The general form of the equation of a plane progressive wave is

$$y = a \sin \frac{2\pi}{\lambda} (vt - x)$$

$$\therefore \frac{dy}{dx} = \frac{2\pi}{\lambda} a \cos \frac{2\pi}{\lambda} (vt - x)$$

Substituting this value in eqn. (7.2), we have

$$p = \frac{2\pi a v^2 \rho}{\lambda} \cos \frac{2\pi}{\lambda} (vt - x)$$

The maximum excess pressure,

$$p_{\max} = \frac{2\pi a v^2 \rho}{\lambda}$$

$$\therefore p = p_{\max} \cos \frac{2\pi}{\lambda} (vt - x)$$

$$\text{Again } p_{\max} = 2\pi a \rho v \cdot \frac{v}{\lambda}$$

$$= 2\pi a \rho v n \quad (8.2)$$

From (7.1) and (7.3), we have

$$I = 2\pi^2 \rho n^2 a^2 v$$

$$= \frac{(2\pi \rho n a v)^2}{2\rho v} \quad (8.3)$$

$$= \frac{p_{\max}^2}{2\rho v}$$

The root mean square value of pressure,  $P_{\text{rms}}$  is given by

$$P_{\text{rms}} = \frac{p_{\max}}{\sqrt{2}}$$

Eqn. 8.3 therefore becomes

$$I = \frac{P_{\max}^2}{2\rho v} = \frac{(P_{\max} \cdot \sqrt{2})^2}{\rho v} = \frac{P_{\text{rms}}^2}{\rho v} \quad (8.4)$$

Eqn. (8.3) or (8.4) shows that the intensity of sound varies directly as the square of the excess pressure. To obtain the value of intensity of sound in acoustics, it is, therefore, important to measure the excess pressure. For ordinary conversation,  $P_{\max} = 0.1$  newton/m<sup>2</sup> and the sound output per square metre is  $1.13 \times 10^{-5}$  watt. Human ear is an extremely sensitive organ and can detect intensities as low as  $10^{-12}$  watt/m<sup>2</sup>.

## 8.2 Measurement of intensity of sound

The intensity of sound is the quantity of sound energy that flows through a unit area in unit time, the direction of flow being perpendicular to the area. Its unit in SI unit is joule/m<sup>2</sup>-s or watts/m<sup>2</sup>.

The physiological sensation termed loudness depends on the intensity of sound. But the loudness of sound is just an aural sensation and is a physiological phenomenon rather than a physical phenomenon. The intensity of sound refers to the external or the objective measurement while loudness refers to the internal or subjective aspect. Intensity of sound is a definite physical quantity and loudness is merely a degree of sensation. Though loudness depends on intensity, it does not increase linearly with the intensity of sound. The increase is in accordance with Weber-Fecher law in physiology, according to which, the loudness produced is proportional to the logarithm of intensity. If  $S$  stands for loudness when the intensity is  $I$ , then,

$$S \propto \log I$$

$$\text{or, } S = K \log I$$

where  $K$  is a constant.

$$\text{or, } \frac{dS}{dI} = \frac{K}{I}$$

The quantity  $\frac{dS}{dI}$  is referred to as the sensitiveness of the ear. As

can be seen, the sensitiveness of the ear decreases with the increase in the intensity of sound.

In all cases of practical measurements, it is the relative intensity rather than its absolute value which is more important. Hence the intensity of sound is often measured as its ratio to a standard intensity  $I_0$ . The standard value of  $I_0$  is taken as  $10^{-12}$  watt/m<sup>2</sup>, which corresponds to the lowest audible sound at a frequency of 100 Hz. It is referred to as the *threshold* of hearing.

### 8.3 Bel

Alexander Graham Bell introduced an arbitrary scale of loudness. If a sound is ten, hundred or thousand times more intense than another, he called the former to be one unit, two units or three units higher than the latter. This unit was later called a *bel*. In more general term, if a sound is  $10^p$  times more intense than another, the former is  $p$  bels higher than the latter. In other words, if a sound is  $q$  times more intense than another, the former is  $\log_{10} q$  bels higher than the latter.

Suppose  $dS$  is the change perceived in loudness when  $dI$  is the change produced in original intensity  $I$ , then we have

$$dS \propto \frac{dI}{I}$$

$$\text{or, } dS = K \frac{dI}{I} \text{ where } K \text{ is a constant.}$$

$$\text{or, } S = K \log_{10} I + A.$$

Let  $S$  be the loudness for an intensity  $I$  and  $S_0$  that for an intensity  $I_0$ . Then

$$S = K \log_{10} I + A$$

$$\text{and } S_0 = K \log_{10} I_0 + A$$

The intensity level  $L$  is the difference in loudness.

Then

$$\begin{aligned} L &= S - S_0 \\ &= K \log_{10} I - K \log_{10} I_0 \end{aligned}$$



$$= K \log_{10} \left( \frac{I}{I_0} \right)$$

when  $K = 1$ ,

$$L = \log_{10} \left( \frac{I}{I_0} \right) \quad (8.5)$$

Here the unit of  $L$  is bel.

In practice bel is a large unit and hence  $\frac{1}{10}$  th of a bel, called *decibel* (written as *db*) is generally used. The intensity level  $L$  when measured in decibels, becomes

$$L = 10 \log_{10} \left( \frac{I}{I_0} \right) \text{ decibel.}$$

Suppose the intensity level changes by 1 decibel,

$$\text{then } 1 = 10 \log_{10} \left( \frac{I}{I_0} \right)$$

$$\text{or, } \frac{I}{I_0} = 1.26$$

It means that the intensity level alters by 1 decibel when the intensity of sound changes by 26%.

The lowest changes in intensity level that can be detected by the human ear is 1 decibel. The range of audibility, also referred to as dynamic range, for the human ear is  $10^{12}$  in terms of intensity. Hence, in terms of bel, the dynamic range of the audibility of the human ear is 12 bels or 120 decibels.

The intensity levels of different sound is given below:

Source of sound	Intensity level in decibels
Threshold of hearing	0
Rustle of leaves	10
Whisper	15 - 20
Normal conversation	60 - 65
Heavy traffic	70 - 80
Roaring of a lion (at a distance of 6m)	90
Thunder	100 - 110
Painful sound	130 and above

### 8.4 Phon

The intensity levels mentioned in the table of Art. 8.3 refer to the loudness in decibels with the assumption that the loudness is independent of the pitch (or frequency). Actually loudness depends upon intensity and frequency. Hence intensity level will be different at different frequencies even for the same value of  $I/I_0$ . The curves a, b, and c in Fig. 8.1 represent respectively three values of equal loudness. As can be seen from the curves, the intensity at X, when the frequency is 1000 Hz, is less than the intensity at Y, when the frequency is 500 Hz, although the loudness is the same.

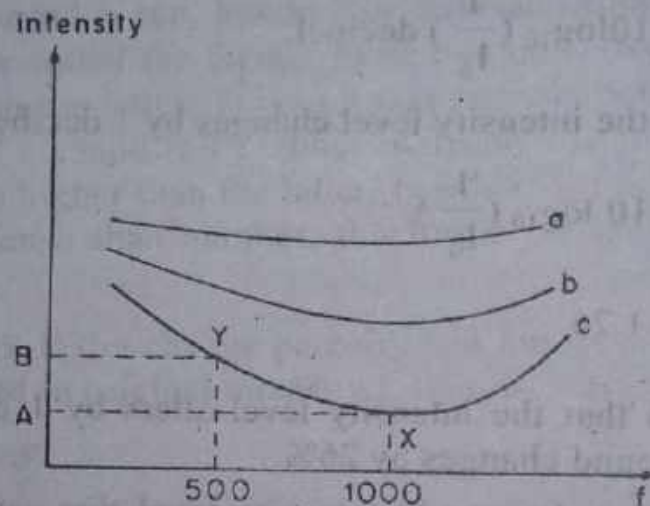


Fig. 8.1

Therefore, to measure loudness, scientists have adopted a new unit called *phon*. The measure of loudness in phons of any sound is equal to the intensity level in decibels of an equally loud pure tone of frequency 1000 Hz. Thus the decibel scale and the phon scale agree for a frequency of 1000 Hz but the two values differ at other frequency.

Suppose the intensity level of a note of frequency 480 Hz is to be determined. A standard source of frequency 1000 Hz is sounded and then its intensity adjusted so that it is equal to the loudness of the given note of frequency 480 Hz. The intensity level of the standard source in decibels is numerically equal to the loudness of the given source in phons. For example, let a frequency of 3000 Hz and intensity level 70 decibels give the same loudness as a standard source



of frequency 1000 Hz at intensity level 67 db. The intensity level of the note at 3000 Hz is then 67 phons.

### 8.5 Limits of audibility

The sensation of sound depends both on frequency and intensity within certain approximate range, which varies from person to person, and also upon the age of the particular person. In order that a sound is audible; it must have a certain minimum intensity and a certain minimum frequency. There is a minimum intensity of sound below which it is not audible. There is also a maximum intensity limit, beyond which the sound produces a sensation of pain in the ear. Similarly there are lower and upper limits of frequency below and above which sound is not audible.

The frequency range of audibility extends from 20 *cps* to 20000 *cps*. some persons can hear vibration of frequency below 20 *cps*. Young persons can hear vibration upto 20000 *cps*. But with age audibility reduces to a frequency of about 12000 *cps* or less.

Besides frequency range, there are intensity range in the detection of sound. An intense sound produces a painful sensation in the ear. Above certain level the ear fails to recognize it as a sound. For a particular frequency within the range there is a minimum intensity for audibility. This is called *threshold of audibility*.

The range of response of average human ear with respect to intensity and frequency can be understood with the help of an audiogram drawn by Wegel (Fig. 8.2). The lower curve represents the lower limit of amplitude at which audibility of sound can be detected. The upper curve represents more or less arbitrary upper limit to the range of audibility sensation. It is regarded as the threshold of feeling at which the loudness is so great that the effect is somewhat painful and any further increase in amplitude cannot be perceived as increase of loudness.

The area enclosed by the two curves, when extrapolated, represents the auditory sensation area. Outside this area sound is not perceived by the ear whatever be the intensity.

The human ear is sensitive for frequency in the range of 500 to 7000 Hz, which represents the range of ordinary speech. The peak sensitiveness of ear ranges from 2000 to 2500 Hz.



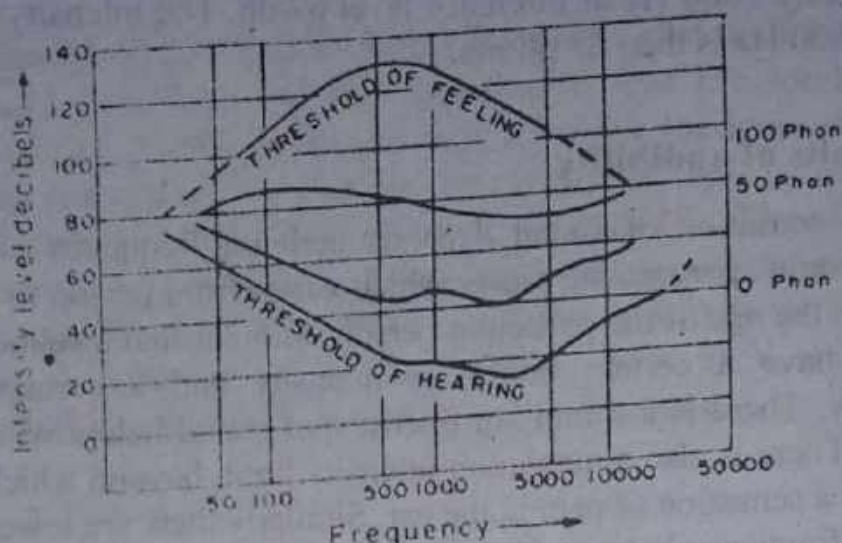


Fig. 8.2

### 8.6 Acoustic measurements

A logarithmic scale is used for measuring acoustic intensity, acoustic power and acoustic pressure. The logarithmic scale is called a *decibel scale*. The quantity to be measured in a decibel scale is always referred logarithmically to some standard reference. Since it is a ratio of two similar quantities, the decibel is a dimensionless quantity.

The decibel value is equal to 10 times the logarithm of the ratio of the quantities measured, to the base 10.

$$1 \text{ bel} = 10 \text{ decibels} = 10 \text{ db}$$

#### Acoustic intensity level

Acoustic intensity level is written as

$$IL = 10 \log \left( \frac{I}{I_0} \right) \text{ db} \quad (8.6)$$

with reference to  $I_0 \text{ watt m}^{-2}$

The standard acoustic intensity reference

$$I_0 = 10^{-12} \text{ watt m}^{-2}$$

$$\therefore IL = 10 \log \left( \frac{I}{10^{-12}} \right) \text{ db}$$

with reference to  $10^{-12}$  watt  $m^{-2}$

$$= 10 [\log I + \log 10^{12}] \text{ db}$$

$$= [10 \log I + 120] \text{ db} \quad (8.7)$$

### Acoustic pressure level

Acoustic pressure level is written as

$$PL = 10 \log_{10} \left( \frac{P}{P_0} \right)^2 \text{ db} \quad (8.8)$$

with reference to a pressure of  $P_0$  newton  $m^{-2}$ .

$$\therefore PL = 20 \log_{10} \left( \frac{P}{P_0} \right) \text{ db}$$

The standard acoustic pressure reference

$$P_0 = 2 \times 10^{-5} \text{ newton } m^{-2}$$

$$\therefore PL = 20 \log \left( \frac{P}{2 \times 10^{-5}} \right) \text{ db}$$

with reference to a pressure of  $2 \times 10^{-5}$  newton  $m^{-2}$ .

$$\begin{aligned} \therefore PL &= 20 [\log P + \log 5 \times 10^4] \text{ db} \\ &= [20 \log P + 20 \log 5 \times 10^4] \text{ db} \\ &= [20 \log P + 20 \times 4.6990] \text{ db} \\ &= [20 \log P + 94] \text{ db (app.)} \end{aligned} \quad (8.9)$$

**Example 8.1.** Calculate the change in intensity level when the intensity of sound increases by  $10^5$  times its original intensity.

**Soln.**

Increase in intensity level.

$$L = 10 \log_{10} \frac{I}{I_0}$$

Here, initial intensity =  $I_0$

final intensity =  $I = 10^5 I_0$ .

$$\begin{aligned}\therefore L &= 10 \log_{10} \frac{10^5 I_0}{I_0} \\ &= 10 \log_{10} 10^5 \\ &= 5 \times 10 \log_{10} (10) \\ &= 50 \text{ decibels.}\end{aligned}$$

**Example 8.2.** Compare the acoustic intensities in air and in water for the same acoustic pressure. Given that

velocity of sound in air = 350 m/se.

velocity of sound in water = 1470 m/sec.

density of air = 1.29 kg/m<sup>3</sup>

density of water = 1000 kg/m<sup>3</sup>

**Soln.**

Acoustic intensity,

$$I = \frac{P_{\text{rms}}^2}{\rho C}$$

$$\text{Hence } \frac{I_{\text{air}}}{I_{\text{water}}} = \frac{(P_{\text{rms}}^2 / \rho C)_{\text{air}}}{(P_{\text{rms}}^2 / \rho C)_{\text{water}}}$$

$$\text{But } (P_{\text{rms}})_{\text{air}} = (P_{\text{rms}})_{\text{water}}$$

$$\begin{aligned}\therefore \frac{I_{\text{air}}}{I_{\text{water}}} &= \frac{(\rho C)_{\text{water}}}{(\rho C)_{\text{air}}} \\ &= \frac{1000 \times 1470}{1.29 \times 350} \\ &= 3256.\end{aligned}$$

**Example 8.3.** A plane acoustic wave in air has an intensity of 21 watts/m<sup>2</sup>. The wave strikes a wall at right angles to its surface. If the area of the wall is 50 m<sup>2</sup>, calculate the force exerted on the wall.



Velocity of the wave in air = 350 m/s.

**Soln.**

Acoustic intensity is the power per unit area.

$$\therefore I = \frac{\text{Power}}{\text{Area}} = \left( \frac{\text{Force}}{\text{Area}} \right) \times \text{Velocity}$$

$$\text{or, } I = Pc \text{ watts/m}^2$$

$$\therefore P = \frac{I}{c}$$

$$\text{Here } I = 21 \text{ watts/m}^2$$

$$\text{and } c = 350 \text{ m/s.}$$

$$\therefore P = \frac{21}{350} = 0.06 \text{ newton/m}^2.$$

$$\text{Area} = 50 \text{ m}^2.$$

$$\text{Hence force} = \text{Pressure} \times \text{area}$$

$$= 0.06 \times 50$$

$$= 3 \text{ newton.}$$

**Example 8.4.** Calculate the increase in the acoustic intensity level when the sound intensity is doubled.

**Soln.**

Intensity level,

$$IL = 10 \log \left( \frac{I}{I_0} \right) \text{ db}$$

In the first case,

$$IL_1 = 10 \log \left( \frac{I_1}{I_0} \right) \text{ db}$$

In the second case,

$$IL_2 = 10 \log \left( \frac{I_2}{I_0} \right) \text{ db}$$

Increase in the acoustic intensity level,

$$\begin{aligned} IL_2 - IL_1 &= 10 \left[ \log \frac{I_2}{I_0} - \log \left( \frac{I_1}{I_0} \right) \right] \\ &= 10 \log \frac{I_2}{I_1} \end{aligned}$$

But  $\frac{I_2}{I_1} = 2,$

$$\begin{aligned} \therefore IL_2 - IL_1 &= 10 \log_{10} 2 \\ &= 10 \times 0.3010 \\ &= 3.01 \text{ db.} \end{aligned}$$

**Example 8.5.** Calculate (i) the acoustic intensity, (ii) acoustic pressure of a plane acoustic wave in air of intensity level of 100 decibels with reference to  $10^{-12}$  watt/m<sup>2</sup>.

**Soln.**

(i)  $IL = 10 \log \left( \frac{I}{I_0} \right) \text{ db.}$

Here,  $I_0 = 10^{-12}$  watt/m<sup>2</sup>.

$$\begin{aligned} \therefore IL &= 10 \log \left( \frac{I}{10^{-12}} \right) \text{ db with reference to } 10^{-12} \text{ watt/m}^2. \\ &= 10 \left[ \log I + \log_{10} 10^{12} \right] \text{ db} \\ &= 10 \left[ \log I + 12 \right] \text{ db} \\ &= \left[ 10 \log I + 120 \right] \text{ db} \\ &= \left[ 10 \log I + 120 \right] \text{ db} \end{aligned} \tag{i}$$

But  $IL = 100 \text{ db}$

(ii)

$\therefore$  from (i) and (ii)

$$100 = 10 \log I + 120$$

or,  $10 \log I = -20$

$$\log I = -2$$

$$\therefore I = 10^{-2} \text{ watt/m}^2.$$

(ii) Acoustic pressure

$$\text{Acoustic intensity, } I = \frac{p^2}{\rho C}$$

$$\text{or, } p = \sqrt{I \rho C}$$

$$\text{Here } I = 10^{-2} \text{ watt/m}^2$$

$$\text{and density of air, } \rho = 1.29 \text{ kg/m}^3$$

Velocity of sound in air,

$$C = 350 \text{ m/s.}$$

$$\therefore p = \sqrt{10^{-2} \times 1.29 \times 350}$$

$$= 2.124 \text{ newton/m}^2.$$

**Example 8.6.** Two sources of sound A and B emit sound waves of different frequencies. The pressure levels due to these sources as recorded at a place C are 80 db and 75 db respectively. Calculate the resultant sound pressure level at C due to the combined effect.

**Soln.**

In the first case,

$$PL_1 = 20 \log \left( \frac{P_1}{P_0} \right) = 80 \text{ db}$$

$$\text{or, } \log \left( \frac{P_1}{P_0} \right) = 4$$

$$\text{or, } P_1 = P_0 (\text{antilog } 4)$$

$$= 10^4 P_0 \text{ newton/m}^2.$$

In the second case,

$$PL_2 = 20 \log \frac{P_2}{P_0} = 75 \text{ db}$$

$$\text{or, } \log \frac{P_2}{P_0} = 3.75$$



$$\begin{aligned}\text{or, } P_2 &= P_0 (\text{antilog } 3.75) \\ &= 5.6 \times 10^3 P_0 \text{ newton/m}^2.\end{aligned}$$

The total sound pressure at C due to the combined effect,

$$\begin{aligned}P &= P_1 + P_2 \\ &= (10^4 + 5.6 \times 10^3) P_0 \\ &= 15.6 \times 10^3 P_0 \text{ newton/m}^2\end{aligned}$$

$$\text{or, } \frac{P}{P_0} = 15.6 \times 10^3$$

The combined sound pressure level,

$$\begin{aligned}\text{PL} &= 20 \log \left( \frac{P}{P_0} \right) \text{ db} \\ &= 20 \log (15.6 \times 10^3) \text{ db} \\ &= 20 (4.195) \text{ db} \\ &= 83.90 \text{ db}.\end{aligned}$$

Therefore, the total sound pressure level at C is **83.90 decibels**.

## 8.7 Architectural Acoustics

Acoustics is the branch of physics that deals with the process of generation, reception and propagation of sound. This branch of physics is closely related to various branches of engineering and in fact covers many fields. Some of the important fields covered are (i) *architectural acoustics* dealing with the design and construction of buildings, recording studios in radio and television broadcasting stations, music halls, operas, etc. (ii) *electro-acoustics* relating to the production and recording of sound (microphones, amplifiers, loudspeakers, etc). (iii) *design of acoustical instruments* and (iv) *musical acoustics* dealing with the design of musical instruments.

### Reverberation

Prior to 1900, architects or building engineers paid very little or no consideration at all to the acoustic properties of rooms or halls. Very often, buildings designed and built as theatre or concert halls were found to be unsatisfactory for the purpose for which they were

built. No planned effort was made to ensure the best reproduction of speech and music. The Fogg Art Museum Hall of Harvard University provides the best example. When it was built sometimes around 1895, it was found to be so defective acoustically that the speaker could hardly make his words intelligible to his audience. Wallace C. Sabine, Professor of Physics, Harvard University was called in to suggest remedial measures. It was from then on that the problem of acoustics of building began to be studied and investigated on a truly scientific and systematic basis by many other scientists in addition to Sabine.

Sabine felt that by far the most important factor contributing to the acoustic badness of a building is excessive reverberation. By *reverberation* is meant the prolonged reflection of sound from the walls, floor and ceiling of a room. When sound is generated in a room, the waves travel towards the walls, etc. and are reflected again with a small loss in energy. A wave may suffer two to three hundred reflections before it becomes inaudible. Thus a listener receives (i) *direct* wave and (ii) several sets of *reflected* waves although of progressively diminishing intensity. The quality of the sound received by the listener is the combined effect of direct and reflected waves. There is a time gap between the direct wave received by the listener and the waves received by successive reflection. Thus so far as the listener is concerned the sound continues to persist for sometime even if it is cut off at the source. *This persistence of audible sound even when the sound source has stopped is called reverberation.* Clearly, if the reverberation of a syllable persists so as to continue its sound while several successive syllables are being pronounced, the distinctness will suffer and the acoustic condition will be bad.

It is quite obvious that if we have a source of sound in an enclosure with perfectly reflecting walls, there will be practically no dissipation of energy and hence the energy density (or maximum intensity) of sound in the room will go on increasing indefinitely so long as the source continues to emit sound. If, on the other hand, the walls are 100 per cent completely absorbing, no sound will be reflected and the energy density will diminish from the source outwards; the intensity will vary inversely as the square of the distance from the source and the situation inside the enclosure will



be same as if the boundary walls have been removed. In reality we have to deal with a situation in between the two extreme cases – an enclosure whose walls reflect fairly copiously but showing at the same time a definite amount of absorption which limits the maximum to which the intensity of sound in the enclosure can rise when the source is continuously sounded.

### **Growth and decay of sound intensity inside a room – Sabine's reverberation formula**

In order to derive an analytical expression for the growth and decay of sound intensity inside a room, let us make the following assumptions :

(i) *The distribution of sound energy within the room is sufficiently uniform in all parts and is equally transmitted in all directions.*

(ii) *The rate at which energy is emitted by the source is constant and is independent of the energy level in the enclosure.*

(iii) *The dissipation of energy in the air in the enclosure is negligible. Energy dissipation is confined to the bounding surfaces of the enclosure.*

(iv) *The absorption coefficient of the surfaces is independent of intensity.*

(v) *Superposition effects may be neglected.*

The various steps involved in the derivation are as follows :

(i) *In the first step, the rate at which energy is incident upon the walls and hence the rate at which it is being absorbed is calculated in terms of the average energy density  $E$ .*

(ii) *In the second step the final steady value of  $E$  is calculated in terms of the rate of emission or power  $P$  of the sound emitting source.*

(iii) *The third and the final step is to formulate and solve a differential equation based on the principle that rate of change of  $E$  is equal to the excess of  $P$  over the rate of absorption of energy.*

Let  $E$  be the average energy density i.e., the average energy per unit volume inside the enclosure. Since the energy is homogeneously distributed inside the enclosure, then in any element



of volume  $dV$ , the amount of energy is  $E \cdot dV$  (Fig. 8.3). Out of this amount of energy, the fraction moving in a direction which will ultimately pass through the element of area  $dS$  is  $\left(\frac{d\omega}{4\pi}\right) \cdot E \cdot dV$ , where  $d\omega$  is the solid angle subtended by  $dS$  at the element of volume  $dV$ ;  $4\pi$  being the total solid angle that can be subtended at a point.

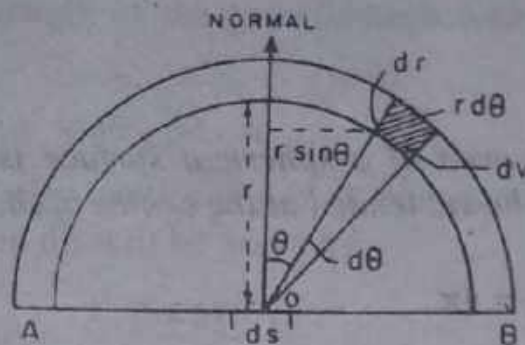


Fig. 8.3

Now the solid angle  $d\omega$  is defined as

$$d\omega = \frac{dS \cos \theta}{r^2}$$

where  $\theta$  is the angle between the normal to  $dS$  and the direction from  $dS$  to  $dV$  and  $r$  is the distance from the element of volume to the element of area (Fig. 8.4).

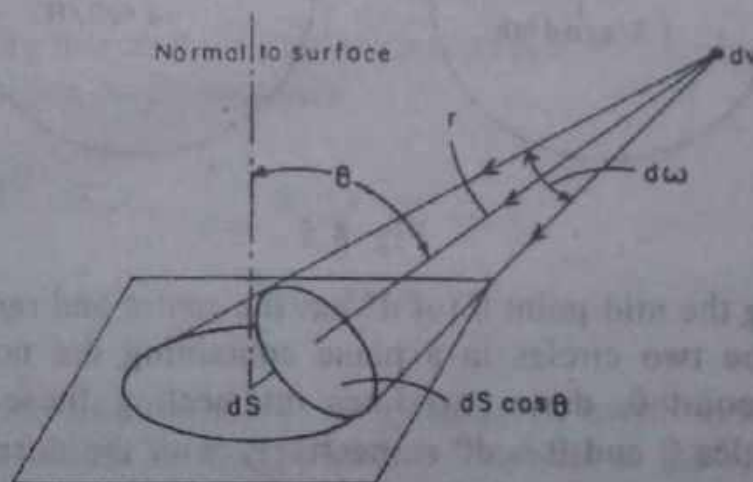


Fig. 8.4

[**Note :** The solid angle subtended at the centre of a sphere by an area of arbitrary shape on the surface of the sphere is given by the ratio of the area of the surface subtending the solid angle to the square of the radius of the sphere.

In general, if  $dS$  is the area of an element of surface which subtends a solid angle at a point and  $R$  is the perpendicular distance of the surface from the point, then the solid angle subtended.

$$dw = \frac{dS}{R^2}$$

Since the total area of a spherical surface is  $4\pi R^2$ , the total solid angle that can be subtended at the centre of the sphere is

$$w = \frac{4\pi R^2}{R^2} = 4\pi$$

The unit of solid angle is steradian. One steradian is the solid angle subtended at a point by a surface whose area is equal to the square of the normal to the surface from the point.

Or,  $dw = 1$  steradian when  $dS = R^2$  (Fig. 8.5)].

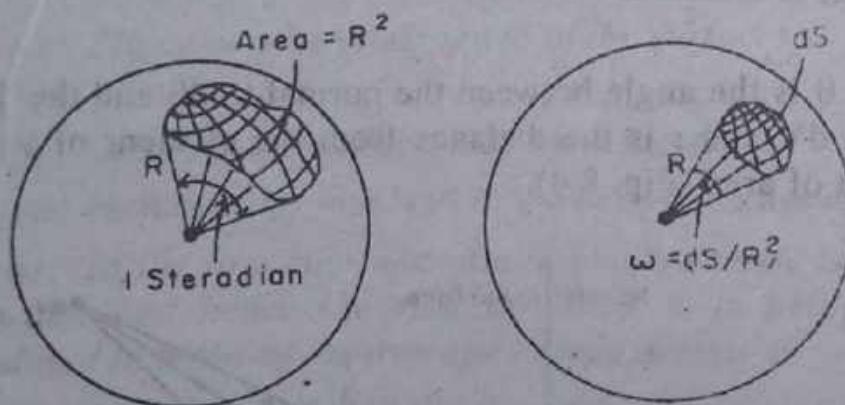


Fig. 8.5

Taking the mid-point  $O$  (of  $dS$ ) as the centre and radii  $r$  and  $(r + dr)$ , describe two circles in a plane containing the normal to  $dS$ . From the point  $O$ , draw two lines intersecting these circles and making angles  $\theta$  and  $\theta + d\theta$  respectively with the normal to  $dS$ . If the figure is now rotated about the normal the small area lying between the two circular arcs and the two lines will sweep out an element of volume which we may take as  $dV$ .

We then have,

$$dV = \text{Area} \times \text{length of the path}$$

$$= rd\theta \, dr \times 2\pi r \sin\theta$$

where  $dr$  = radial length of the area

$rd\theta$  = circular or arc length of the area and  $2\pi r \sin\theta$  = circumferential length of the path through which the area has been rotated.

$$\text{or, } dV = 2\pi r^2 \sin\theta \cdot d\theta \cdot dr$$

Thus the energy contained within this element of volume which ultimately falls on  $dS$  will be given by

$$\begin{aligned} \frac{dw}{d\pi} \cdot E \cdot dV &= \frac{E \cdot dS \cdot \cos\theta \cdot dV}{4\pi r^2} \\ &= \frac{E \cdot dS \cdot \cos\theta \cdot 2\pi r^2 \cdot \sin\theta \cdot d\theta \cdot dr}{4\pi r^2} \\ &= \frac{E \cdot dS \cdot \cos\theta \cdot \sin\theta \cdot d\theta \cdot dr}{2} \\ &= \frac{E \cdot dS \cdot 2 \cos\theta \cdot \sin\theta \cdot d\theta \cdot dr}{4} \\ &= \frac{E \cdot dS \cdot \cos 2\theta \cdot d\theta \cdot dr}{4} \end{aligned}$$

Extending this to the whole shell between the two hemispheres, the energy falling on  $dS$  becomes

$$\begin{aligned} \frac{1}{4} E \cdot dS \cdot dr \cdot \int_0^{\pi} \sin 2\theta \cdot d\theta \\ = \frac{1}{4} E \cdot dS \cdot dr \end{aligned}$$

In order to find the total energy received by  $dS$  in one second, this expression must be integrated for the whole of the volume lying within a distance  $c$  of  $dS$ ,  $c$  being the velocity of sound. Thus, integrating with respect to  $r$  whose value varies between  $r = 0$  to  $r = c$ , we have energy striking  $dS$  per second



$$= \frac{E \cdot dS}{4} \int_0^c dr = \frac{1}{4} E \cdot c \cdot dS$$

let  $a$  be the coefficient of absorption *i.e.*, the fraction of the energy absorption, of the wall AB of which  $ds$  is a part. Then the energy absorbed by the element of surface per second is

$$\frac{1}{4} E \cdot c \cdot a \cdot dS$$

Therefore, energy absorbed in whole enclosure per second (total rate of energy absorption)

$$= \frac{1}{4} E \cdot c \cdot \sum a \cdot dS = \frac{1}{4} E \cdot c \cdot A$$

where  $A = \sum a \cdot dS$  is the total absorption of all the surfaces.

Let  $P$  be the power, *i.e.*, rate of emission of energy of the source,  $V$  the volume of the enclosure and  $EV$  the total energy in the enclosure at a given instant.

The rate of growth of energy is then given by

$$\frac{d}{dt} (EV) = V \frac{dE}{dt}$$

The rate of absorption of energy at the same time as found above

$$= \frac{1}{4} E \cdot c \cdot A$$

rate of growth  
of energy inside  
the room

+

rate of absorption  
of energy by  
the walls

=

rate of supply  
of energy by  
the source

or,  $V \frac{dE}{dt}$

+

$$\frac{1}{4} E \cdot c \cdot A$$

=

$$P$$

or,  $V \frac{dE}{dt} = P - \frac{1}{4} E \cdot c \cdot A$

putting  $\frac{cA}{4V} = b$ , we have

$$\frac{V dE}{dt} = P - b.V.E$$

$$\text{or, } \frac{b.V.dE}{b.dt} = P - b.V.E$$

$$\text{or, } \frac{bV dE}{P - bVE} = b . dt$$

Integrating both sides, we have

$$\log (P - bVE) = -bt + K \quad (8.10)$$

where  $K$  is a constant of integration.

If we measure  $t$  from the instant when the source starts, then for

$$t = 0, E = 0,$$

$$\text{or, } K = \log P$$

$$\therefore \log (P - bVE) = -bt + \log P$$

$$\text{or, } \log \left(1 - bV \frac{E}{P}\right) = -bt$$

$$\text{or, } 1 - bV \frac{E}{P} = e^{-bt}$$

$$\text{or, } E = \frac{P}{bV} (1 - e^{-bt})$$

$$= \frac{4P}{cA} (1 - e^{-Act/4V}) \quad (8.11)$$

For the steady state we put  $t = \infty$  and the maximum energy density  $E_m$  is given by

$$E_m = \frac{P}{bV} = \frac{4P}{cA}$$

Hence eqn. (8.11) becomes

$$E = E_m (1 - e^{-Act/4V}) \quad (8.12)$$

Eqn. (8.11) above gives the expression for the growth of  $E$  with time in a room of volume  $V$  and total absorption  $A$ . (Fig. 8.6) is a graphical representation of eqn. (8.11) which is a rising exponential function.

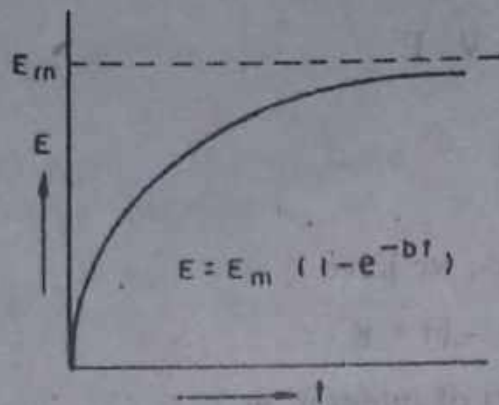


Fig. 8.6

### Decay

For decay of intensity we have as before the relation

$$\log (P - bVE) = -bt + K$$

If the source is cut off when  $E$  has reached the maximum value of  $E_m$ , then in this case  $E = E_m$  and  $P = 0$  when  $t = 0$ .

$$\therefore \log (-bVE_m) = K$$

Hence eqn. (8.10) becomes

$$\frac{E}{E_m} = e^{-bt}$$

$$\text{or, } E = E_m e^{-bt}$$

$$= \frac{4P}{cA} e^{-Act/4V} \quad (8.13)$$

Eqn. (8.13) represents the decay of  $E$  after the source has been cut off. It is a decaying exponential function as shown in (Fig. 8.7).



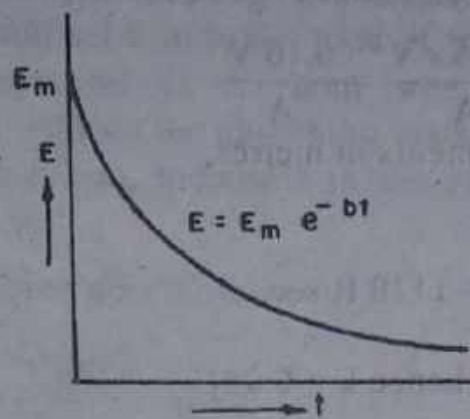


Fig. 8.7

### Reverberation time

The persistence of audible sound after the source has ceased to operate is referred to as reverberation and the *time of reverberation* is defined as the time taken for the energy density to fall to the minimum audible value from an initial value  $10^6$  times as great i.e., a range of 60 decibels. This particular definition was adopted by W.C. Sabine partly owing to the fact that the standard source of sound – an organ pipe of frequency 512 vibrations per second, which he used in his experiments gave an initial energy level in the rooms in which he tried it of about  $10^6$  times the minimum audible.

Now putting  $E = E_m / 10^6$  and  $t = T$ , we get

$$E_m \times 10^{-6} = E_m e^{-bT}$$

$$\text{or, } e^{-bT} = 10^{-6}$$

$$\text{or, } e^{bT} = 10^6$$

Taking logarithm,

$$bT = 6 \log_e 10$$

Substituting the value of  $b = \frac{cA}{4V}$ , we get

$$\frac{cA}{4V} \cdot T = 6 \log_e 10$$

$$\text{or, } T = \frac{24 \log_e 10 \cdot V}{cA} = \frac{kV}{A} \quad (8.14)$$

Taking the velocity of sound  $c = 340$  m/sec, we have,

$$T = \frac{24 \times 2.3026 \times V}{340 \times A} = \frac{0.16 V}{A}$$

Hence, for measurements in metres,

$$k = 0.16$$

[In F.P.S. units,  $v = 1120$  ft/sec,

$$T = \frac{0.05 V}{A} \quad ; \text{ hence } k = 0.05]$$

The expression for the reverberation time was first experimentally determined by W.C. Sabine and the close agreement between his experimental results and theoretically obtained values according to the relation  $T = kV/A$  seem to justify the assumptions made.

### Eyring equation

Sabine's formula is valid for large enclosures. Experiments show that the formula is valid for absorption coefficient,  $a \leq 0.2$  only.

However C.F. Eyring, basing on the methods of images, derived another equation for reverberation time. This is given by

$$T = \frac{0.16 V}{S [-\log (1 - a)]}$$

In the limiting case  $[-\log (1 - a)]$  approaches  $a$ , and the formula becomes

$$T = \frac{0.16 V}{aS}$$

where  $a$  is the mean absorption coefficient and  $S$  is the total surface area. This, in fact, is the Sabine's formula. Thus Sabine's formula can be regarded as a special case of the more general Eyring formula.

### 8.8 Determination of absorption coefficient

The absorption coefficient or coefficient of absorption ( $a$ ) of a surface is defined as the reciprocal of its area which absorbs the same sound energy as absorbed by a unit area of an open window

and can be determined by measuring reverberation times. By using a standard source (Sabine used an organ pipe of frequency 512 hertz), the time of reverberation of a room was measured with a chronograph, (i) first without the absorbing material in the room and (ii) then with the absorbing materials in the room. Let these two times be  $T_1$  and  $T_2$ .

According to Sabine's formula,

$$\frac{1}{T_1} = \frac{A}{kV} = \frac{\sum aS}{kV}$$

and

$$\frac{1}{T_2} = \frac{\sum aS + \sum a_1 S_1}{kV}$$

where  $a_1$  is the absorption coefficient of the material of area  $S_1$ . Subtracting one from the other, we get

$$\frac{a_1 S_1}{kV} = \left( \frac{1}{T_2} - \frac{1}{T_1} \right)$$

Knowing the values of  $k$ ,  $V$  and  $S_1$ , the value of  $a_1$  can be calculated.

The *absorption coefficient of a given material is different at different frequencies*. It is generally higher at higher frequencies. The absorption coefficient of some common materials calculated with source of frequency 512 hertz is given in the table below.

**Table**

Material	Absorption coefficient
Acoustic Felt	0.45
Acoustic Plaster	0.30
Asbestos	0.26
Carpet	0.30
Common Plaster	0.03
Concrete	0.17
Cork	0.23
Fibre Board	0.50



Fibre Glass	0.75
Glass	0.027
Hair or Felt	0.58
Hard Plaster	0.01 – 0.02
Heavy Curtains	0.51
Lime Plaster	0.03 – 0.04
Marble	0.01
Perforated Cellulose Fibre Tiles	0.85
Unpainted brick	0.03
Wood Blocks	0.06

### 8.9 Requisites for good acoustics

Reverberation is one of the important factor that affects the acoustics of a room or a hall. The reverberation time must be correct for a building to be acoustically correct. It should be neither too long nor too short. If the reverberation time is too long it causes confusion and renders speech unintelligible and music dissonant. If the reverberation time of a room is, say, 5 seconds, then even if a most deliberate and slow speaker would have uttered about a dozen syllables, due to long time of reverberation, all these would be sounding together thus making the utterance hopelessly confused. Increasing the loudness of sound will make matters worse instead of improving them. The reverberation time can be reduced by the following methods:

(i) The walls and ceiling of the room should be covered with materials of high absorption coefficient *i.e.*, with perforated card boards, felt, asbestos, fiber glass, etc.

(ii) The walls should be engraved and made rough with decorative material to increase the absorption.

(iii) Heavy curtains should be hanged.

(iv) The room must have few open windows.

(v) In case of a auditorium, there should be a good audience. Each person is equivalent to about 0.50 sq. metre area of an open window.

(vi) While planning a concert hall or theatre, upholstered seats should be provided so that the absorption is approximately the same with or without the audience.

(vii) The curved walls and corners bounded by two walls should be avoided. This is done to avoid (a) concentration of sound and (b) dead spaces.

### Reverberation time – optimum value

However, the room should not be over-corrected for reverberation. If the reverberation time is too short, most of the sound energy is absorbed and the room appears acoustically *dead*. The sensation of speaking in such a room is similar to that experienced when speaking from the top of an isolated tower or building – there will be a *deadening* effect on the voice of the speaker or the singer.

What then is the optimum value of time of reverberation? The best value for reverberation time depends on the use for which the building is designed. Within broad limits a time of 0.5 second is found acceptable by most listeners while for music the value is between 1 and 2 seconds. For theatres or auditorium, the optimum value of time of reverberation varies with the volume; it varies from 1.1 to 1.5 seconds for small ones to upto 2.3 seconds for larger ones.

**Example 8.7.** The volume of a room is  $600 \text{ m}^3$ . The wall area of the room is  $220 \text{ m}^2$ , the floor area is  $120 \text{ m}^2$  and the ceiling area is  $120 \text{ m}^2$ . The average sound absorption coefficient, (i) for the walls is 0.03; (ii) for the ceiling is 0.80, and (iii) for the floor is 0.06. Calculate the average sound absorption coefficient and the reverberation time.

**Soln.**

The average sound absorption coefficient,

$$a = \frac{\sum aS}{\sum S}$$

$$= \frac{a_1 S_1 + a_2 S_2 + a_3 S_3}{S_1 + S_2 + S_3}$$



$$\text{Here } S_1 = 220 \text{ m}^2 \quad a_1 = 0.03$$

$$S_2 = 120 \text{ m}^2 \quad a_2 = 0.80$$

$$S_3 = 120 \text{ m}^2 \quad a_3 = 0.06$$

$$\begin{aligned} \therefore a &= \frac{0.03 \times 220 + 0.8 \times 120 + 0.06 \times 120}{220 + 120 + 120} \\ &= \frac{109.8}{460} = 0.2389 \end{aligned}$$

The total sound absorption in the room

$$= a \sum S = 0.2389 \times 460$$

$\therefore$  Reverberation time,

$$t = \frac{0.158V}{a \sum S} = \frac{0.158 \times 600}{109.8}$$

$$= 0.8634 \text{ sec.}$$

**Example 8.8.** Calculate the reverberation time of a  $2400 \text{ m}^3$  hall having a seating capacity of 600 (a) when empty and (b) when full of audience from the following data:

Surface	Area or number	Coeff. of absorption
Plaster ceiling	$500 \text{ m}^2$	0.02
Plastered walls	$600 \text{ m}^2$	0.03
Wood floor	$500 \text{ m}^2$	0.06
Wood doors	$20 \text{ m}^2$	0.06
Seats, cushion	400	0.1 sabine/chair
Seats, cane	200	0.01 sabine/chair

Take the absorption of each member of the audience as 0.45 sabine and neglect the effect of absorption by the performers and their instruments.

**Soln.**

(a) The total absorption when the hall is empty



$$\begin{aligned}
 aS &= (500 \times 0.02) + (600 \times 0.03) + (500 \times 0.06) + (20 \times 0.06) \\
 &\quad + (400 \times 0.1) + (200 \times 0.01) \\
 &= 101.2 \text{ sabines}
 \end{aligned}$$

$$\text{Reverberation time, } T = \frac{0.16V}{aS}$$

$$= \frac{0.16 \times 2400}{101.2} = 3.79 \text{ seconds.}$$

(b) Increase in absorption when the hall is full,

$$\begin{aligned}
 &= 400 (0.45 - 0.1) + 200 (0.45 - 0.01) \\
 &= 228 \text{ sabines}
 \end{aligned}$$

$$\text{Total absorption } aS = 101.2 + 228$$

$$= 329.2 \text{ sabines}$$

$\therefore$  Reverberation time,

$$\begin{aligned}
 T &= \frac{0.16 V}{aS} \\
 &= \frac{0.16 \times 2400}{329.2} = 1.12 \text{ seconds.}
 \end{aligned}$$

**Example 8.9.** A room has dimensions  $6 \times 4 \times 5$  metres. Calculate (i) the mean free path of the sound wave in the room, (ii) the number of reflections made per second by the sound wave with the walls of the room. Velocity of sound in air = 350 m/s.

**Soln.**

(i) The mean free path of the sound waves is defined as the average distance traveled by a sound wave through air between any two consecutive encounters with the walls of the room.

Hence,

$$\text{Mean free path, } L = \frac{4 (\text{volume of the room})}{\text{Total surface area}}$$

$$\text{Here, volume of the room} = 6 \times 4 \times 5 = 120 \text{ m}^3$$

$$\begin{aligned}\text{Total surface area} &= 2 [ 6 \times 4 + 4 \times 5 + 6 \times 5 ] \\ &= 148 \text{ m}^2\end{aligned}$$

$$\begin{aligned}\therefore L &= \frac{4 \times 120}{148} \\ &= 3.243 \text{ m}\end{aligned}$$

(ii) The number of reflections made per second,

$$\begin{aligned}N &= \frac{\text{Velocity of sound}}{\text{Mean free path}} \\ &= \frac{350}{3.243} \\ &= 107.9\end{aligned}$$

**Example 8.10.** Find the reverberation time of a room 10 m wide by 20 m long by 3 m high. The ceiling is acoustic, the walls are plaster, the floor is concrete, and there are 36 people in the room. [sound absorption coefficient are : acoustic ceiling – 0.60, plaster – 0.03, concrete – 0.02. Take the absorbing power per person to be 0.5]

**Soln.**

$$T = \frac{0.16 V}{A}$$

$$\text{Here, } V = (3) (20) (10) = 600 \text{ m}^3$$

$$\begin{aligned}A &= a_1 S_1 + a_2 S_2 + a_3 S_3 + a_4 S_4 \\ &= 200 (0.6) + 200 (0.02) + (30 + 30 + 60 + 60) (0.03) + 36 (0.5) (1) \\ &= 120 + 4 + 5.4 + 18 = 147.4,\end{aligned}$$

$$\begin{aligned}\text{Hence, } T &= \frac{(0.16) (600)}{147.4} \\ &= 0.65 \text{ s.}\end{aligned}$$


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## EXERCISES

- 1] Define the intensity and energy density at a point in a plane wave of sound. How are they related? Obtain an expression for the intensity of a plane wave in terms of acoustic pressure.
- 2] Explain what is meant by bel and phon. Discuss the factors influencing loudness.
- 3] Give the theory of growth and decay of sound inside a room. Hence obtain an expression for Sabine's reverberation formula.
- 4] What are reverberation and reverberation time? Obtain an expression for the reverberation time.
- 5] Discuss the theory of reverberation. What are the acoustic requirement of a good auditorium? What is the order of magnitude of optimum reverberation?
- 6] How would you measure the absorption coefficient of a material?
- 7] Calculate the change in intensity level when the intensity of sound increases 100 times its original intensity. [20 decibels].
- 8] Calculate the amplitude of the displacement wave, given that for a frequency of 400 Hz, the feeblest sound that could be heard corresponds to a pressure amplitude of  $8 \times 10^{-5}$  newton/m<sup>2</sup>. The density of air is 1.29 kg/m<sup>3</sup> and velocity of sound in air is 345 m/s. [ $7.15 \times 10^{-11}$  m]
- 9] Calculate the increase in sound pressure level when the sound pressure is doubled. [6.02 db]
- 10] Calculate the acoustic intensity level in each case, at a distance of 10 metres from a source which radiates energy at the rate of 3.14 watts. Use reference intensities of (i) 100 watts/m<sup>2</sup> (ii) 1 watt/m<sup>2</sup> and (iii)  $10^{-12}$  watt/m<sup>2</sup>. [(i) 46.21 db (ii) 26.021 db (iii) 93.979 db]
- 11] An air-conditioning unit operates at a sound intensity level of 75 db. If it is operated in a room with an existing sound intensity level of 70 db, what will be the resultant intensity level? [79.025 db]
- 12] What is the amplitude of motion for the air in the path of a 60 -dB, 800 - Hz sound wave? Assume that  $\rho = 1.29$  kg/m<sup>3</sup> and  $v = 330$  m/s. [13.6 nm]
- 13] Calculate the reverberation time in a hall measuring 40 x 100 x 20 ft with the following parameters : (i) 7500 sq.ft. of plaster,  $a_1 = 0.03$ , (ii) 600 sq. ft. of wood and floor, etc,  $a_2 = 0.06$ , (iii) 400 sq. ft. of glass,  $a_3 = 0.025$ , (iv) 600 seats,  $a_4 = 0.3$ , (v) audience of 500 persons,  $a_5 = 4.3$  per person. What will be the reverberation time if the hall is without audience? [1.44 sec, 5.16 sec.]



## CHAPTER IX

## ELASTICITY

*General properties of matter-Elasticity-Load-Stress-Strain-Hooke's law-Elastic limit-Stress-strain diagram of a material-Different types of elasticity-Poisson's ratio-Shear is equivalent to compression and extension-Equivalence of a shearing stress to an equal tensile and an equal compressive stress at right angles to one another-Relations connecting the elastic constants-Limiting values of Poisson's ratio-Work done in a strain-Twisting couple on a cylinder-Determination of Young's modulus by Searle's apparatus-Determination of the rigidity modulus of a wire-Deformation by bending-Bending moment-Depression of a beam fixed at two ends and loaded in the middle-Solved problems-Exercises.*

**9.1 General properties of matter**

Matter can exist in three basic states namely the solid state, the liquid state and the gaseous state. Under right conditions, a given substance can exist in any of the three states. For example water can come in the form of ice and water vapour. The chemical composition of the substance is the same in all three states. In the gaseous state the molecules are far apart and they are not arranged in a regular pattern. Because the molecules are far apart, the density and viscosity of the gaseous state is low making the substance highly compressible in this state. The gaseous substance can not withstand applied forces and hence it has no rigidity.

The molecules in the liquid state are much closer together than in the gaseous state but the structural pattern is disordered and continually changing. Thus the rigidity of a liquid is still very low. However, as the molecules are now more closely packed the viscosity is about 100 times that of the gaseous state and the density is about 1000 times that of the gaseous phase. The high density gives rise to a low compressibility.

In the solid state, the atoms are closely packed and the density is comparable to that of the liquid phase and hence it is also difficult to compress a solid. However, the atoms are more well arranged in this state thus imparting a high rigidity to solids. Thus, solids have a definite shape and volume at a given temperature and the shape cannot be easily changed. Solids can be divided into *crystalline and*

*amorphous* (non-crystalline) solids. The atoms in a crystalline solid are arranged in a regular three dimensional array with different types of packing. In crystalline solids a sharp melting point exists. Amorphous solids on the other hand do not melt at a sharply defined temperature but they melt slowly.

There are different properties of matter associated with its different states. However, these properties do not depend only on the different states of matter-different matters in the same state may possess different properties. Among these properties of matter, mention may be made of density, elasticity, surface tension, viscosity, thermal expansion, specific heat, different electrical properties such as conductivity, semi-conductivity, etc., different magnetic properties such as ferro-magnetism, para magnetism, etc., reflecting properties of light, etc.

The properties associated with matter are so many that it is not possible to discuss them in detail in a single book. Different books discuss specific types of properties. Of these properties, elasticity, surface tension, viscosity and fluid mechanics are discussed in detail in many books under the heading of *general properties of matter*. This practice will also be followed in this book.

## 9.2 Elasticity

A rigid body is defined as a solid body in which the distance between any two particles is small and fixed and remains unaltered whatever the external forces applied to it and in whatever manner they may vary, so that it remains undeformed *i.e.*, the shape, size and volume of the body remain unaffected.

In actual practice, however, there is no such body as a perfectly rigid body. Real material always yield to some extent under the influence of applied forces *i.e.*, they are found to get deformed to a greater or smaller extent under suitably applied external forces. The force tends to displace the particles within the body from their equilibrium positions. However, this is opposed by a restoring force which builds up to a value that balances the external force. When the applied force is released, the restoring force brings the particles back to their equilibrium positions. This property of a material body



of being able to regain its initial state of condition on removal of the external forces to which it may be subjected is called *elasticity*.

### Some definitions connected with elasticity

#### (i) Perfectly elastic body -

If the deformation of a body produced by a given deforming force at a given temperature remains unchanged (*i.e.*, the deformation neither increases nor decreases by the prolonged application of that force) and if the body completely regains its original state on the removal of the force, the body is said to be *perfectly elastic*.

#### (ii) Perfectly plastic body

If, on the other hand, the body remains deformed and shows no tendency to regain its original condition on the removal of the deforming force, the body is said to be *perfectly plastic*.

**Note :** It should, however, be pointed out that there exists no such body which is either perfectly elastic or perfectly plastic. The nearest approach to the former is a *quartz fibre* and to the latter, ordinary *putty*. Steel is a highly elastic material; it returns closely to original dimensions even after being subjected to relatively large forces. Materials like dough, and lead are some other examples of highly plastic materials.

### 9.3 Load

Any combination of external forces acting on a body, *e.g.*, its own weight along with forces connected with it, whose net effect is to deform the body, is referred to as a *load*.

### 9.4 Stress

When a body is in statical equilibrium under the influence of its internal forces, the body is said to be in its natural state. But when the body is subjected to external forces, the particles are displaced from their equilibrium positions giving rise to internal forces of reaction which tends to oppose and balance the opposing forces, until a certain limit, called the *elastic limit*, is reached and the body gets permanently deformed. The body is then said to be under *stress*.



If this opposite force is uniform, *i.e.*, proportional to the area, then the stress on the body is defined as

$$\text{stress} = \frac{\text{force applied}}{\text{area over which the force acts}}$$

Thus if  $F$  be the deforming force applied uniformly over an area  $A$ , then  $\text{stress} = F/A$ .

If the deforming force be inclined to the surface, *i.e.*, if the force  $F$  makes an angle  $\alpha$  with the normal to the surface (Fig. 9.1), then the stress on the surface can be resolved into two components, one parallel to the surface called *tangential (or shearing) stress* and one perpendicular to the surface called *normal stress*. These two stress components are

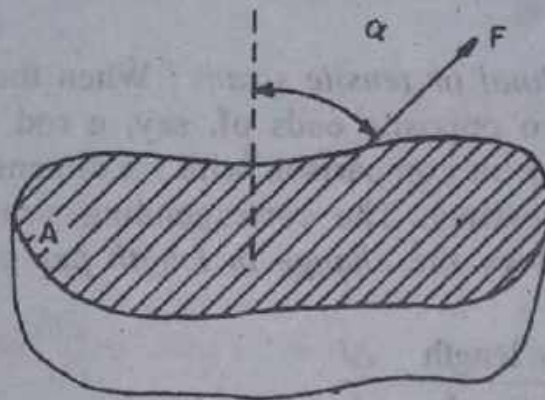


Fig. 9.1

$$\text{tangential or shearing stress,} = \frac{F \cos \alpha}{A}$$

$$\text{and normal stress,} = \frac{F \sin \alpha}{A}$$

Each of these components has different effect on the surface  $A$ . The perpendicular component tends to pull the body apart while the parallel component will result in a *shear* on the body. The tangential stress is, therefore, called the *shear stress* while the normal stress is called the *tensile stress*. If  $F$  were to be in the opposite direction, the perpendicular component will tend to compress the body. The perpendicular component is then called a *compressive stress*.

Tensile (or compressive) stress cause a change in length or volume and shape of the body while a shear stress results in a change in shape but not the volume of the body.

Stress, like pressure, is force per unit area and hence its dimensions are the same as those of pressure, viz,  $ML^{-1}T^{-2}$ . It is measured in the same units as those of pressure, viz.,  $\text{dynes/cm}^2$  and  $\text{Newton/m}^2$  in the C.G.S. and M.K.S. system of units respectively.

### 9.5 Strain

When a body is under stress, i.e., under a system of forces or couples in equilibrium, it undergoes a deformation in respect of length or volume or shape. This change in the dimension and/or shape of the body is described by a quantity called the *strain*. Since there are three types of stresses, there are three corresponding types of strains.

(i) *Longitudinal or tensile strain* : When the force is applied to the body at two opposite ends of, say, a rod or a wire whose length is very large in comparison with its dimension, the stress is *tensile (or compressive)*. The corresponding strain or the tensile strain is measured by the *change in length per unit length* and is given by

$$\frac{\text{change in length}}{\text{original length}} = \frac{\Delta l}{l}$$

where  $\Delta l$  is the increase or decrease in length and  $l$  is the original length.

(ii) *Volume strain* : When a body is subjected to uniform pressure acting normally all over the surface of the body, a change in volume without any change in shape is produced in all the three states of matter. The corresponding strain or the *volume strain* is measured by the *change in volume per unit volume* of the body and is given by

$$\frac{\text{change in volume}}{\text{original volume}} = \frac{\Delta V}{V}$$

(iii) *Shearing strain* : A change in shape or deformation of the body may be produced in various ways viz., (a) by a shearing or tensile stress, (b) by flexure or bending (c) by torsion or twisting. Fig 9.2 shows the effect of applying a shearing stress to a body. The



original body ABCD is now ABEG, and the shearing angle is given by the angle  $\theta$ . If  $\theta$  is small (which is usually the case) then

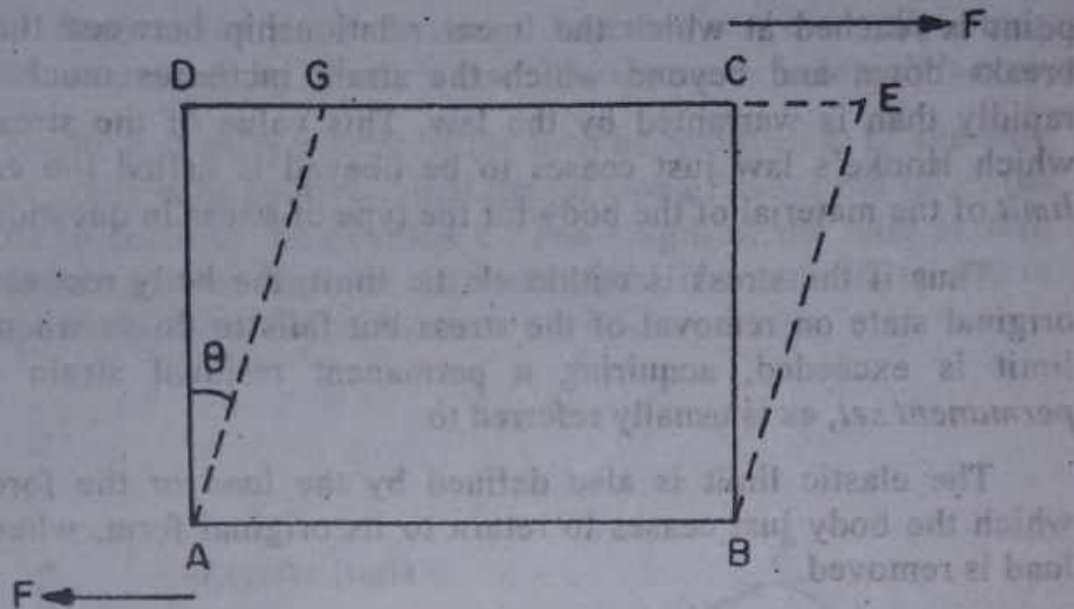


Fig. 9.2

$$\tan \theta = \theta$$

$$\text{But } \tan \theta = \frac{DG}{AD}$$

$\therefore$  shearing strain is given by,

$$\theta = \frac{DG}{AD}$$

### 9.6 Hooke's law

As a result of large number of experiments Hooke, in 1678, announced his famous law of electricity. This is the fundamental law of elasticity and states that, *provided the strain is small (i.e., within a certain limit called the elastic limit), the stress is proportional to strain*; so that in such a case, the ratio of stress to strain is a constant. This constant is called the *modulus of elasticity* (a term first introduced by T. Young) or the *coefficient of elasticity*. Hooke's concise phrase stating the law was : *ut tensio sic vis* i.e., *as the tension, so the strain*.



### 9.7 Elastic limit

In the case of solid, if the stress is gradually increased, the strain too increases with it in accordance with Hooke's law until a point is reached at which the linear relationship between the two breaks down and beyond which the strain increases much more rapidly than is warranted by the law. This value of the stress for which Hooke's law just ceases to be obeyed is called the *elastic limit* of the material of the body for the type of stress in question.

Thus if the stress is within elastic limit, the body recovers its original state on removal of the stress but fails to do so when this limit is exceeded, acquiring a permanent residual strain or a *permanent set*, as is usually referred to.

The elastic limit is also defined by the load or the force at which the body just ceases to return to its original form, when the load is removed.

### 9.8 Stress-strain diagram of a material

If a wire (or a bar) is subjected to gradually increasing stress and a graph is plotted between the stress applied and the corresponding strain produced, then a curve of the form shown in Fig. 9.3 is obtained. This curve, usually referred to as the *stress-strain diagram* of the material, has the following general characteristics:

(i) *Up to the point A*, the wire is perfectly elastic, i.e., Hooke's law governing the linear relationship between stress and strain is fully obeyed, as shown by straight portion OA of the curve. On removal of the stress (i.e., the load) at any point in this portion of the curve, the wire recovers its original condition of zero strain i.e., original length represented by the point O.

*From A to B*, stress and strain are not proportional, but nevertheless, if the load is removed at any point between O and B, the curve will be retraced and the material will return to its original length. In the region OB, the material is said to be *elastic* or to exhibit *elastic behaviour*. Obviously, the tensile stress corresponding to B, therefore, gives the *elastic limit* of the wire. This stress is also referred to as *elastic strength* of the wire.





(iii) *Beyond the point D*, there starts a large but irregular increase in strain with little or no increase in stress and continues upto the point E, where the stress is just a little smaller than that at D. This is represented by the wavy portion ED of the curve. The point D, where the large, erratic increase in strain just starts, is called the *yield point* of the material of the wire. The stress corresponding to the point D is naturally referred to as the *yielding stress*. The points D and E are sometimes referred to as the *upper* and *lower yield points* respectively.

(iv) *Beyond the point E*, the yielding comes to a stop. Stress must now be increased for producing any further increase in strain, which is now purely plastic in nature. *As the strain increases, there is a corresponding decrease in the area of cross-section of the wire, so that its volume remains unaffected.* This continues upto the point F, beyond which the extension goes on even without any addition to the load applied: the wire being in a state of virtual *flow*. The cross-section of the wire decreases faster at some section of it and a local constriction, commonly referred to as a *neck* or a *waist* is developed there. This results in an automatic increase in stress until a point G is reached at which *fracture* takes place (*i.e.*, the wire snaps or breaks). The point G represents the *fracture* or *breaking point* of the wire. If large plastic deformation takes place between the elastic limit and the fracture point, the metal is said to be *ductile*. If, however, fracture occurs soon after the elastic limit is passed, the metal is said to be *brittle*.

The stress corresponding to the point G represents the maximum stress to which the wire can be *subjected* and is *measured by the maximum load or force applied divided by the original cross-section of the wire*. This is fittingly called the *breaking stress* and measures the *tensile or ultimate strength* of the wire.

In actual practice no part of a machinery is, as a safe-guard, subjected to a stress as high as this. The maximum stress to which it is subjected is very much less and always within the elastic limit. This is called the *working stress*. The ratio between the breaking stress and the working stress is called the *factor of safety*. This is also referred to as *factor of ignorance*.



### 9.9 Different types of elasticity

Corresponding to three different types of strain, we have three different types of elasticity, viz.,

(i) linear elasticity or elasticity of length, called *Young's modulus*, corresponding to linear (or tensile strain);

(ii) elasticity of volume or *Bulk modulus*, corresponding to volume strain;

and (iii) elasticity of shape, called *shear modulus* or *modulus of rigidity* corresponding to shear strain.

#### (i) Young's modulus (or elasticity of length)

When the deforming force is applied to the body only along a particular direction, the change per unit length in that direction is called longitudinal, linear or tensile-strain, and the force applied per unit cross-section is called longitudinal or tensile stress. *The ratio of tensile stress to tensile strain*, within the elastic limit, is called *Young's modulus*, and is usually denoted by the letter  $Y$ .

If a force  $F$ , applied normally to a cross-sectional area  $A$ , produces a change in length  $l$  in the original length  $L$ , then

$$\text{Tensile stress} = F/A$$

$$\text{and tensile strain} = l/L$$

$$\therefore \text{Young's modulus, } Y = \frac{\text{tensile stress}}{\text{tensile strain}}$$

$$= \frac{F/A}{l/L} = \frac{F.L}{l.A} \quad (9.1)$$

#### (ii) Bulk modulus (or elasticity of volume)

If a force is applied *normally* and *uniformly* to the whole surface of a body; so that there is a change in volume without any change in shape, then the force applied per unit area (or pressure) gives the *stress* and the change in volume per unit volume the *strain*, their ratio giving the *bulk modulus* for the body. It is usually denoted by the letter  $K$ .

If  $F$  be the force applied normally and uniformly on a surface area  $A$ , then the

*stress = force applied per unit area*

*= pressure,  $P = F/A$ .*

Let  $v$  be the resultant change in volume in an original volume  $V$ . Then the

$$\text{strain} = - \frac{v}{V}$$

minus sign indicates that if pressure increases volume decreases and *vice versa*.

Hence, the

*bulk modulus,*

$$K = \frac{F/A}{-v/V} = - \frac{F.V}{v.A} = - \frac{PV}{v} \quad (9.2)$$

$K$  is also sometime referred to as *incompressibility* of the material of the body.

The reciprocal of bulk modulus, *i.e.*,  $1/K$  is, therefore, called *compressibility*.

As liquids and gases can permanently sustain only hydrostatic pressure, *the only elasticity they possess is bulk modulus*.

### (iii) Modulus of rigidity (torsion modulus or elasticity of shape)

In this case, while there is no change in the volume of the body, there is a change in the shape of the body. This occurs due to the movement of the layers of the body, one over the other, under a tangential force (pack of cards).

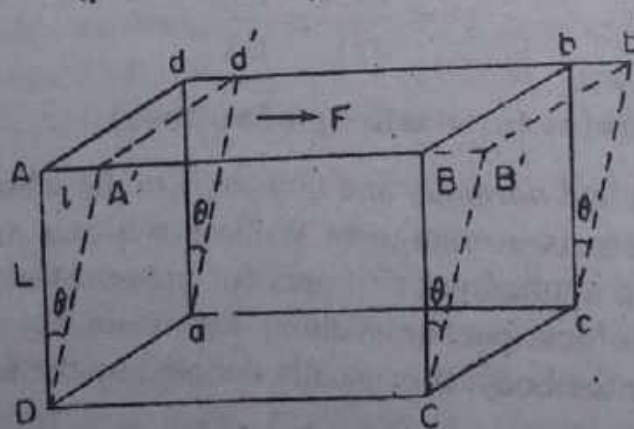


Fig. 9.4



Let the lower face aDCc of the rectangular solid cube be fixed (Fig. 9.4). A tangential force  $F$  be applied to its upper face in the direction shown. The couple so produced makes the layers, parallel to the two faces, move over one another, such that the point  $A$  shifts to  $A'$ ,  $B$  to  $B'$ ,  $d$  to  $d'$  and  $b$  to  $b'$  i.e., the lines joining the two faces turn through an angle  $\theta$ . The face  $ABCD$  is then said to be sheared through an angle  $\theta$ .

This angle  $\theta$  (expressed in radians), through which a line originally perpendicular to the fixed face is turned, gives the *strain*, or the *shear strain*, or the *angle of shear*, as it is often called. The applied force  $F$  divided by the area  $A$  of the face  $ABbc$ , i.e.,  $F/A$  is the *stress* or the *tangential stress*. The *coefficient of rigidity* or the *modulus of rigidity* expressed by  $\eta$ , is then given by

$$\eta = \frac{\text{tangential stress}}{\text{tangential strain}} = \frac{F/A}{\theta}$$

If  $L$  is the length of the side  $AD$  and  $l$  is the displacement  $AA'$ , then  $\tan \theta = l/L$ . Since  $\theta$  is very small,  $\tan \theta = \theta$ . Thus the shear strain  $\theta = l/L$ . Hence

$$\eta = \frac{F/A}{l/L} = \frac{F.L}{l.A} \quad (9.3)$$

As can be seen from relations (9.1) and (9.3), modulus of rigidity is similar to Young's modulus with the only difference that, in case of modulus of rigidity,  $F$  is the tangential stress, not a linear one, and the displacement  $l$  take place at right angles to  $L$ , not along it.

Obviously, if  $A = 1$  and  $\theta = 1$  (radian), we have  $\eta = F$ .

Thus the *modulus of rigidity of a material may be defined as the tangential or shearing stress per unit shear*, i.e., an angle of shear equal to one radian, assuming of course, Hooke's law to be valid for such a large shear. In practice, Hooke's law in the case of metals ceases to be valid for a shear exceeding only  $1/200^{\text{th}}$  of a radian.

### 9.10 Poisson's ratio

It is common knowledge that on being stretched, a wire becomes *longer but thinner* i.e., although its length increases, its diameter or cross-section decreases. This means that a *linear* or



*tangential strain* i.e., a strain in the direction of the applied force is always accompanied by a *lateral strain* or a strain in a direction at right angles to the applied force. This linear and lateral strains are sometimes referred to as the *primary* and *secondary* strains respectively. This is true not only for a wire but for all bodies in general.

Within the elastic limit, the ratio between the contraction or lateral strain to the elongation or linear strain for a given tensile stress is constant for a body of a given material and is called the *Poisson's ratio* for the material. Poisson's ratio is usually denoted by the symbol  $\sigma$  or, in engineering practice, by  $1/m$ . Thus

*Poisson's ratio  $\sigma$  (or  $1/m$ )*

$$\begin{aligned} &= \frac{\text{lateral strain}}{\text{linear (or tangential) strain}} \\ &= \frac{\text{secondary strain}}{\text{primary strain}} \end{aligned}$$

Linear (or tangential) and lateral strains per unit stress are usually denoted by  $\alpha$  and  $\beta$  respectively. Thus,

$$\sigma (= 1/m) = \beta/\alpha. \quad (9.4)$$

### 9.11 Shear is equivalent to compression and extension

We know that when a cube is sheared, its shape is altered but its volume as well as its thickness remain unchanged.

$ABCD$  in Fig. 9.5 represent a section of a cube, the length of whose side is  $l$ . Let a tangential force  $F$  be applied to the upper face  $AD$  of the cube. As the lower face  $BC$  is fixed,  $AD$  is sheared through a small angle  $\theta$  into its new position  $A'D'$ . As a result the diagonal  $BD$  is increased to length  $BD'$  while the diagonal  $AC$  is shortened to  $A'C$ .

Since the amount of shear is extremely small, this extension and compression may be expressed in terms of the angle  $\theta$  ( $=\angle ABA' = \angle DCD'$ ). Let  $A'M$  and  $DL$  be the perpendiculars dropped from  $A'$  and  $D$  on to  $AC$  and  $BD'$  respectively.

As the shear is very small, the triangles  $AMA'$ , and  $DLD'$  may be assumed to be right-angled isosceles triangles so that  $LD' = DD' \cos 45^\circ = DD' / \sqrt{2}$ .

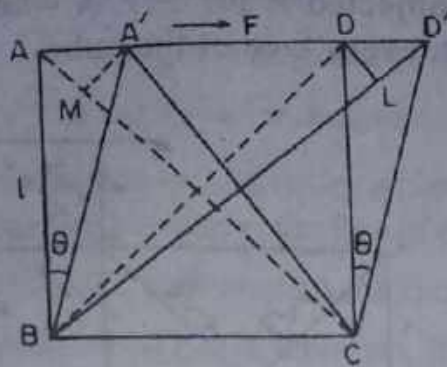


Fig. 9.5

Since  $AD = l$ , we have

$$BD = BL = \sqrt{l^2 + l^2} = l\sqrt{2}$$

$\therefore$  extension or tensile strain along the diagonal BD

$$= \frac{LD'}{BD} = \frac{DD'}{\sqrt{2}} \times \frac{1}{l\sqrt{2}} = \frac{DD'}{2l}$$

But  $\frac{DD'}{l} = \frac{DD'}{CD} = \theta$

$\therefore$  extension along BD =  $\frac{\theta}{2}$

Similarly the compression along AC

$$= \frac{AM}{AC} = \frac{AA'}{\sqrt{2}} \cdot \frac{1}{AC} = \frac{AA'}{\sqrt{2}} \cdot \frac{1}{l\sqrt{2}} = \frac{AA'}{2l} = \frac{\theta}{2}$$

*Thus, the simple shear  $\theta$  is equivalent to a compression and an extension at right angles to each other, each of value  $\theta/2$ .*

### 9.12 Equivalence of a shearing stress to an equal tensile and an equal compressive stress at right angles to one another

Let a tangential force  $F$  be applied to the upper face  $AD$  of the unit cube (Fig. 9.6), the lower face  $BC$  being fixed. If  $l$  is the length of each side of the cube, then the tangential or shearing stress to

which the face AD is subjected is  $F/l^2 = F/A$  where  $A$  is the area of the face AD, and in fact, every face of the cube.

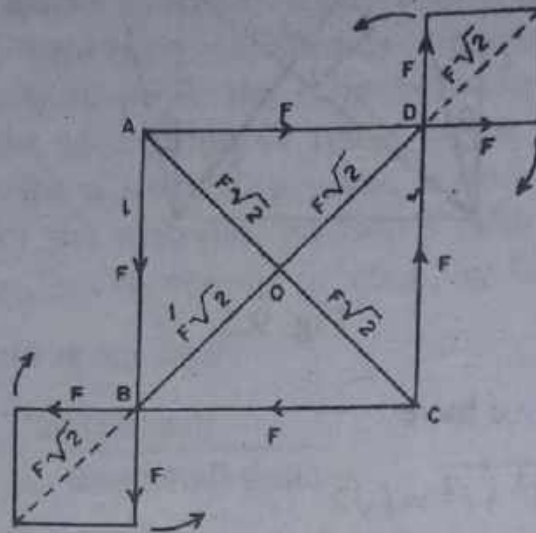


Fig. 9.6

If the cube were free to move, it would have moved along the direction of the force  $F$ . But as the lower face is fixed, an equal and opposite force comes into play on this face, to form a couple  $F.l$  which tends to rotate the cube clockwise as shown in the figure. But since the cube does not rotate, it means that there must be an equal and opposite couple acting on it due to forces  $F$  and  $F$  acting along the sides  $AB$  and  $CD$ . This couple is also equal to  $F.l$  and tends to rotate the cube in an anticlockwise direction in order to keep the cube in equilibrium.

It can thus be seen that a tangential force  $F$  applied to one face of the cube gives rise to an equal tangential force on all other faces of the cube in appropriate directions as shown in the figure.

Now the resultant of the forces  $F$  and  $F$  along  $AD$  and  $CD$  is  $F.\sqrt{2}$  along  $OD$  while the resultant of the forces  $F$  and  $F$  along  $AB$  and  $CB$  is  $F.\sqrt{2}$  along  $OB$  where  $O$  is the point of intersection of the diagonals  $BD$  and  $AC$  of the face  $ABCD$ . Thus, there is an outward pull on diagonal  $BD$ , tending to extend its length.

Similarly, there is an equal but inward pull on diagonal  $AC$ , tending to shorten its length.



Thus, a tangential force ( $F$ ) applied to one face of the cube brings into play a tensile force  $F \cdot \sqrt{2}$  along one diagonal ( $BD$  in this case) and an equal compressive force  $F \cdot \sqrt{2}$  along the other diagonal  $AC$ , perpendicular to it.

If the cube is now cut into two halves by a plane through diagonal  $BD$  and perpendicular to the plane of the paper, the face of each half parallel to  $BD$  will have an area  $l \cdot \sqrt{2} \cdot l = l^2 \cdot \sqrt{2}$ , with an outward force  $F \cdot \sqrt{2}$  acting normally on it. The tensile stress acting along  $BD$  is, therefore, equal to  $F \cdot \sqrt{2} / l^2 \sqrt{2} = F/l^2 = F/A =$  the tangential stress applied to the upper face of the cube.

Similarly, if the cube is cut into two halves along the diagonal  $AC$ , a compressive force  $F \cdot \sqrt{2}$  will be found to be acting normally on the face of each half, parallel to  $AC$ . The compressive stress along  $AC$  will, therefore, be equal to  $F \cdot \sqrt{2} / l^2 \cdot \sqrt{2} = F/l^2 = F/A =$  the tangential stress applied to the upper face of the cube.

It can, therefore, be seen that a tangential or a shearing stress is equivalent to an equal tensile stress and an equal compressive stress at right angles to each other.

### 9.13 Relations connecting the elastic constants

The elastic constants  $Y$ ,  $K$ ,  $\eta$ , as also the Poisson's ratio  $\sigma$ , are all interconnected. Before establishing the relations connecting these constants, the constants will be first expressed in terms of  $\alpha$  (the linear strain per unit stress) and  $\beta$  (the lateral strain per unit stress).

#### Young's modulus

Let us imagine a cube of unit edge, acted upon by a unit tension along one edge. If  $\alpha$  be the increase per unit length per unit tension along the direction of the force, then,

$$\text{Stress} = \text{force per unit area} = 1$$

$$\text{and strain} = \frac{\alpha}{1} = \alpha$$

Therefore, Young's modulus

$$Y = \frac{\text{stress}}{\text{strain}} = \frac{1}{\alpha} \quad (9.5)$$

### Bulk modulus

Let ABCDEFGH be a unit cube and let  $T_x$ ,  $T_y$  and  $T_z$  be the forces per unit area acting perpendicular to the faces BEHC and AFGD, ABCD and EFGH, and ABEF and CHGD respectively as shown in Fig. 9.7. Let  $\alpha$  be the increase per unit length per unit tension along the direction of the force, and  $\beta$ , the contraction produced per unit length per unit tension in a direction perpendicular to the force (Art. 9.10). Then the elongation produced in the edge AB due to the force  $T_x$  (acting along AB) is  $\alpha T_x$  and the contractions produced by the forces  $T_y$  and  $T_z$  (acting in a direction perpendicular to AB) are  $\beta.T_y$  and  $\beta.T_z$ . Similar will be the cases for the other two edges BE and BC.

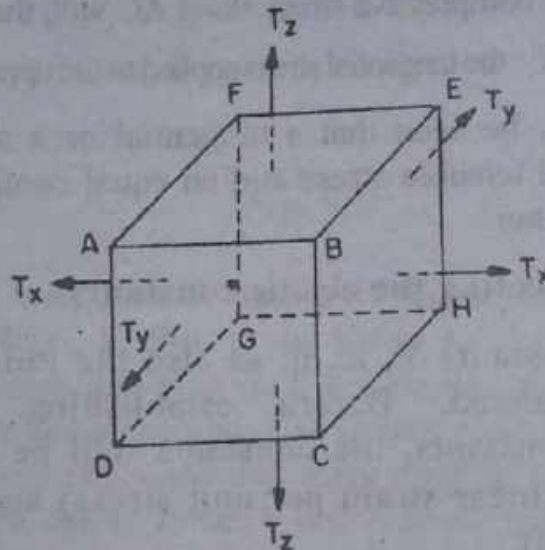


Fig. 9.7

Thus the lengths of the edges become

$$AB = 1 + \alpha.T_x - \beta.T_y - \beta.T_z$$

$$BE = 1 + \alpha.T_y - \beta.T_z - \beta.T_x$$

$$BC = 1 + \alpha.T_z - \beta.T_x - \beta.T_y$$

Hence the volume of the cube will now become

$$\begin{aligned} & (1 + \alpha.T_x - \beta.T_y - \beta.T_z) (1 + \alpha.T_y - \beta.T_z - \beta.T_x) \\ & (1 + \alpha.T_z - \beta.T_x - \beta.T_y) \\ & = 1 + (\alpha - 2\beta) (T_x + T_y + T_z) \end{aligned}$$

neglecting squares and products of  $\alpha$  and  $\beta$ , which are very small compared with the other quantities involved.

If  $T_x = T_y = T_z = T$

The volume of the cube becomes

$$1 + (\alpha - 2\beta) 3T.$$

The increase in volume is, therefore,

$$1 + 3T (\alpha - 2\beta) - 1 = 3T (\alpha - 2\beta)$$

Thus

$$\begin{aligned} \text{Bulk modulus, } K &= \frac{\text{stress}}{\text{volume strain}} \\ &= \frac{T/1}{2T (\alpha - 2\beta)} = \frac{1}{3 (\alpha - 2\beta)} \end{aligned} \quad (9.6)$$

Compressibility, which is the reciprocal of the bulk modulus, is, therefore, equal to

$$\frac{1}{K} = 3 (\alpha - 2\beta)$$

### Modulus of rigidity

Let the top face  $ABHG$ , of a cube the length of whose side is  $L$ , be sheared by a shearing force  $F$ , the lower face of the cube being fixed. As a result of the *shear*  $A$  takes up the position  $A'$  and  $B$ , the position  $B'$ , the angle  $ADA'$  being equal to the angle  $BCB' = \theta$  (Fig. 9.8). As a result the diagonal  $DB$  is increased to  $DB'$  and the diagonal  $AC$  shortened to  $A'C$ .

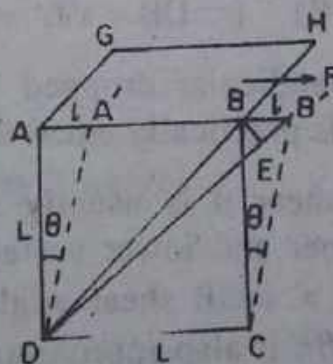


Fig. 9.8



Then,

$$\begin{aligned}\text{The stress} &= \frac{F}{\text{area of the face ABHG}} = \frac{F}{L^2} \\ &= T \text{ (say)}\end{aligned}$$

Let the displacement  $AA' = BB' = l$ ,

then

$$\text{shear strain} = \frac{l}{L} = \theta$$

Thus the *coefficient or modulus of rigidity*

$$\eta = \frac{\text{stress}}{\text{strain}} = \frac{T}{\theta}$$

As discussed in Art. 9.12, a shearing stress along AB is equivalent to a tensile stress along the diagonal DB and an equal compressive stress along the diagonal AC perpendicular to it. The former clearly increases the length of the diagonal DB. Now a compressive strain along AC is accompanied by a lateral strain of the opposite type in a direction perpendicular to it, i.e., the compressive strain along AC also increases the length of DB.

The, extension of the diagonal DB, due to tensile stress along DB is  $\alpha.T$ , and that due to compressive strain along AC is  $\beta.T$  where  $\alpha$  and  $\beta$  are the linear and lateral strain per unit stress respectively. Therefore, total extension in length of the diagonal DB is

$$\begin{aligned}\text{DB} \times (T.\alpha + T.\beta) &= \text{DB} \times T (\alpha + \beta) \\ &= L\sqrt{2} \cdot T (\alpha + \beta) \quad (\because \text{DB} = \sqrt{L^2 + L^2} = \sqrt{2}.L)\end{aligned}$$

Let BE be the perpendicular dropped from B on DB'. Then, increase in length of DB is practically equal to EB'.

Now the angle of shear  $\theta$  is usually very small so that the distance between the upper and lower surfaces practically remains the same. Therefore, for a small shear angle  $\theta$ ,  $\angle BB'C \simeq 90^\circ$  and  $\angle BB'E \simeq 45^\circ$ . Hence  $\angle B'BE$  is also approximately equal to  $45^\circ$ .

$$\text{Thus } EB = EB' = BB' \cos 45^\circ = l \cdot \frac{1}{\sqrt{2}} = \frac{l}{\sqrt{2}}$$

Hence

$$L\sqrt{2} T(\alpha + \beta) = \frac{l}{\sqrt{2}}$$

$$\text{or, } \frac{LT}{l} = \frac{1}{2(\alpha + \beta)}$$

$$\text{or, } \frac{T}{l/L} = \frac{1}{2(\alpha + \beta)}$$

$$\text{or, } \frac{T}{\theta} = \frac{1}{2(\alpha + \beta)}$$

$$\text{But } \frac{T}{\theta} = \frac{\text{stress}}{\text{strain}} = \eta$$

Hence, the modulus of rigidity

$$\eta = \frac{1}{2(\alpha + \beta)} \quad (9.7)$$

#### Relation between Y, K and $\eta$

From eqns. (9.5), (9.6) and (9.7) we get respectively,

$$Y = \frac{1}{\alpha} \quad (i)$$

$$K = \frac{1}{3(\alpha - 2\beta)} \quad (ii)$$

$$\text{and } \eta = \frac{1}{2(\alpha + \beta)} \quad (iii)$$

Multiplying (iii) by 2 we get

$$2\eta = \frac{1}{(\alpha + \beta)}$$

$$\text{or, } \frac{1}{\eta} = 2\alpha + 2\beta \quad (iv)$$

Rearranging (ii), we get

$$\frac{1}{3K} = \alpha - 2\beta \quad (v)$$

Adding eqns. (iv) and (v), we get

$$3\alpha = \frac{1}{\eta} + \frac{1}{3K} = \frac{3K + \eta}{3K\eta}$$

$$\text{or, } \alpha = \frac{3K + \eta}{9K\eta} \quad (vi)$$

$$\text{But } \alpha = \frac{1}{Y}$$

Hence

$$\frac{1}{Y} = \frac{3K + \eta}{9K\eta}$$

$$\text{or, } \frac{9}{Y} = \frac{3K + \eta}{K\eta}$$

$$= \frac{3K}{K\eta} + \frac{\eta}{K\eta}$$

$$= \frac{3}{\eta} + \frac{1}{K} \quad (9.8)$$

This, then, is the relation connecting the three elastic constants.

**Relation connecting Y, K and  $\sigma$**

From eqn. (9.6) we have

$$\begin{aligned} K &= \frac{1}{3(\alpha - 2\beta)} = \frac{1}{3\alpha \left(1 - \frac{2\beta}{\alpha}\right)} \\ &= \frac{1/\alpha}{3 \left(1 - 2 \cdot \frac{\beta}{\alpha}\right)} \end{aligned}$$



But from eqns. (9.4) and (9.5) we have

$$\sigma = \frac{\beta}{\alpha} \quad \text{and} \quad Y = \frac{1}{\alpha}$$

$$\therefore K = \frac{Y}{3(1-2\sigma)}$$

$$\text{or, } Y = 3K(1-2\sigma) \quad (9.9)$$

**Relation between  $Y$ ,  $\eta$  and  $\sigma$**

From eqn. (9.4) we have

$$\begin{aligned} \eta &= \frac{1}{2(\alpha + \beta)} = \frac{1}{2\alpha \left(1 + \frac{\beta}{\alpha}\right)} \\ &= \frac{1/\alpha}{2 \left(1 + \frac{\beta}{\alpha}\right)} \\ &= \frac{Y}{2(1 + \sigma)} \quad [\text{from (9.4) and (9.5)}] \end{aligned}$$

$$\text{or, } Y = 2\eta(1 + \sigma) \quad (9.10)$$

**Relation between  $K$ ,  $\eta$  and  $\sigma$**

From eqn. (9.9) we have

$$Y = 3K(1 - 2\sigma)$$

Again from eqn. (9.10) we have

$$Y = 2\eta(1 + \sigma)$$

Hence,

$$3K(1 - 2\sigma) = 2\eta(1 + \sigma)$$

$$\text{or, } 3K - 2\eta = \sigma(6K + 2\eta)$$

$$\text{or, } \sigma = \frac{3K - 2\eta}{6K + 2\eta} \quad (9.11)$$

### 9.14 Limiting values of Poisson's ratio ( $\sigma$ )

From eqns. (9.9) and (9.10) we have

$$Y = 3K(1 - 2\sigma) \quad \text{and} \quad Y = 2\eta(1 + \sigma)$$

$$\text{Hence } 2\eta(1 + \sigma) = 3K(1 - 2\sigma) \quad (i)$$

$$\text{or, } \frac{3K}{2\eta} = \frac{1 + \sigma}{1 - 2\sigma} \quad (ii)$$

Both  $\eta$  and  $K$  are positive quantities; hence

$$\frac{3K}{2\eta} \text{ is positive; therefore,}$$

$$\frac{1 + \sigma}{1 - 2\sigma} \text{ is also positive.}$$

If  $\sigma > \frac{1}{2}$ , the numerator is positive but the denominator will be negative, so that  $\frac{3K}{2\eta}$  becomes negative which is untenable. Again if  $\sigma < -1$ , the denominator will be positive but the numerator will be negative, so that  $\frac{3K}{2\eta}$  becomes negative, which is again untenable.

Hence  $\frac{3K}{2\eta}$  will be positive only when  $\sigma$  lies between  $-1$  and  $\frac{1}{2}$ .

Thus the limiting values of  $\sigma$  are  $-1$  and  $\frac{1}{2}$ .

**Example 9.1.** A copper wire 2 metres long and 0.5 mm in diameter, supports a mass of 10 kilograms. It is stretched by 2.33 mm. Calculate the Young's modulus of the wire.

**Soln.**

$$Y = \frac{\text{tensile stress}}{\text{tensile strain}} = \frac{F/A}{l/L} = \frac{FL}{A.l}$$

$$F = mg = 10 \times 10000 \times 981$$

$$A = \pi r^2 = 3.14 \times (0.025)^2$$

$$l = 2.38 \text{ mm} = 0.238 \text{ cm}$$

$$L = 2 \text{ metres} = 200 \text{ cm}$$

$$\begin{aligned} \therefore Y &= \frac{10 \times 1000 \times 981 \times 200}{3.14 \times (25 \times 10^{-3})^2 \times 0.238} \\ &= \frac{1962 \times 10^6}{467.1 \times 10^{-6}} = 4.2 \times 10^{12} \text{ dynes/cm}^2. \end{aligned}$$

**Example 9.2.** Find the greatest length of a steel wire that can hang vertically without breaking. Breaking stress for steel =  $7.9 \times 10^8 \text{ N/m}^2$ . Density of steel  $7.9 \times 10^3 \text{ kg/m}^3$ .

**Soln.**

Let  $l$  = length of the wire that can hang vertically without breaking. Then the stretching force on it is clearly equal to its own weight.

If  $A$  is the area of cross-section of the wire and  $\rho$  is the density of the material of the wire, then the mass of the wire = volume  $\times$  density =  $l.A.\rho$ .

Hence the tensile stress applied

$$= F/A = mg/A = \frac{lA.\rho g}{A} = l\rho g.$$

Since this is equal to the breaking stress, we have

$$l\rho g = 7.9 \times 10^8$$

$$\begin{aligned} \text{or, } l &= \frac{7.9 \times 10^8}{7.9 \times 10^3 \times 9.81} \\ &= 0.102 \times 10^5 \text{ m} \\ &= 1.02 \times 10^6 \text{ cm.} \end{aligned}$$

**Example 9.3.** A metal wire of length 3 metres and diameter 1 mm is stretched by a weight of 10 kg. If young's modulus and Poisson's ratio for its material are  $12.5 \times 10^{11} \text{ dynes/cm}^2$  and 0.26 respectively, calculate the lateral compression produced.

**Soln.**

Let  $l$  and  $d$  be the length and diameter of the wire and  $\Delta l$  and  $\Delta d$  the longitudinal extension and lateral compression produced respectively.



Then

$$\text{longitudinal strain} = \Delta l/l$$

$$\text{and lateral strain} = \Delta d/d$$

Now

$$\text{Young's modulus, } Y = \frac{\text{stress}}{\text{strain}}$$

$$= \frac{F/A}{\Delta l/l}$$

$$\text{or, } \Delta l/l = \frac{F}{AY}$$

$$\text{Poisson's ratio, } \sigma = \frac{\text{lateral strain}}{\text{longitudinal strain}}$$

$$= \frac{\Delta d/d}{\Delta l/l} = \frac{\Delta d/d}{F/AY}$$

$$\text{or, } \Delta d/d = F\sigma/AY$$

$$\text{or, } \Delta d = \frac{F\sigma d}{AY}$$

$$d = 1 \text{ mm} = 0.1 \text{ cm}$$

$$F = mg = 10 \times 1000 \times 981$$

$$= 981 \times 10^4$$

$$\sigma = 0.26$$

$$A = \pi r^2 = 3.14 \times (0.05)^2$$

$$= 7.85 \times 10^{-3} \text{ sq. cm.}$$

$$\therefore \Delta d = \frac{981 \times 10^4 \times 0.26 \times 0.1}{7.85 \times 10^{-3} \times 12.5 \times 10^{11}}$$

$$= \frac{25.506 \times 10^4}{98.125 \times 10^8} = 0.26 \times 10^{-4} \text{ cm}$$

$$= 2.6 \times 10^{-5} \text{ cm.}$$

**Example 9.4.** You are given 200 cc of air at a pressure due to 760 mm of mercury. On increasing the pressure by that due to 1 mm of mercury without change of temperature, the volume is observed to decrease by 0.263 cc. Find the coefficient of volume elasticity of the gas. Density of mercury = 13.6 gm/cc.

**Soln.**

Bulk modulus or, coefficient of volume elasticity  $K = \frac{P}{dv/v}$ .

$$p = h\rho g$$

$$= 0.1 \times 981 \times 13.6$$

$$v = 200 \text{ cc}, dv = 0.263$$

$$\therefore K = \frac{0.1 \times 981 \times 13.6 \times 200}{0.263}$$

$$= \frac{266832}{0.263} = 1014570.3$$

$$= 1.01457 \times 10^6 \text{ dynes/cm}^2.$$

**Example 9.5.** Show that a volume strain is equivalent to three mutually perpendicular linear strains, each equal in magnitude to one-third of the volume strain.

**Soln.**

Let us consider a cube of unit edge. Let this cube be compressed uniformly and equally in all the three directions so that the length of each edge decreases by  $l$  and the volume decrease by  $V$ .

Then,

the volume strain

$$= \frac{\text{decrease in volume}}{\text{original volume}} = \frac{V}{1}$$

and the linear strain along each edge

$$= \frac{\text{decrease in length}}{\text{original length}} = \frac{l}{1}$$

Since the length of each edge decreases by  $l$ , the reduced length of each edge  $= 1 - l$ .

Hence the reduced volume  $= (1 - l)^3$ .

Therefore, the decrease in volume,  $V = 1 - (1 - l)^3 = 3l$

(neglecting higher power of  $l$ )

$$\text{or, } l = \frac{V}{3}$$

Thus a volume strain  $V$  is equivalent to three linear strains, each equal to  $V/3$ , along the three edges of the cube *i.e.*, along three mutually perpendicular directions.

**Example 9.6.** What will be the density of lead under a pressure of 20,000 newton/cm<sup>2</sup>? Density of lead = 11.4 gm/cm<sup>3</sup>; bulk modulus of lead =  $0.80 \times 10^{10}$  newton/m<sup>2</sup>.

**Soln.**

$$\text{Bulk modulus, } K = - \frac{dp}{dV/V}$$

where  $dp$  = pressure or force per unit area (stress) and  $dV/V$  = volume strain produced.

$$\therefore -dV = dp \cdot V / K$$

$$\text{Now } dp = 2 \times 10^4 \text{ N/cm}^2,$$

$$K = 0.80 \times 10^{10} \text{ N/m}^2$$

$$= (0.80 \times 10^{10}) / 10^4 \text{ N/cm}^2$$

$$= 8 \times 10^5 \text{ N/cm}^2.$$

$$\therefore -dV = \frac{2 \times 10^4 V}{8 \times 10^5} = \frac{V}{40}$$

$$\text{Thus the new volume} = V - \frac{V}{40} = \frac{39}{40} V.$$

If  $\rho$  is the new density of lead, its mass

$$= \frac{39}{40} V \rho \text{ gm.}$$



Since the mass remains unchanged, we have

$$\frac{39}{40} V\rho = V \times 11.4$$

$$\text{or, } \rho = \frac{V \times 11.4 \times 40}{39 \times V} = 11.69 \text{ gm/cm}^3.$$

So the new density of lead  
 $= 11.69 \text{ gm/cm}^3.$

**Example 9.7.** Calculate the Poisson's ratio for a material given  $Y = 12.25 \times 10^{10} \text{ N/m}^2$  and  $\eta = 4.55 \times 10^{10} \text{ N/m}^2$ .

**Soln.**

We have

$$Y = 2\eta (1 + \sigma)$$

$$\text{or, } 1 + \sigma = \frac{Y}{2\eta} = \frac{12.25 \times 10^{10}}{2 \times 4.55 \times 10^{10}} = 1.346$$

$$\therefore \sigma = 1.346 - 1 \\ = 0.346.$$

**Example 9.8.** Calculate  $\eta$  and  $\sigma$  for silver, given  $Y$  and  $K$  for silver  $= 7.25 \times 10^{11} \text{ dynes/cm}^2$  and  $11 \times 10^{11} \text{ dynes/cm}^2$ .

**Soln.**

We have

$$Y = 3K (1 - 2\sigma)$$

$$\text{or, } 1 - 2\sigma = \frac{Y}{3K} = \frac{7.25 \times 10^{11}}{3 \times 11 \times 10^{11}} = 0.219$$

$$\therefore 2\sigma = 1 - 0.219 = 0.781$$

$$\text{or, } \sigma = 0.3905$$

$$\text{Again, } Y = 2\eta (1 + \sigma)$$

$$\text{or, } 7.25 \times 10^{11} = 2\eta (1 + 0.3905) \\ = 2.78\eta$$

$$\text{or, } \eta = \frac{7.25 \times 10^{11}}{2.78} = 2.61 \times 10^{11} \text{ dynes/cm}^2.$$

**Example 9.9.** A uniform rod of 2 sq. mm cross-sectional area is heated from  $0^{\circ}\text{C}$  to  $20^{\circ}\text{C}$ . Find the compressive force which must be exerted to prevent it from expanding. Coefficient of linear expansion of the material of the wire =  $12 \times 10^{-6}$  per  $^{\circ}\text{C}$ .  $Y = 10^{11}$  newton/ $\text{m}^2$ .

**Soln.**

Let  $l$  be the length of the rod at  $0^{\circ}\text{C}$  and  $l'$  be the length at  $20^{\circ}\text{C}$ .  
So the increase in length,

$$\begin{aligned}\Delta l &= l' - l = l(1 + 12 \times 10^{-6} \times 20) - l \\ &= l \times 12 \times 10^{-6} \times 20\end{aligned}$$

Young's modulus,

$$Y = \frac{F/A}{\Delta l/l}$$

$$\text{or, } F/A = Y \times \frac{\Delta l}{l}$$

$$\begin{aligned}&= 10^{11} \times \frac{l \times 12 \times 10^{-6} \times 20}{l} \\ &= 10^{11} \times 12 \times 10^{-6} \times 20\end{aligned}$$

Now,  $A = 2 \text{ sq. mm}$

$$= 2 \times 10^{-6} \text{ sq. metre.}$$

$\therefore$  compressive force,

$$\begin{aligned}F &= 10^{11} \times 12 \times 10^{-6} \times 20 \times 2 \times 10^{-6} \\ &= 48 \text{ newtons.}\end{aligned}$$

**Example 9.10.** A platform is suspended by four wires at its corners. The wires are 3m long and have a diameter of 2.0 mm. Young's modulus for the materials of the wires is 180000 MPa. How far will the platform drop (due to elongation of the wires) if a 50 Kg load is placed at the centre of the platform?

**Soln.**

$$\text{From } Y = \frac{F/A}{l/L}, \text{ we have}$$

$l = \frac{LF}{AY}$  where  $Y$  is the Young's modulus of the material of the wires,  $L$  is the original length of the wires,  $A$  is the area of cross-section of the wires and  $F$  is the load applied. Since each wire supports one-quarter of the load,

$$F = \frac{(50 \text{ Kg}) (9.8 \text{ m/s}^2)}{4} = 123 \text{ N.}$$

$$\begin{aligned} \text{Now } Y &= 180000 \text{ MPa} = 180000 \times 10^6 \text{ N/m}^2 \\ &= 1.8 \times 10^{11} \text{ N/m}^2. \end{aligned}$$

$$L = 3 \text{ m}$$

$$A = \pi r^2 = (3.14) (1.0 \times 10^{-3} \text{ m})^2 = 3.14 \times 10^{-6} \text{ m}^2.$$

$$\begin{aligned} \therefore l &= \frac{LF}{AY} = \frac{(3 \text{ m}) (123 \text{ N})}{(3.14 \times 10^{-6} \text{ m}^2) (1.8 \times 10^{11} \text{ N/m}^2)} \\ &= 65 \times 10^{-5} \text{ m} = 0.65 \text{ mm.} \end{aligned}$$

**Example 9.11.** A pressure of 84 KPa decreases the volume of 200 L of water by 0.004 per cent. Compute the compressibility of water.

**Soln.**

$$\text{Compressibility (K)} = \frac{1}{\text{Bulk modulus}} = \frac{1}{K}$$

$$\text{Now } K = - \frac{p}{dv/v}$$

$$\therefore K = \frac{1}{K} = - \frac{1}{p} \frac{dV}{V}$$

$$\text{Here } p = 84 \text{ KPa} = 84 \times 10^3 \text{ N/m}^2$$

$$dV = (2004) (0.00004) = 0.0084.$$

$$V = 2004.$$

$$\therefore K = \frac{(0.0084)}{(84 \times 10^3 \text{ N/m}^2) (2004)}$$

$$= 4.76 \times 10^{-10} \text{ m}^2/\text{N.}$$



**Example 9.12.** Two parallel and opposite forces, each 400 N, are applied tangentially to the upper and lower faces of a cubical metal block 25cm on a side. Find the angle of shear and the displacement of the upper surface relative to the lower surface. The shear modulus for the metal is 80 LPa.

**Soln.**

The shear modulus or modulus of rigidity is given by

$$\eta = \frac{\text{stress}}{\text{strain}} = \frac{F/A}{l/L}$$

Since the angle of shear  $\theta$  is very small

$$\tan \theta = \theta = l/L.$$

$$\therefore \eta = \frac{F/A}{\theta}; \quad \text{or, } \theta = \frac{F/A}{\eta}$$

$$\text{or, } \theta = \frac{F/A}{\eta}$$

$$= 8.0 \times 10^{-7} \text{ rad.}$$

The displacement  $l$  of the upper surface is given by

$$\begin{aligned} l &= \theta \cdot L = (8.0 \times 10^{-7} \text{ rad}) (25 \text{ cm}) \\ &= 2.0 \times 10^{-5} \text{ cm.} \end{aligned}$$

**Example 9.13.** A block of gelatin is 60mm by 60 mm by 20 mm when unstressed. A force of 0.245N is applied tangentially to the upper surface, causing a 5 mm displacement relative to the lower surface. Find (a) shearing stress, (b) the shearing strain, and (c) the shear modulus.

**Soln.**

$$\begin{aligned} \text{(a) shearing stress} &= \frac{F}{A} = \frac{0.245 \text{ N}}{(6 \times 10^{-2} \text{ m})^2} \\ &= 86.1 \text{ N/m}^2 \end{aligned}$$

$$(b) \text{ shearing strain, } \theta = \tan \theta = \frac{l}{L}$$

$$\text{here } l = 5 \text{ mm} = 5 \times 10^{-3} \text{ m}$$

$$L = 20 \text{ mm} = 20 \times 10^{-3} \text{ m}$$

$$\therefore \text{ strain} = \frac{5 \times 10^{-3} \text{ m}}{20 \times 10^{-3} \text{ m}} = 0.25$$

$$(c) \text{ shear modulus} = \frac{\text{stress}}{\text{strain}} = \frac{F/A}{l/L}$$

$$= \frac{68.1 \text{ N/m}^2}{0.25} = 272.4 \text{ N/m}^2.$$

**Example 9.14.** What is the minimum diameter of a brass rod if it is to support a 400-N load without exceeding the elastic limit? Assume that the stress for the elastic limit is 379 MPa.

**Soln.**

To find the minimum diameter ( $d$ ), and hence minimum cross-sectional area ( $A$ ), let us assume that the force  $F = 400 \text{ N}$  brings us to the elastic limit.

Then from the stress  $F/A$ , we get

$$\frac{(400 \text{ N})}{A} = 379 \times 10^6 \text{ N/m}^2$$

$$\text{or, } A = \frac{400 \text{ N}}{379 \times 10^6 \text{ N/m}^2} = 1.0554 \times 10^{-6} \text{ m}^2.$$

$$\begin{aligned} \text{Now } A &= \pi \frac{D^2}{4}; \quad \text{or, } D^2 = \frac{4A}{\pi} = \frac{4 \times 1.0554 \times 10^{-6} \text{ m}^2}{3.14} \\ &= 1.344 \times 10^{-6} \text{ m}^2 \end{aligned}$$

$$\begin{aligned} \therefore D &= \sqrt{1.344 \times 10^{-6} \text{ m}^2} = 1.16 \times 10^{-3} \text{ m} \\ &= 1.16 \text{ mm.} \end{aligned}$$

### 9.15 Work done in a strain

Work has to be done by the applied force in *deforming* a body, to whatever type of strain it might be subjected. The work done is stored up in the body in the form of potential energy which, in this case, may be called *elastic potential energy*, *energy of strain* or simply, *strain energy*.

The work done per unit volume, i.e., the strain energy per unit volume is calculated below for three different cases of strain.

(i) *longitudinal strain* : Let  $F$  be the stretching force applied (within the elastic limit) to a vertical wire, fixed at the upper end, by supporting a load at its lower end. If  $l$  is the resulting stretch or elongation produced in the wire whose original length is  $L$  and area of cross-section is  $A$ , then

**Soln.**

$$Y = \frac{F/A}{l/L} = \frac{F.L}{l.A}$$

$$\text{or, } F = \frac{YlA}{L} \quad (9.12)$$

where  $Y$  is the Young's modulus of the material of the wire.

If there is an additional small increase  $dl$  in the length of the wire, then the work done for this increase is

$$F \cdot dl = \frac{YAl}{L} \cdot dl$$

Hence the total work done during the stretch of the wire from 0 to  $l$ , say,  $W$  is given by

$$\begin{aligned} W &= \int_0^l F \cdot dl = \int_0^l \frac{YAl}{L} \cdot dl = \frac{1}{2} \frac{YAl^2}{L} \\ &= \frac{1}{2} \left( \frac{YAl}{L} \right) \cdot l = \frac{1}{2} F \cdot l \end{aligned} \quad (9.13)$$

$$= \frac{1}{2} \text{ stretching force} \times \text{stretch or elongation}$$



Now the volume of the wire of length  $L$  and cross-sectional area  $A = L \times A$ .

$\therefore$  Strain energy per unit volume

$$= \frac{1}{2} F \cdot l \div L \cdot A$$

$$= \frac{1}{2} \frac{F}{A} \cdot \frac{l}{L} \quad (9.14)$$

$$= \frac{1}{2} \text{ stress} \times \text{strain}$$

$$= \frac{1}{2} \frac{mg}{\pi r^2} \times \frac{l}{L}$$

Where  $F = mg$  and  $A = \pi r^2$ ;  $m$  is the mass of the load supported,  $g$  the acceleration due to gravity and  $r$  is the radius of the wire.

From (9.13) and (9.14), the strain energy per unit volume

$$= \frac{1}{2} \frac{Y A l}{L \cdot A} \cdot \frac{l}{L}$$

$$= \frac{1}{2} Y \left( \frac{l}{L} \right)^2 \quad (9.15)$$

$$= \frac{1}{2} \text{ Young's modulus} \times (\text{longitudinal strain})^2$$

(ii) *volume strain* : Let  $p$  be the stress which is force per unit area or pressure which is applied normally over an area  $A$  of a body of volume  $V$  such that its volume decreases by  $v$ . Then the bulk modulus,  $K$ , for the material of the body is given by

$$K = p \cdot \left( -\frac{v}{V} \right)$$

$$\text{or, } p = \frac{K \cdot v}{V}$$

Omitting the minus sign which merely indicates that volume decreases as pressure increases and vice versa.

If  $dx$  is a small movement in the direction of  $p$ , then the work done  
 $= p \cdot A \cdot dx$

where  $p \cdot A$  is the force exerted on the area  $A$ . But  $A \cdot dx = dv$  is the small change produced in the volume. Thus the work done for a small change in volume  $dv$  is  $= p \cdot dv$

Hence  $W$ , the total work done for the whole change in volume from 0 to  $v$  is given by

$$\begin{aligned} W &= \int_0^v p \cdot dv = \int_0^v \frac{Kv}{V} \cdot dv \\ &= \frac{K}{V} \cdot \frac{1}{2} v^2 \\ &= \frac{1}{2} \frac{Kv}{V} \cdot v \\ &= \frac{1}{2} p \cdot v \quad (9.16) \\ &= \frac{1}{2} \text{ stress} \times \text{change in volume} \end{aligned}$$

Therefore, work done per unit volume (or strain energy per unit volume)

$$\begin{aligned} &= \frac{1}{2} \frac{pv}{V} = \frac{1}{2} p \cdot \frac{v}{V} \quad (9.17) \\ &= \frac{1}{2} \text{ stress} \times \text{strain} \end{aligned}$$

Again strain energy per unit volume

$$\begin{aligned} &= \frac{1}{2} \frac{Kv^2}{V} \div V \\ &= \frac{1}{2} K \cdot \left(\frac{v}{V}\right)^2 \quad (9.18) \\ &= \frac{1}{2} \text{ Bulk modulus} \times (\text{volume strain})^2 \end{aligned}$$

(iii) *shearing strain* : Let the upper face of a cube, the length of each side of which is  $L$ , be sheared through an angle  $\theta$  under a tangential force  $F$ , its lower face being fixed (Fig. 9.9). Then the modulus of rigidity of the material of the cube,  $\eta = F/A\theta$  where  $A$  is the area of the upper (or for that matter any face) of the cube.

Now  $A = L^2$  and  $\theta = l/L$  where  $l$  is the displacement of each vertical edge of the cube. Therefore,

$$F = \eta A \theta = \eta \cdot L^2 \cdot \frac{l}{L} = \eta l$$

If  $dl$  is a small additional displacement of the vertical edge, then the work done for this displacement,

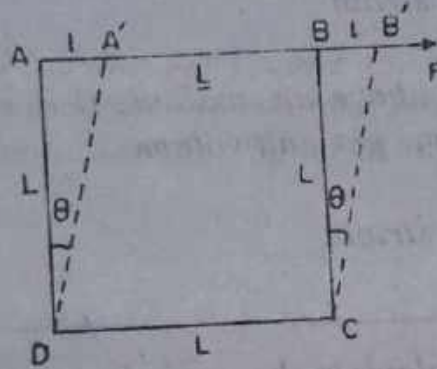


Fig. 9.9

$$F \cdot dl$$

$$= \eta l \cdot dl$$

Hence the work done for the whole displacement from 0 to  $l$ , i.e., during the whole shearing strain  $\theta$ ,

$$W = \int_0^l \eta l \cdot dl = \eta L \cdot \frac{1}{2} l^2$$

$$= \frac{1}{2} \frac{F}{L} \cdot L l^2 = \frac{1}{2} F l \quad (9.19)$$

$$\left[ \because \eta = \frac{F}{A\theta} = \frac{F}{L^2} \cdot \frac{L}{l} = \frac{F}{Ll} \right]$$



$$= \frac{1}{2} \text{ tangential force} \times \text{displacement}$$

Then the work done (or strain energy) per unit volume

$$= \frac{1}{2} \frac{F \cdot l}{L^3}$$

$$[\because \text{volume of the cube} = L^3]$$

$$= \frac{1}{2} \frac{F}{L^2} \cdot \frac{l}{L}$$

$$= \frac{1}{2} \frac{F}{A} \cdot \theta \quad (9.20)$$

$$= \frac{1}{2} \text{ stress} \times \text{strain}$$

**Note :** From the above discussions, it is evident that in every type of strain, work done per unit volume

$$= \frac{1}{2} \text{ stress} \times \text{strain}$$

**Example 9.14.** Calculate the work done in stretching a uniform metal wire of area of cross-section  $10^{-6} \text{ m}^2$  and length 1.5 m through  $4 \times 10^{-3} \text{ m}$ . Given  $Y = 2 \times 10^{11} \text{ N/m}^2$ .

**Soln.**

$$\text{strain} = \frac{\Delta l}{l} = \frac{4 \times 10^{-3}}{1.5}$$

$$\text{stress} = Y \times \text{strain}$$

$$= \frac{2 \times 10^{11} \times 4 \times 10^{-3}}{1.5} \text{ N/m}^2$$

Work done per unit volume

$$= \frac{1}{2} \times \text{stress} \times \text{strain}$$

Hence, total work done

$$= \frac{1}{2} \times \text{stress} \times \text{strain} \times \text{volume}.$$

volume = area of cross-section  $\times$  length

$$= A \times l = 10^{-6} \times 1.5.$$

$\therefore$  Total work done

$$= \frac{1}{2} \times \frac{2 \times 10^{11} \times 4 \times 10^{-3}}{1.5} \times \frac{4 \times 10^{-3}}{1.5} \times 10^{-6} \times 1.5$$

$$= 10.66 \times 10^{-1} \text{ J}$$

$$= 1.066 \text{ Joules.}$$

**Example 9.15.** Find the work done in Joules in stretching a uniform wire of cross-section 1 sq.mm. and length 2 metres through 1mm. Young's modulus for the material of wire is  $2 \times 10^{12}$  dynes/cm<sup>2</sup>.

**Soln.**

$$Y = \frac{\text{stress}}{\text{strain}} = \frac{F/A}{l/L}$$

$$\therefore F = \frac{Y \cdot l A}{L} = \text{stretching force.}$$

$$\text{Work done} = \frac{1}{2} \times \text{stretching force} \times \text{stretch or elongation}$$

$$= \frac{1}{2} F \times l = \frac{1}{2} \frac{Y l A}{L} \cdot l$$

$$= \frac{1}{2} \frac{2 \times 10^{12} \times (0.1) \times (0.01) \times (0.1)}{200}$$

$$= 5 \times 10^5 \text{ ergs}$$

$$= \frac{5 \times 10^5}{10^7} = 5 \times 10^{-2} \text{ Joules.}$$

**Example 9.16.** The Young's modulus of a metal is  $2 \times 10^{11} \text{ N/m}^2$  and its breaking stress is  $1.078 \times 10^9 \text{ N/m}^2$ . Calculate the maximum amount of energy per unit volume which can be stored in the metal when stretched.

**Soln.**

Here  $Y = 2 \times 10^{11} \text{ N/m}^2$

Maximum stress =  $1.078 \times 10^9 \text{ N/m}^2$ .

$$Y = \frac{\text{stress}}{\text{strain}}$$

$$\text{or, strain} = \frac{\text{stress}}{Y} = \frac{1.078 \times 10^9}{2 \times 10^{11}}$$

$$\begin{aligned} \text{Energy stored per unit volume} &= \text{work done per unit volume} \\ &= \frac{1}{2} \times \text{stress} \times \text{strain} \end{aligned}$$

$$= \frac{1}{2} \frac{1.078 \times 10^9 \times 1.078 \times 10^9}{2 \times 10^{11}}$$

$$= 2.90 \times 10^6 \text{ J/m}^2.$$

**Example 9.17.** Find the energy stored up in a wire 5 metres long and 1 mm in diameter which has been stretched by 3/10 mm, due to a load of 10 kg. ( $g = 300 \pi$ ).

**Soln.**

Energy stored in the wire

$$= \frac{1}{2} \times \text{stress} \times \text{strain} \times \text{volume}.$$

$$\text{stress} = \frac{F}{A} = \frac{10 \times 1000 \times 300\pi}{\pi \times (0.05)^2}$$

$$\text{strain} = \frac{l}{L} = \frac{0.03}{500}$$

Volume = area  $\times$  length

$$= \pi \times (0.05)^2 \times 500$$



∴ energy stored in the wire

$$\begin{aligned}
 &= \frac{1}{2} \times \frac{10^4 \times 300\pi}{\pi \times (0.05)^2} \times \frac{0.03}{500} \times \pi \times (0.05)^2 \times 500 \\
 &= \frac{0.2826 \times 10^6}{2} \\
 &= 1.413 \times 10^5 \text{ ergs.}
 \end{aligned}$$

### 9.16 Twisting couple on a cylinder (or wire)

Let a solid cylinder (or wire) of radius  $r$  be fixed at its upper end and let a couple be applied to its lower end in a plane perpendicular to its length (with its axis coinciding with that of the cylinder) such that it is twisted through an angle  $\theta$ . This naturally brings into play a resisting or restoring couple tending to oppose the twisting couple applied, the two balancing each other in the position of equilibrium. The restoring couple is due to the shear elasticity of the cylinder and is called the *shearing couple*.

To calculate the moment of the shearing couple, let us imagine the cylinder to consist of a large number of hollow, co-axial cylinder, one inside the other. Let us consider one such cylinder of radius  $x$  and width  $dx$  situated at a distance  $l$  from the fixed upper end (Fig. 9.10). Let an element at  $P$  on the cylinder in the undisplaced position be shifted to  $P'$  through angle  $\theta$ . Then the linear shift  $PP' = x\theta$  and the shearing strain  $= \frac{x\theta}{l}$ . Hence, the shearing

stress  $= \eta \cdot \frac{x\theta}{l}$  where  $\eta$  is the modulus of rigidity of the material of the cylinder. [ It may be readily seen that each radius of the base of the cylinder will turn through the same angle  $\theta$ , but the displacement  $PP'$  will be maximum at the rim, i.e., when  $x = r$ , progressively decreasing to zero at the centre  $O$  indicating that the stress is not uniform all over ].

Now the shearing force on the whole of the cylinder (of radius  $x$ )  
 $= \frac{\eta x \theta}{l} \times 2\pi x \cdot dx$  where  $2\pi x \cdot dx$  is the area of the cylinder of radius  $x$   
 and width  $dx$ .

The moment of this force about the axis  $OO'$  of the cylinder

$$= 2\pi x \cdot dx \cdot \frac{\eta x \theta}{l} \cdot x$$

$$= \frac{\eta 2\pi \theta}{l} \cdot x^3 \cdot dx$$

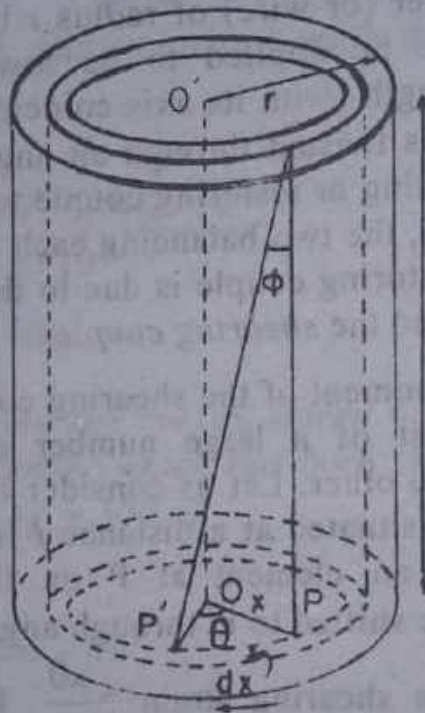


Fig. 9.10

Therefore, the moment of the shearing or twisting couple on the whole cylinder

$$= \int_0^r \frac{\eta 2\pi \theta}{l} \cdot x^3 \cdot dx$$

$$= \frac{1}{4} \frac{\eta 2\pi \theta}{l} r^4$$

$$= \frac{\pi \eta r^4 \theta}{2l} \quad (9.21)$$

If  $\theta = 1$  radian, we have, twisting couple per unit twist of the cylinder or wire

$$= \frac{\pi \eta r^4}{2l} \quad (9.22)$$

The twisting couple per unit twist of the wire is also called the *torsional rigidity* of the material of the cylinder.

**Example 9.18.** A wire 0.32 mm in diameter elongates by 1 mm, when stretched by a force of 330 gms-wt and twists through 1 radian when equal and opposite torques of 145 dynes-cm are applied to its ends. Find the value of Poisson's ratio for the wire.

**Soln.**

$$F = 330 \text{ gms-wt} = 330 \times 981 \text{ dynes}$$

$$\text{elongation in length, } l = 1 \text{ mm} = 0.1 \text{ cm.}$$

$$A = \pi r^2 = 3.14 \left( \frac{0.032}{2} \right)^2$$

$$= 3.14 \times (0.016)^2$$

Let  $L$  be the original length of the wire. Then

$$Y = \frac{F/A}{l/L} = \frac{F.L}{l.A}$$

$$= \frac{330 \times 981 \times L}{3.14 \times (0.016)^2 \times 0.1} \text{ dynes/cm}^2 \quad (i)$$

Couple acting on the wire

$$= 145 \text{ dynes-cm.}$$

Angle of twist = 1 radian.

$$\text{Hence, couple per unit twist} = \frac{145}{1}$$

$$= 145 \text{ dynes-cm.}$$



If  $\eta$  is the modulus of rigidity, then the couple per unit twist is also given by  $\frac{\pi\eta r^4}{2L}$  where  $r$  is the radius of the wire.

$$\therefore \frac{\pi\eta r^4}{2L} = 145$$

$$\text{or, } \eta = \frac{145 \times 2L}{3.14 \times (0.016)^4} \quad \text{(ii)}$$

From eqns. (i) and (ii), we have

$$\begin{aligned} \frac{Y}{\eta} &= \frac{300 \times 981 \times L}{3.14 \times (0.016)^2 \times 0.1} \times \frac{3.14 \times (0.016)^4}{145 \times 2L} \\ &= \frac{300 \times 981 \times (0.016)^2}{2 \times 0.1 \times 145} \times \frac{82.875}{29} \simeq 2.86 \end{aligned}$$

$$\text{But } \frac{Y}{\eta} = 2(1 + \sigma)$$

$$\text{or, } (1 + \sigma) = \frac{2.86}{2} = 1.43$$

$$\therefore \sigma = 1.43 - 1 = 0.43$$

**Example 9.19.** A wire of length 1m and diameter  $10^{-3}$  m is stretched by  $6 \times 10^{-4}$  m by a load 10 kg. The wire is twisted by  $70^\circ$  by a force of  $5 \times 10^3$  kg-wt applied to each end of 0.2 m length rod soldered at its mid point to the end of the wire. Calculate (i)  $Y$  (ii)  $\eta$  (iii)  $K$  and (iv)  $\sigma$ .

**Soln.**

$$(i) \quad Y = \frac{F/A}{l/L} = \frac{FL}{lA}$$

$$F = mg; m = 10 \text{ kg}; g = 9.8 \text{ m/sec}^2$$

$$A = \pi r^2; r = 5 \times 10^{-4} \text{ m}$$

$$L = 1 \text{ m}; l = 6 \times 10^{-4} \text{ m}$$

$$\therefore Y = \frac{10 \times 9.8 \times 1}{6 \times 10^{-4} \times \pi (5 \times 10^{-4})^2}$$

$$= 20.79 \times 10^{10} \text{ N/m}^2.$$

(ii) Total torque applied on the rod,

$$\tau = \frac{\pi \eta r^4 \theta}{2l}$$

when  $\eta$  = modulus of rigidity

$$\text{or, } \eta = \frac{\tau \cdot 2l}{\pi r^4 \theta}$$

$$\tau = F \times l = 5 \times 10^{-3} \times 9.8 \times 0.2$$

$$= 9.8 \times 10^{-3} \text{ N-m}$$

$$l = 1 \text{ m}, r = 5 \times 10^{-4} \text{ m}$$

$$\theta = 70^\circ = \frac{70 \times 2\pi}{360} \text{ radian.}$$

$$\therefore \eta = \frac{2 \times 9.8 \times 10^{-3} \times 1 \times 360}{\pi \times (5 \times 10^{-4})^4 \times 140\pi}$$

$$= 8.798 \times 10^{10} \text{ N/m}^2.$$

(iii) We have

$$\frac{9}{Y} = \frac{3}{\eta} + \frac{1}{K}$$

$$\text{or, } \frac{9}{20.79 \times 10^{10}} = \frac{3}{8.798 \times 10^{10}} + \frac{1}{K}$$

$$\text{or, } 0.4329 \times 10^{-10} = 0.3409 \times 10^{-10} + \frac{1}{K}$$

$$\text{or, } \frac{1}{K} = (0.4329 - 0.3409) \times 10^{-10}$$

$$= 0.092 \times 10^{-10}$$

$$\text{or, } K = \frac{1}{0.092 \times 10^{-10}}$$

$$= 10.869 \times 10^{10} \text{ N/m}^2.$$

(iv) We have

$$\sigma = \frac{3K - 2\eta}{6K + 2\eta}$$

$$= \frac{3 \times 10.869 \times 10^{10} - 2 \times 9.798 \times 10^{10}}{6 \times 10.869 \times 10^{10} + 2 \times 8.798 \times 10^{10}}$$

$$= \frac{15.011 \times 10^{10}}{82.910 \times 10^{10}}$$

$$= 0.181$$

### 9.17 Determination of Young's modulus by Searle's apparatus

Searle's apparatus consists of two metal frame-works AB and CD supported by two similar wires W and W' hanging from the same support P (Fig. 9.11). The metal frames are loosely fitted by a cross-piece K so that there is no twisting of the suspended wires and that any one of the frames can slide vertically through a small range. One end of a sensitive spirit level L rests on an arm pivoted to the frame AB while the other end rests on a platform supported by a micrometer screw S which moves against a short linear scale R fixed at this arm. Two hooks are attached to the lower parts of the two frames AR and CD for supporting the stretching weight M and a scale pan (not shown). Sufficient load is also put on the scale pan to keep the wire attached to it stretched and hence straight. This load which may be from 1-2 kg, depending on smaller or larger thickness of the wire, is called *dead load*. In calculating results the dead load is taken as *zero load*. The micrometer screw is then



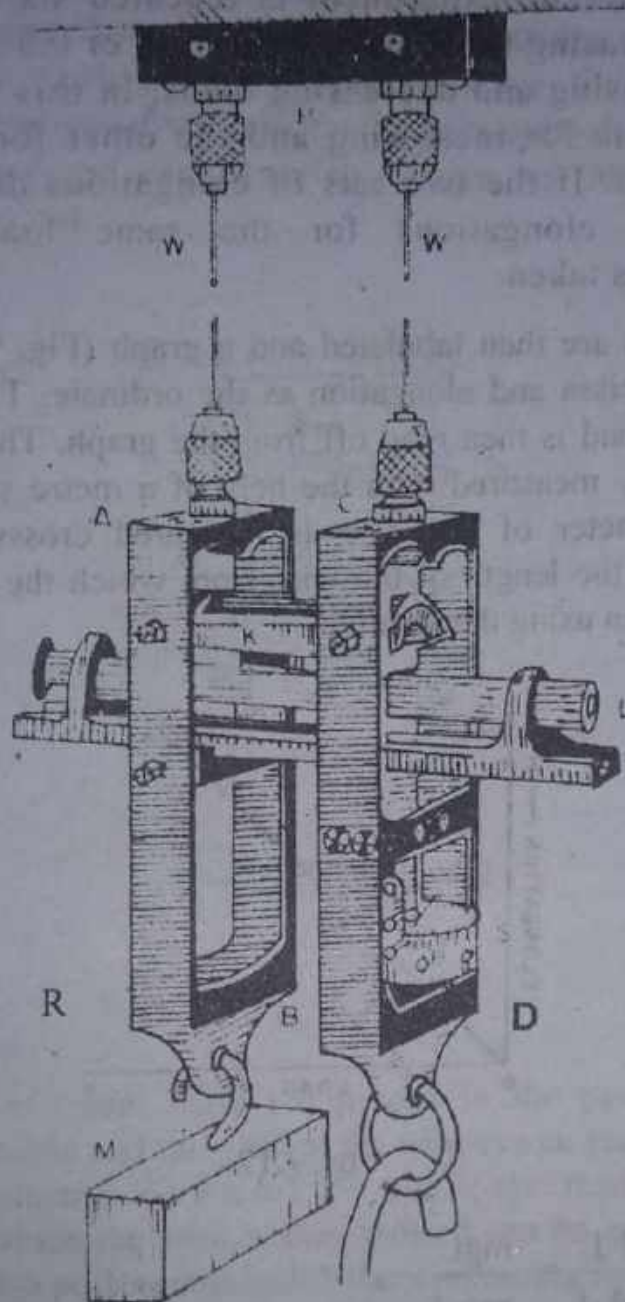


Fig. 9.11

the dead load is taken as *zero load*. The micrometer screw is then worked to bring the bubble of the spirit level to the centre. The load on the scale pan is then increased by say about 1 or 0.5 kg, so that the wire carrying the scale pan is stretched. As a result the air bubble is displaced. The micrometer screw is adjusted till the bubble is brought back to the centre. The difference between the two micrometer readings gives the elongation of the wire due to

the load added. The experiment is repeated six or seven times, each time increasing the load by steps of 1 or 0.5 kg. This is done both for increasing and decreasing loads. In this way two sets of elongations, one for increasing and the other for the decreasing loads are taken. If the two sets of elongations do not agree, the mean of the elongations for the same load in the two observations, is taken.

The results are then tabulated and a graph (Fig. 9.12) drawn with load as the abscissa and elongation as the ordinate. The elongation for any particular load is then read off from the graph. The length  $L$  of the wire is carefully measured with the help of a metre scale. By a screw gauge the diameter of the wire is measured crosswise at different positions along the length of the wire from which the mean radius  $r$  is determined. Then using the formula

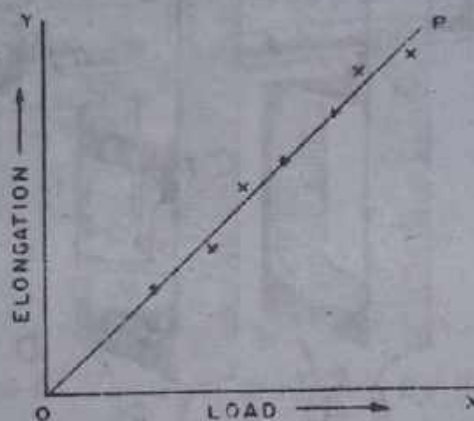


Fig. 9.12

$$Y = \frac{F.L}{A.l} = \frac{mgL}{\pi r^2 . l} \quad (9.23)$$

where  $m$  is the mass of the load and  $g$  the acceleration due to gravity, the Young's modulus of the material of the wire can be determined.

### 9.18 Determination of the rigidity modulus of a wire (statical method)

The apparatus used is shown in (Fig. 9.13). The wire  $W$ , the value of  $\eta$  for the material of which is to be determined is fixed at

its upper end at the torsion head H, the free end of it being firmly attached to a heavy metal cylinder C. Two strings are wound round the cylinder in such a way that they leave the cylinder at the opposite extremities of a diameter in two opposite direction, pass over two pulleys and carry two scale pans (or hangers) on which loads are to be placed.

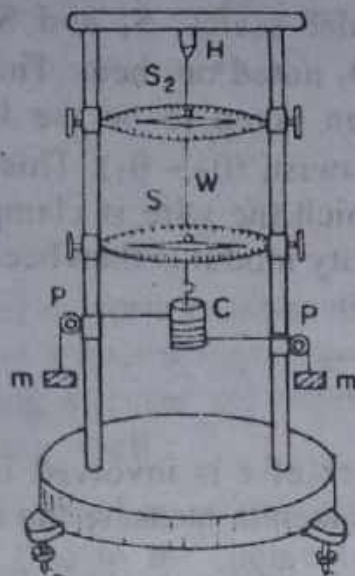


Fig. 9.13

As loads of equal mass are placed in the pans, the cylinder experiences a couple and turns about the wire as axis (coincident with its own axis of symmetry). As a result the wire is also made to turn and the angle through which the wire is thus twisted can be read on a circular scale over which a pointer attached to the wire moves freely. The circular scale is graduated in degrees and placed above the cylinder.

Let  $l$  be the length of the wire from the support to the top of the cylinder,  $m$  the mass of the load (including the scale pan) and  $\theta$ , the deflection or twist corresponding to the load  $m$ .

If  $d$  is the diameter of the cylinder, then the couple exerted by the string on the cylinder =  $mgd$ . This is also the twisting couple applied to the wire. The restoring couple set up in length  $l$  of the

wire =  $\frac{\pi \eta r^4}{2l} \cdot \theta$  as given by eqn. (9.21), Art 9.17.



Since in the position of equilibrium, the two couples balance each other, we have

$$\frac{\pi \eta r^4}{2l} \cdot \theta = mgd$$

or,  $\eta = \frac{2lmgd}{\pi r^4 \theta}$  (9.24)

Actually two circular scales,  $S_1$  and  $S_2$  are arranged and the angles of twist  $\theta_1$  and  $\theta_2$  noted on them. The effective length of the wire used, *i.e.*,  $l$  is then taken to be the length between the two scales and the angle of twist,  $(\theta_2 - \theta_1)$ . This eliminates any error as to the exact point at which the wire is clamped at the torsion head. The expression for rigidity modulus then becomes,

$$\eta = \frac{2lmgd}{\pi r^4 (\theta_2 - \theta_1)} \quad (9.25)$$

As the fourth power of  $r$  is involved in the expression for  $\eta$ , particular care must be taken to measure the radius.

If  $(\theta_2 - \theta_1)$  be measured in degrees, then  $(\theta_2 - \theta_1)$  in radian =  $(\theta_2 - \theta_1)^\circ \times \frac{\pi}{180}$

Then

$$\eta = \frac{360lmgd}{\pi^2 r^4} \times \frac{1}{(\theta_2 - \theta_1)} \quad (9.26)$$

### 9.19 Deformation by bending

A metal or any solid may be deformed by bending as illustrated in Fig. 9.14. In Fig. 9.14 (a) one end A of the bar AB is fixed while a force  $P = mg$  is applied downwards to the other free end B. Due to the force, B bends down,  $d$  being the amount of depression. In Fig. 9.14 (b), the bar AB is supported at two points equidistant from its middle point C. When a downward force  $P = mg$  is applied at C, the bar bends down at the middle,  $d$  being the amount of depression. In each case  $d$  is proportional to  $P$  approximately in accordance with Hooke's law.

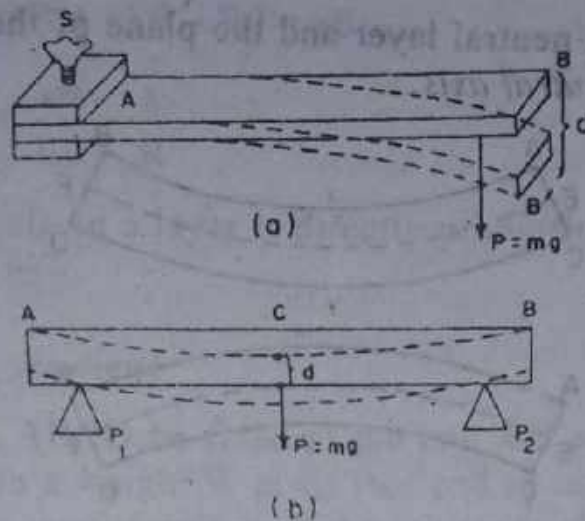


Fig. 9.14

A rod or a bar of uniform cross-section, circular or rectangular, the length of which is very much greater than its thickness is called a beam. The shearing stresses set up over any section of such a structure are negligibly small.

When a beam is fixed at one end and a load is applied at the other end, it bends due to the moment of the applied force. The plane of bending is the same in which the couple acts. The deformation of the beam sets up restoring forces and for the condition of equilibrium, the bending couple will be equal and opposite to the restoring or resisting couple. The beam, so bent from its normal shape, will of course return to its original shape once the deforming forces are withdrawn provided it has not been strained beyond the elastic limit.

In whatever manner the beam may be deformed by the force applied to it, one of the surfaces becomes concave while the other convex. As can be seen from Fig. 9.15, the layers near the concave side [AB in (a) and CD in (b)] become shortened or compressed while the layers near the convex side [CD in (a) and AB in (b)] become lengthened or extended. In between these two portions, there is a layer or surface, in which the material of the beam is neither compressed nor extended, the original length EF remaining constant. This surface layer undergoing no deformation is called the *neutral surface* of the beam. The line EF which is the line of



intersection of the neutral layer and the plane of the applied couple is known as the *neutral axis*.

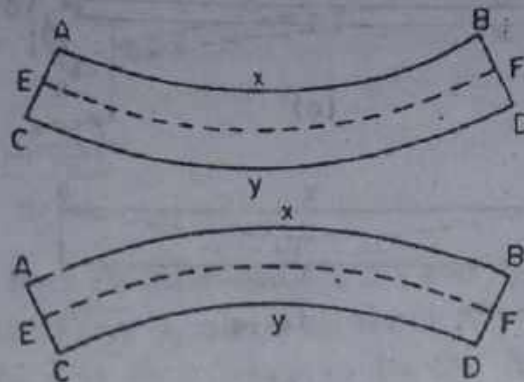


Fig. 9.15

When there is no strain in the beam, the neutral surface remains a plane surface. The filament of this surface lying in the plane of symmetry of the bent beam is called the *neutral filament*.

When the beam is bent, the amount by which the length of any filament changes is proportional to its distance from the neutral surface. This can be proved as follows :

Let a small portion of the bent beam along the neutral axis form the arc  $mn$  of a circle of radius  $R$  and let it subtend an angle  $\theta$  at the centre of curvature  $O$  (Fig. 9.16). Let  $m'n'$  be the length of an element at a distance  $x$  from the neutral axis. In the unstrained condition of the beam  $m'n' = mn$ .

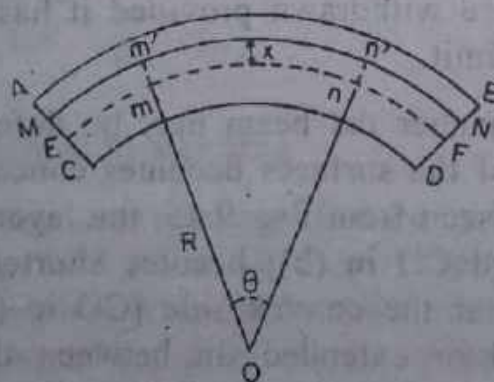


Fig. 9.16

Now  $mn = R\theta$  and  $m'n' = (R + x)\theta$ .

$\therefore$  increase in length,



$$m'n' - mn = (R + x) \theta - R\theta = x\theta.$$

$$\therefore \text{strain} = \frac{x\theta}{R\theta} = \frac{x}{R} \quad (9.27)$$

Thus the strain in a layer is directly proportional to its distance from the neutral axis.

## 9.20 Bending moment

Let a beam ABCD be fixed at the end AD and loaded (within elastic limit) with a weight W at its free end so as to be bent a little and remaining in equilibrium in that position as shown in Fig. 9.17. PBCP' is a section of this bent beam, cut by a plane at right angles to its length and its plane of bending.

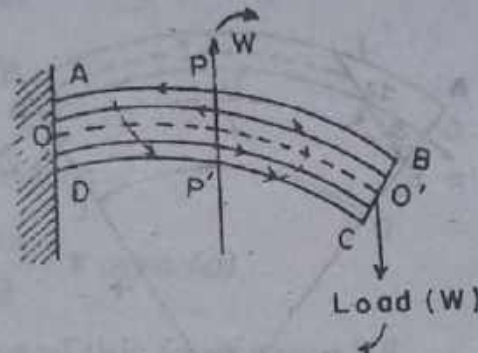


Fig. 9.17

It is clear that the load W acting vertically downwards at its end gives rise to an equal and opposite reactional force W at P but acting vertically upwards. The two forces thus form a couple tending to bend the section clockwise as indicated by full line arrows. This couple, due to the applied load W, is called the *bending couple*.

Since the section is in equilibrium, there must be an equal and opposite couple in the section. This is brought into play due to tensile and compressive stresses set up in the upper and lower halves of the section respectively. The filaments above the neutral axis OO' being elongated and hence in a state of tension, exert an inward pull on the filaments next to them towards the fixed end. Similarly the filaments below the neutral axis being shortened and hence in a state of compression exert an outward pull on the filaments next to them towards the loaded end. These pulls are indicated by arrowheads in the two halves respectively.

The inward and the outward pull form a pair of equal and opposite forces or a couple which tend to bend the section in a direction opposite to that due to the applied couple, *i.e.*, in an anticlockwise direction, as indicated by the dotted arrows. As this couple opposes or resists the bending of the beam by the applied couple and since in the equilibrium position its moment is equal in magnitude, though opposite in direction, to that of the bending couple, the moment of this couple is referred to as the *bending moment* ( $M$ ).

Let us consider a small element ABDC of the beam bent in the form of an arc of a circle of radius  $R$  (Fig. 9.18). The neutral surface EF subtends an angle  $\theta$  at the centre of curvature  $O$ . Evidently  $EF = R\theta$ .

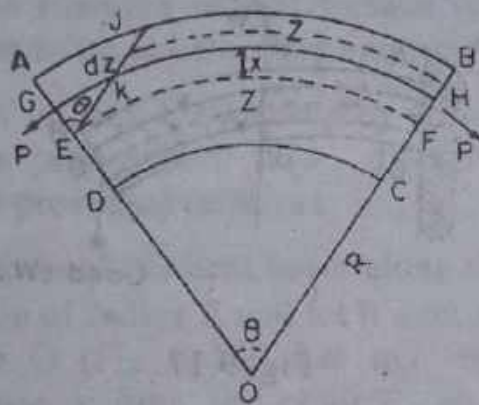


Fig. 9.18

Let EJ be drawn parallel to FB cutting GH at K. Now GH is the length of the stretched filament HK lying at a distance  $x$  above the neutral surface so that  $KH = EF = Z$  where  $Z$  is the normal or unstretched length. Then the extension of length  $KH = KG = dz$  (say).

$$\therefore \text{tensile strain} = \frac{\text{increase in length}}{\text{original length}} = \frac{dz}{Z}$$

If  $\alpha$  is the area of cross-section of the filament, then the tensile stress =  $P/\alpha$  where  $P$  is the magnitude of the applied force and hence that of the internal force. As the change of shape is not appreciable, shearing stress may be neglected.



Now the Young's modulus for the material of the beam is given by

$$Y = \frac{\text{tensile stress}}{\text{tensile strain}}$$

$$= \frac{P/\alpha}{dz/Z}$$

$$\text{or, tensile stress, } P/\alpha = Y \cdot \frac{dz}{Z}$$

$$= \text{young's modulus} \times \text{strain}$$

$$\text{But } dz = x\theta \text{ and } Z = R\theta$$

$$\therefore P/\alpha = Y \cdot \frac{x\theta}{R\theta} = Y \cdot \frac{x}{R}$$

$$\text{or, } P = \frac{Yx\alpha}{R}$$

$$\text{or, force on the area } \alpha$$

$$= \text{stress} \left( \frac{Yx}{R} \right) \times \text{area} (\alpha)$$

Then the moment of this force about EF

$$= P \cdot x = \frac{Yx\alpha}{R} \cdot x = \frac{Yx^2\alpha}{R}$$

Since moment of all the forces on filaments like GH both in the upper and lower halves of the section act in the same direction, we have the total moment of the forces acting on the filaments in the element ABDC

$$= \sum \frac{Yx^2}{R} \cdot \alpha$$

$$= \frac{Y}{R} \sum \alpha x^2$$

The term  $\sum \alpha x^2$  is analogous to the moment of inertia about the neutral axis and is called the *geometric moment* of the cross-section (or area)  $\alpha$  about that axis. This is denoted by  $I_g$  to



distinguish it from the mechanical moment of inertia for which we use the symbol  $I$ . It is also called the *second moment of inertia*.

Therefore, the total moment of all the forces about EF, or the bending moment of the beam,

$$M = \frac{YI_g}{R} \quad (9.28)$$

This bending moment obviously balances the moment of the external force acting at the section.

The quantity  $YI_g$ , measures the resistance of the beam to bending and is called the *flexural rigidity* of the beam. It may be defined as the external bending moment which will produce a bending of unit radius of curvature ( $R = 1$ )

Thus,

bending moment of a beam

$$= \frac{\text{flexural rigidity}}{R}$$

For a beam of rectangular cross-section of breadth  $b$  and depth

$$d, I_g = \frac{bd^3}{12}. \text{ Thus the bending moment } (M) = \frac{Ybd^3}{12R} \quad (9.29)$$

And for a beam of circular cross-section of radius  $r$ ,  $I_g = \frac{\pi r^4}{4}$

Therefore,

$$\text{bending moment } (M) = \frac{Y\pi r^4}{4R} \quad (9.30)$$

### 9.21 The cantilever

A beam rigidly fixed at one end and loaded at the other is called a *cantilever*.

Let AB be a cantilever of length  $l$ , having one end rigidly fixed at A and the other end B loaded (within the elastic limit) with a weight  $w$ , so that the end B is bent and, therefore, depressed into the dotted position (Fig. 9.19). As the bending is small, the end B may be taken to be practically vertically below its earlier position. The neutral

axis  $OO'$  will now take up the position  $OO''$  so that the depression (or deflection) of the loaded end is  $O'O''$ .

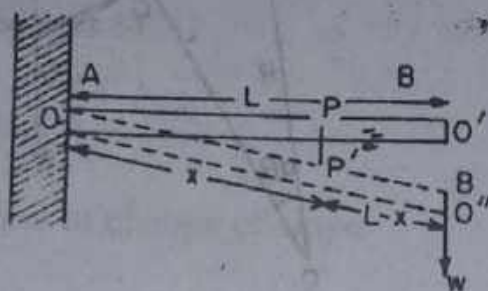


Fig. 9.19

Consider a section PB of the beam by a plane passing through P at right angles to its length and the plane of bending, a distance  $x$  from the fixed end A. Then the external bending moment acting on the section due to load  $w = w \times P'O'' = w(l-x)$ .

For equilibrium of the beam, this must be balanced by the internal bending moment  $YI_g/R$  (vide Art. 9.21) due to tensile and compressive stresses set up inside the beam. Here  $Y$  is the Young's modulus for the material of the beam,  $R$  is the radius of curvature of the neutral axis of the section at P.

We, therefore, have

$$w(l-x) = \frac{YI_g}{R} \quad (9.31)$$

The expression for  $1/R$  may be obtained as follows :

As the moment of the load increases from P to the fixed end A, the radius of curvature varies from point to point as one proceeds from P to A. Let Q be a point at very small distance  $\delta x$  from P' on a flat curve  $AO''$  so that the radius of curvature of the portion P'Q is  $R$  with C as the centre of curvature (Fig. 9.20). Let DP' and EQ be tangents to the curve at P' and Q respectively. And let the angle subtended by P'Q at C ( $\angle P'CQ$ ) be  $d\theta$ . Then the angle  $\angle EE'D$  subtended by P'Q at C is evidently  $d\theta$  since CP' and CQ are perpendicular to the tangents at P' and Q.

Thus, if the curvature ( $1/R$ ) be small (as it usually is in the case of bent bars or beams) its value at any point on the curve is given by the rate of change of slope at that point.

Eqn. (9.31), then reduces to

$$w(l-x) = YI_g \frac{d^2y}{dx^2}$$

where  $\frac{d^2y}{dx^2}$  is the rate of change of slope.

$$\text{or, } \frac{d^2y}{dx^2} = \frac{w}{YI_g} (l-x)$$

Integrating, we have

$$\frac{dy}{dx} = \frac{w}{YI_g} \left( lx - \frac{x^2}{2} \right) + C_1 \quad (9.32)$$

where  $C_1$  is a constant of integration.

Since the end A is fixed we have  $\frac{dy}{dx} = 0$  at  $x = 0$ . Therefore,

$$C_1 = 0.$$

Eqn. (9.32) then becomes

$$\frac{dy}{dx} = \frac{w}{YI_g} \left( lx - \frac{x^2}{2} \right)$$

Integrating once again, we have

$$y = \frac{w}{YI_g} \left( l \cdot \frac{x^2}{2} - \frac{x^3}{6} \right) + C_2 \quad (9.33)$$

where  $y$  is the depression or deflection of the beam at a distance  $x$  from the fixed end and  $C_2$  is yet another constant of integration.

Now the depression  $y$  of the beam at A is zero, i.e.,  $y = 0$  at  $x = 0$ , so that  $C_2 = 0$ .

Eqn. (9.33) then becomes



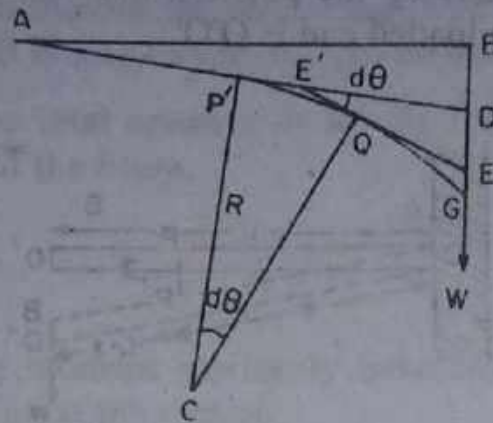


Fig. 9.20

Then, we have

$$P'Q = \delta x = R \cdot d\theta ; \quad \text{or, } R = \frac{\delta x}{d\theta}$$

Since  $d\theta$  is the change in slope of the tangents from  $P'$  to  $Q$ , we have

$$d\theta = \left( \frac{dy}{dx} \right)_Q - \left( \frac{dy}{dx} \right)_P$$

Since the rate of change of slope with distance is  $\frac{d}{dx} \left( \frac{dy}{dx} \right)$  or

$$\frac{d^2y}{dx^2}, \text{ the change in slope from } P' \text{ to } Q \text{ is equal to } \frac{d^2y}{dx^2} \cdot \delta x.$$

We, therefore, have

$$d\theta = \frac{d^2y}{dx^2} \cdot \delta x$$

$$\text{and } \delta x = R \cdot d\theta = R \cdot \left( \frac{d^2y}{dx^2} \right) \cdot \delta x$$

$$\text{whence } R \cdot \left( \frac{d^2y}{dx^2} \right) = 1.$$

$$\text{or, } \frac{1}{R} = \frac{d^2y}{dx^2}$$

$$y = \frac{w}{YI_g} \left( l \frac{x^2}{2} - \frac{x^3}{6} \right) \quad (9.34)$$

Since the free (or the loaded) end B of the beam is at a distance  $l$  from A, we have  $x = l$  and, therefore,

*Depression of the loaded end B is given by*

$$y = \frac{w}{YI_g} \left( \frac{l^3}{2} - \frac{l^3}{6} \right) = \frac{w}{YI_g} \cdot \frac{2l^3}{6}$$

or,  $y = \frac{wl^3}{3YI_g} \quad (9.35)$

The slope of the end B relative to end A (or its inclination to horizontal) is given by

$$\tan \theta = \theta = \left( \frac{dy}{dx} \right)_{x=l} = \frac{wl^2}{3YI_g}$$

If the beam is of rectangular cross-section of breadth  $b$  and depth  $d$ ,  $I_g = \frac{bd^3}{12}$

We, therefore, have

$$y = \frac{wl^3}{3Y(bd^3/12)} = \frac{4wl^3}{Ybd^3} \quad (9.36)$$

and for a beam of circular cross-section of radius  $r$ ,  $I_g = \frac{\pi r^4}{4}$ , so that

$$y = \frac{wl^3}{3Y(\pi r^4/4)} = \frac{4}{3} \frac{wl^3}{Y\pi r^4} \quad (9.37)$$

## 9.22 Depression of a beam fixed at two ends and loaded in the middle

Let a beam AB be supported on two knife-edges at its two ends A and B, distance  $l$  apart. Let it be loaded in the middle at O with a weight  $w$  (Fig. 9.21).

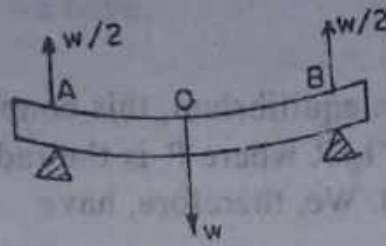


Fig. 9.21

As A and B are equidistant from O, the reaction at each knife-edge will be in the upward direction and equal to  $w/2$ . Since the middle part of the beam being sensibly horizontal, the whole beam may be taken to be a combination of two inverted cantilevers OA and OB, each of effective length  $l/2$ , fixed at O and bending upwards under a load  $w/2$  acting at A and B. The elevation of A and B above O or the depression of O below A and B is clearly given by

$$y = \frac{(w/2)(l/2)^3}{3YI_g}$$

$$= \frac{wl^3}{48YI_g} \quad (9.38)$$

However, the expression for  $y$  can be directly obtained as follows :

Consider the section PB of the cantilever OB, a distant, say  $x$ , from its fixed end O (Fig. 9.22). The moment of the bending couple due to the load  $w/2$

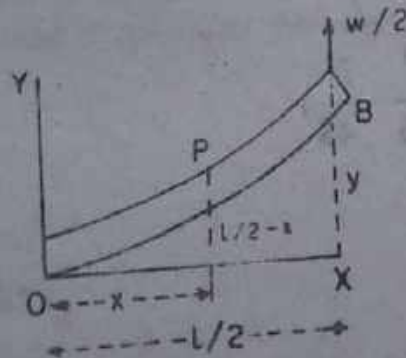


Fig. 9.22



$$= \frac{w}{2} \left( \frac{l}{2} - x \right)$$

The beam being in equilibrium, this couple must be balanced by the bending moment  $YI_g/R$  where  $R$  is the radius of curvature of the section at  $P$  (Art. 9.21). We, therefore, have

$$\frac{YI_g}{R} = YI_g \frac{d^2y}{dx^2} = \frac{w}{2} \left( \frac{l}{2} - x \right)$$

$$\text{or, } \frac{d^2y}{dx^2} = \frac{w}{2YI_g} \left( \frac{l}{2} - x \right)$$

which on integration gives,

$$\frac{dy}{dx} = \frac{w}{2YI_g} \left( l \cdot \frac{x}{2} - \frac{x^2}{2} \right) + C$$

$C$ , being a constant of integration.

Since  $\frac{dy}{dx} = 0$  at  $x = 0$ ,  $C = 0$ .

$$\text{Therefore, } \frac{dy}{dx} = \frac{w}{2YI_g} \left( l \cdot \frac{x}{2} - \frac{x^2}{2} \right)$$

$$\text{or, } dy = \frac{w}{2YI_g} \left( l \cdot \frac{x}{2} - \frac{x^2}{2} \right) \cdot dx \quad (9.39)$$

Eqn. (9.39) on further integration between the limits  $x = 0$  and  $x = l/2$ , gives

$$\begin{aligned} y &= \frac{w}{2YI_g} \left( \frac{l^3}{16} - \frac{l^3}{48} \right) \\ &= \frac{wl^3}{48YI_g} \end{aligned} \quad (9.40)$$

The same expression as eqn. (9.38) above.

If the cross-section of the beam be rectangular of breadth  $b$  and depth  $d$ , then

$$I_g = \frac{bd^3}{12} \text{ and we have,}$$

$$y = \frac{wl^3}{48Y} \left( \frac{12}{bd^3} \right)$$

$$= \frac{wl^3}{4Ybd^3} \quad (9.41)$$

And if the beam be of circular cross-section of radius  $r$ , then

$$I_g = \frac{\pi r^4}{4} \text{ and we have}$$

$$y = \frac{wl^3}{48Y} \left( \frac{4}{\pi r^4} \right)$$

$$= \frac{wl^3}{12Y\pi r^4} \quad (9.42)$$

#### determination of Young's modulus

Eqn. (9.42) can be rearranged as

$$Y = \frac{wl^3}{4ybd^3} \quad (9.43)$$

Knowing  $w$ ,  $l$ ,  $b$ ,  $d$  and  $y$ ,  $Y$ , the Young's modulus can be determined.

**Example 9.20.** A metallic strip of width 2 cms and thickness 3 mm, supported horizontally on knife-edges 80 cms apart, is loaded with 50 gms at its middle. Find by how much the centre of the metallic strip is depressed. Young's modulus for the material is  $20 \times 10^{11}$  dynes/cm<sup>2</sup>.

**Soln.**

$$\text{depression, } y = \frac{wl^3}{48YI_g}$$

$$= \frac{wl^3}{48Y} \times \frac{12}{bd^3} = \frac{wl^3}{4Ybd^3}$$

$$w = 50 \times 980, l = 80 \text{ cm},$$

$$Y = 20 \times 10^{11} \text{ dyne/cm}^2.$$

$$b = 2 \text{ cm}, d = 0.3 \text{ cm}.$$

$$\begin{aligned} \therefore y &= \frac{50 \times 980 \times (80)^3}{4 \times 20 \times 10^{11} \times 2 \times (0.3)^3} \\ &= \frac{250880 \times 10^5}{432 \times 10^9} \\ &= 0.0580 \text{ cm}. \end{aligned}$$

**Example 9.21.** A uniform rod of length 1m is clamped horizontally at one end. A weight of 0.1 kg is attached at the free end. Calculate the depression of the midpoint of the rod. The diameter of the rod is 0.02m.

**Soln.**

$$Y = 10^{10} \text{ N/m}^2.$$

The general formula for depression at a point which is at a distance  $x$  from the clamped end is given by

$$y = \frac{w}{YI_g} \left( \frac{lx^2}{2} - \frac{x^3}{6} \right)$$

$$\text{Here } x = \frac{l}{2}; \therefore y = \frac{w}{2YI_g} \left( \frac{l^3}{8} - \frac{l^3}{4} \right) = \frac{5wl^3}{48YI_g}$$

$$w = mg \text{ and } I_g = \frac{\pi r^4}{4}$$

$$\therefore y = \frac{5mg l^3}{12\pi Y r^4}$$

$$m = 0.1 \text{ kg}, g = 9.8 \text{ m/sec}^2$$

$$l = 1 \text{ m}, Y = 10^{10} \text{ N/m}^2, r = 0.01 \text{ m}$$

$$\begin{aligned} \therefore y &= \frac{5 \times 0.1 \times 9.8 \times (1)^3}{12\pi \times 10^{10} \times (0.01)^4} \\ &= 1.299 \times 10^{-3} \text{ m}. \end{aligned}$$



## EXERCISES

- [1] (a) Explain the terms stress, strain and elastic limit.  
(b) Define Young's modulus, bulk modulus and modulus of rigidity. If  $Y$ ,  $K$  and  $\eta$  represent these moduli respectively then show that  $Y = 9\eta K / (3K + \eta)$ .
- [2] (a) Discuss the general behaviour of material under stress with reference to a stress-strain curve.  
(b) What is Poisson's ratio of a material? Establish a relation connecting Poisson's ratio, Young's modulus and modulus of rigidity of a material.
- [3] Distinguish between elastic and plastic materials. Explain the terms yield point, elastic limit, elastic fatigue and breaking stress.
- [4] Derive the relations
  - (i)  $\eta = \frac{Y}{2(1 + \sigma)}$
  - (ii)  $\sigma = \frac{3K - 2\eta}{6K + 2\eta}$
  - (iii)  $\frac{9}{Y} = \frac{3}{\eta} + \frac{1}{K}$
- [5] What is Poisson's ratio? Show that the value of Poisson's ratio lies between  $-1$  and  $0.5$ .
- [6] Define modulus of rigidity.  
Show that a shear is equivalent to a compression and an extension at right angles to each other.
- [7] Name the four elastic constants and explain what they mean. Deduce expressions to show how they are related with each other for an isotropic medium.
- [8] Discuss the elastic behaviour of a solid metallic wire of uniform cross-section under different loads. Draw the load-extension graph and describe the properties of the distinctive region.
- [9] Describe Searle's method for measuring Young's modulus of a material in the form of a wire. Derive the formula used.
- [10] If  $Y$ ,  $K$ ,  $\eta$  and  $\sigma$  represent Young's modulus, Bulk modulus, modulus of rigidity Poisson's ratio of a substance, then derive the

relationships connecting (i)  $Y$ ,  $K$ , and  $\sigma$ ; (ii)  $Y$ ,  $\eta$  and  $\sigma$ ; and (iii)  $K$ ,  $\eta$  and  $\sigma$ . Hence find the limiting value of  $\sigma$ .

- [11] Define the coefficient of rigidity. Describe a method for determining the modulus of rigidity of a wire. Deduce the formula you would use.
- [12] (a) Explain clearly what you understand by the term rigidity. Calculate the couple required to twist a cylindrical rod of circular cross-section through an angle  $\theta$  at one end, the other end being kept fixed, and hence deduce an expression for the modulus of rigidity of the rod.

(b) Show that the couple required to twist one end of a cylindrical rod through 1 radian, the other end being fixed, is

$$C = \frac{\pi \eta r^4}{2l}$$

- [13] (a) Find the work done in stretching a wire and hence deduce an expression for the energy per unit volume.

(b) Show that in any type of strain, work done per unit volume is equal to  $\frac{1}{2}$  stress  $\times$  strain.

- [14] Prove that a shearing stress is equivalent to a linear tensile stress and an equal compression stress mutually at right angles.

- [15] What is meant by a beam? Explain the terms: neutral surface, neutral axis, flexural rigidity and bending moment of a beam.

- [16] Derive an expression for the couple required to bend a uniform straight metallic strip into an arc of a circle of small curvature.

- [17] What is a cantilever? Obtain an expression for the depression at the free end of a thin light beam clamped horizontally at one end and loaded at the other. Neglect the weight of the cantilever.

- [18] A light beam of circular cross-section is clamped horizontally at one end and a heavy mass is attached at the other end. Find the depression at the loaded end.

- [19] A light uniform beam is supported at the two ends on two knife-edges and loaded at the middle. Derive an expression for the depression of the mid-point of the beam. Neglect the mass of the beam.

- [20] Derive an expression for the restoring couple acting on a cylinder of length  $l$  and radius  $r$  when its top end is clamped and the lower end is twisted through an angle  $\theta$  about a vertical axis. Describe an accurate method for the determination of modulus of rigidity of the material of the wire.



- [21] Derive an expression for the bending moment of a horizontal beam clamped at one end and loaded at the other and hence derive an expression for the depression of a beam supported at the ends and loaded at the centre, neglecting the weight of the beam.
- [22] A wire 300 cm long and 0.625 sq. cm in cross-section is found to stretch 0.3 cm under a tension of 1200 kg. What is the Young's modulus for the material of the wire? ( $2.3 \times 10^{12}$  dynes/cm<sup>2</sup>).
- [23] A steel rod of length 5 m is fixed rigidly between two supports. The coefficient of linear expansion of steel =  $12 \times 10^{-6}$  °C. Calculate the stress in the rod for an increase in temperature of 40°C. The Young's modulus of elasticity of steel is  $2 \times 10^{11}$  N/m<sup>2</sup>. ( $9.6 \times 10^7$  N/m<sup>2</sup>).
- [24] The modulus of rigidity and Poisson's ratio of the material of a wire are  $2.87 \times 10^{10}$  N/m<sup>2</sup> and 0.379 respectively. Find the value of Young's modulus of the material of the wire. ( $7.915 \times 10^{10}$  N/m<sup>2</sup>).
- [25] Find the work done in stretching a wire of 1 sq. mm cross-section and 2m long through 0.1mm.  $Y = 2 \times 10^{11}$  N/m<sup>2</sup>. ( $5 \times 10^{-4}$  J).
- [26] A wire, 4 metres long and 0.3 mm in diameter, is stressed by a force of 800 gm-wt. If the extension in length amounts to 1.5 mm, calculate the energy stored in the wire. ( $5.88 \times 10^4$  ergs).
- [27] Find the load in kilograms to stretch a vertical wire 628 cms long and 2 mm in diameter by 1 mm in length. Young's modulus for steel =  $2 \times 10^{12}$  dynes/cm<sup>2</sup>,  $g = 980$  cm/sec<sup>2</sup>. (10.21 kg).
- [28] Find the amount of work done in twisting a steel wire of radius 1.0 mm and of length 25 cm through an angle of 45°. Given  $\eta$  for steel =  $8 \times 10^{11}$  dynes/cm<sup>2</sup>. ( $15.47 \times 10^5$  ergs).
- [29] A steel wire of 1.00 mm radius is bent in the form of a circular arc of radius 50 cm. Calculate (i) the bending moment (ii) the maximum stress. Young's modulus for steel =  $2 \times 10^{12}$  dynes/cm<sup>2</sup>. [(i)  $3.142 \times 10^6$  dyne-cm, (ii)  $4 \times 10^9$  dynes/cm<sup>2</sup>].
- [30] A cantilever of length 50 cm is depressed by 15.0 mm at the loaded end. Calculate the depression at a distance of 30 cm from the fixed end. (6.48 mm).
- [31] A steel wire of length 100 cm and cross-sectional area  $0.5 \times 10^{-5}$  cm<sup>2</sup> is stretched horizontally between two pillars. A mass of 100 gms is suspended from the middle point of the wire. Calculate the depression of the middle point.  $Y$  for steel =  $2 \times 10^{12}$  dynes/cm<sup>2</sup> (1.227 cm).



## CHAPTER X

FRICTION AND DYNAMICS  
OF RIGID BODIES

*Introduction – Static and Kinetic frictional forces – Laws of friction, Properties of friction – Rotation – Angular velocity – Angular acceleration – Moment of Inertia – Work done by a torque – Theorem of perpendicular axes – Theorem of parallel axes – Moment of inertia of some standard geometric objects – Solved examples – Exercises.*

**10.1** Frictional forces are unavoidable in our daily life. Left to alone, these forces would stop every rolling wheel and bring to a halt every rotating shaft. On the other hand, if friction were totally absent, we would not be able to walk or ride a bicycle. We would not be able to hold a pencil and, if we could, it would not write. Nails and screws will be useless, woven cloth would fall apart, and knots would come undone.

In this chapter, we shall discuss the frictional forces that exist between dry solid surfaces, moving across each other at relatively slow speeds. Let us consider the following two simple experiments :

**First experiment :** A book is sent sliding across a tabletop. The tabletop will exert a frictional force on the bottom of the sliding book. This will slow the book and eventually stop it. If we want to make the book move across the table with constant velocity, we must push or pull it with a steady force, the magnitude of which must match the magnitude of the opposing frictional force.

**Second experiment :** A heavy crate is resting on the floor of a warehouse. The crate is pushed horizontally with a steady force, but it does not move. What happens is that the force applied is exactly balanced by a frictional force, exerted horizontally on the bottom of the crate in a direction opposite to the direction of the push. Remarkably, this frictional force automatically adjusts itself, both in magnitude and direction, to cancel exactly whatever force is applied. Of course, if the crate is pushed hard enough, it will be possible to move the crate (see the first experiment).

**10.2 Static and kinetic frictional forces**

A situation similar to one discussed above is shown in greater detail in Fig. 10.1. In Fig. 10.1 (a), a block rests on a tabletop, its

weight  $W$  balanced by an equal but opposite normal force  $N$ . A force  $F$  is exerted on the block, as shown in Fig. 10.1 (b), attempting to pull it to the left. In response, a frictional force  $f_s$  arises, pointing to the right,

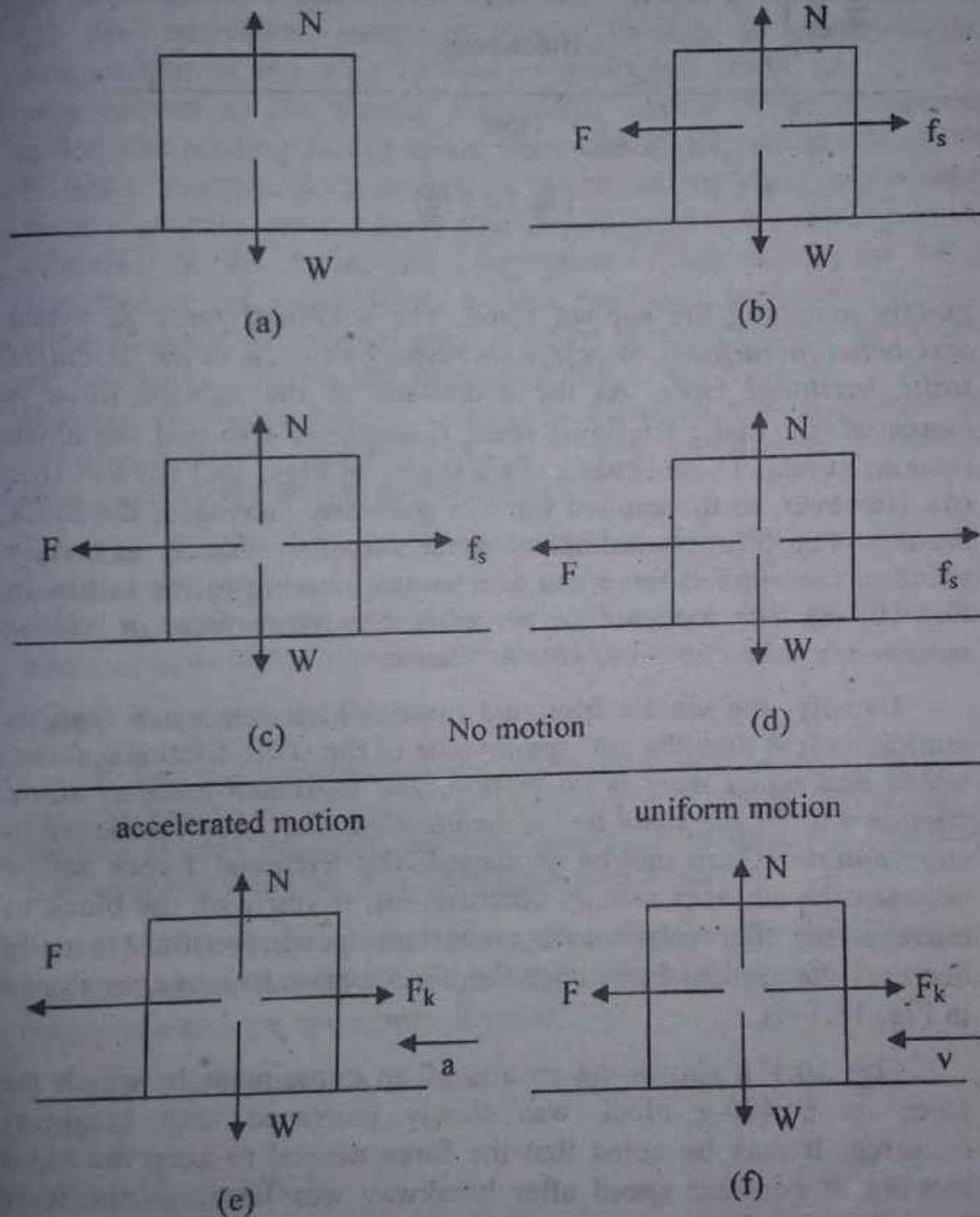


Fig. 10.1



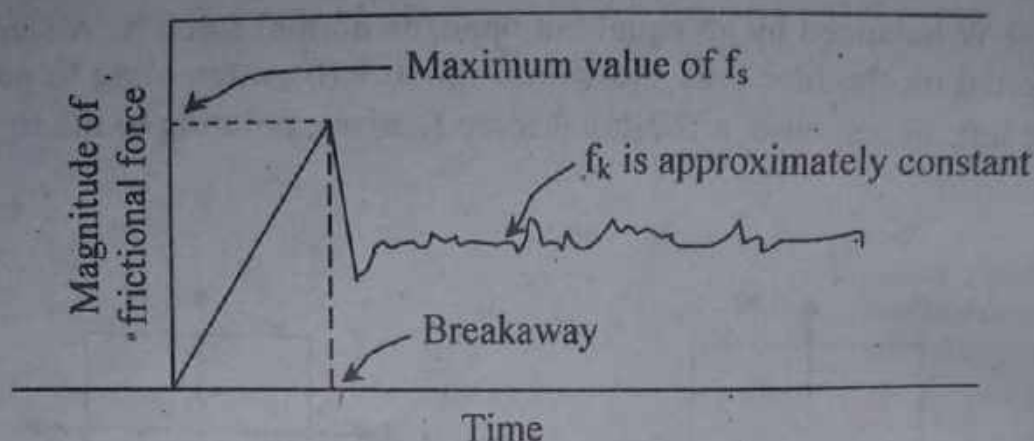


Fig. 10.1 (g)

exactly matching the applied force. The frictional force  $f_s$ , which acts between surfaces at rest with respect to each other, is called static frictional force. As the magnitude of the applied force is increased, the static frictional force  $f_s$  increases also and the block remains at rest. These situations are shown in Figs. 10.1 (c) and 10.1 (d). However, as the applied force is gradually increased, the block *breaks away* from its intimate contact with the tabletop at certain value of the applied force and accelerates, moving to the left as in Fig. 10.1(e). The frictional forces acting between surfaces in relative motion are called forces of kinetic friction.

Usually, the kinetic frictional force, which acts when there is motion, is less than the maximum value of the static frictional force, which acts when there is no motion. The maximum force of static friction will be the same as the smallest applied force necessary to start motion. Once motion is started, the frictional forces acting between the surfaces usually decrease. So, if you wish the block to move across the surface with a constant speed, you must usually decrease the applied force once the block begins to move, as shown in Fig. 10.1 (f).

Fig. 10.1 g shows the results of an experiment in which the force on a 400-g block was slowly increased until breakaway occurred. It may be noted that the force needed to keep the block moving at constant speed after breakaway was less than the force needed for breakaway.



### 10.3 Laws of friction

The maximum force of static friction between any pair of dry unlubricated surfaces follows two empirical laws which were first discovered experimentally by Leonardo da Vinci, about two centuries before Newton developed the concept of force. These are (1) *the maximum force of static friction is approximately independent of the area of contact over wide limits* and (2) *it is proportional to the normal force*. The normal force (sometimes called the loading force) arises from the elastic properties of the bodies in contact. Such bodies are never entirely rigid, and when a force is exerted on one body that is prevented from moving in the direction of the force, the body pushes back to oppose being stretched or deformed. When a block rests on a horizontal table or slides along it, the normal force is equal in magnitude to the weight of the block, which is the downward pull of the earth on the block. Because the block has no vertical acceleration, the table must be exerting a force on the block that is directed upward and is equal in magnitude to the weight of the block.

The force of kinetic friction  $f_k$  between dry, unlubricated surfaces follows the same two laws as those of static friction. It is also reasonably independent of the relative speed with which the surfaces move over each other.

### 10.4 Properties of friction

It has been established experimentally that when a body is pressed against a surface (both being dry and unlubricated) and a force  $F$  attempts to slide the body along the surface, the resulting frictional force has the following three properties :

(1) If the body does not move, then the static frictional force  $f_s$  and the component of  $F$  that is parallel to the surface are equal in magnitude and are oppositely directed.

(2) The magnitude of  $f_s$  has a maximum value  $f_{s, \max}$  and is given by

$$f_{s, \max} = \mu_s N \quad (10.1)$$

where  $\mu_s$  is the *coefficient of static friction* and  $N$  is the magnitude of the normal force. Thus, the ratio of the magnitude of the

*maximum* force of static friction to the magnitude of the normal force is called the *coefficient of static friction* for the surfaces involved. Eqn. (10.1) holds only when  $f_s$  has maximum value. In general eqn. (10.1) can be better represented by the equation

$$f_s \leq \mu_s N \quad (10.2)$$

where  $f_s$  is the magnitude of the force of static friction. The equality sign holds only when  $f_s$  has its maximum value.

If the magnitude of the component of  $F$  that is parallel to the surface exceeds  $f_{s, \max}$  then the body begins to slide along the surface.

(3) When the body begins to slide along the surface, the magnitude of the frictional force rapidly decreases to a value  $f_k$  given by

$$f_k = \mu_k N \quad (10.3)$$

where  $\mu_k$  is the *coefficient of kinetic friction*. The coefficient of kinetic friction can, therefore, be defined as the ratio of the magnitude of the force of kinetic friction to the magnitude of the normal force. During the sliding, the kinetic frictional force  $f_k$  has a magnitude given by eqn. (10.3).

Although the properties 1 and 2 are worded in terms of a single force  $F$ , they also hold for the net or resultant force due to several forces acting on the body.

The coefficients  $\mu_s$  and  $\mu_k$  are dimensionless and must be determined experimentally. Since their values depend on the body and the surface, they are usually referred to with the preposition *between* as in "the value of  $\mu_s$  *between* a sled and asphalt is 0.5". Both  $\mu_s$  and  $\mu_k$  can exceed unity, although commonly they are less than 1. Some representative values of  $\mu_s$  and  $\mu_k$  are shown in the table below. It should, however, be remembered that the values quoted are approximate and are only intended only as estimates. The actual coefficients of friction for any pair of surfaces depend on such conditions as the cleanliness of the surface, the temperature, and the humidity.



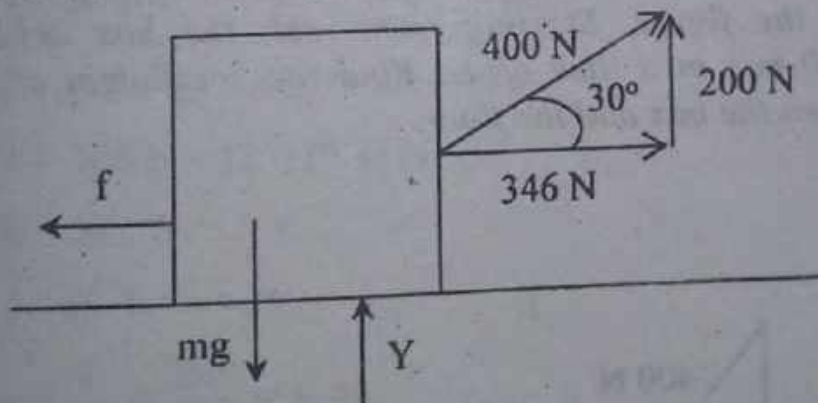
## COEFFICIENTS OF FRICTION

Surfaces	$\mu_s$	$\mu_k$
Wood on wood	0.25 – 0.5	0.2
Glass on glass	0.9 – 1.0	0.4
Steel on steel, clean surfaces	0.6	0.6
Steel on steel, lubricated	0.09	0.05
Rubber on dry concrete	1.0	0.8
Teflon on teflon	0.04	0.04

**Example 10.1.** A 70 kg box is pulled by a 400-N force at an angle of  $30^\circ$  to the horizontal. If the coefficient of sliding friction is 0.50, find the acceleration of the box.

**Soln.**

Since the box does not move vertically  $\sum F_y = ma_y = 0$ . From the figure we see that the forces acting in a vertical direction



are (i) the component of the 400-N force in the vertical direction  
 $= (400\text{-N}) \sin 30^\circ$   
 $= 200\text{ N}.$

(ii) the normal force acting on the body in a vertical direction = Y

(iii) the weight of the box acting vertically downwards.

$$\text{So, } \sum F_y = Y + 200\text{ N} - mg = 0$$



$$\text{But } mg = (70 \text{ Kg}) (9.8 \text{ ms}^{-2}) = 686 \text{ N}$$

$$\therefore Y + 200 \text{ N} - 686 \text{ N} = 0$$

$$\text{or, } Y = 486 \text{ N.}$$

The force of friction acting on the body is given by

$$f = \mu Y = (0.50) (486 \text{ N}) = 243 \text{ N}$$

Now the force acting on the body in the horizontal direction is given by

$$\begin{aligned} \sum F_x &= \text{the component of } 400 \text{ N force in the horizontal} \\ &\text{direction} - \text{the frictional force (which acts in the opposite direction)} \\ &= (400 \text{ N}) \cos 30^\circ - 243 \text{ N} = 346 \text{ N} - 243 \text{ N} \\ &= 103 \text{ N.} \end{aligned}$$

$$\text{But } \sum F_x = ma_x$$

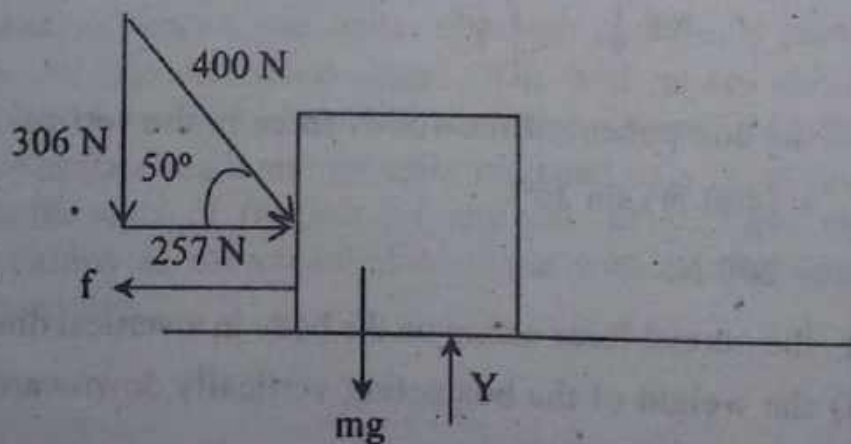
$$\therefore 103 \text{ N} = (70 \text{ kg}) \cdot a_x$$

So, the acceleration of the box,

$$a_x = \frac{103 \text{ N}}{70 \text{ kg}} = 1.47 \text{ ms}^{-2}.$$

**Example 10.2.** A force of 400 N pushes a 25 kg box on a floor as shown in the figure. Starting from rest, the box achieves a velocity of 2.0 m/s in a time of 4s. Find the coefficient of sliding friction between the box and the floor.

**Soln.**



The frictional force  $f$  can be obtained by using the relation  $F = ma$  where  $a$  is the acceleration of the box.  $a$  can be obtained using the relation

$$v_f = v_o + at$$

$$\text{or, } a = \frac{v_f - v_o}{t} = \frac{2 \text{ m/s} - 0}{4 \text{ s}} \quad (\because v_o = 0)$$

$$= 0.50 \text{ m/s}^2.$$

Now, there is no motion in the vertical direction. So for the horizontal direction, we can write

$$\sum F_x = ma_x \quad \text{where } a_x = a = 0.50 \text{ m/s}^2.$$

From the figure,

$$\sum F_x = 257 \text{ N} - f \quad \text{where } 400 \text{ N} \cos 50^\circ = 257 \text{ N}.$$

$$\therefore 257 \text{ N} - f = ma = (25 \text{ kg}) (0.50 \text{ m/s}^2)$$

$$\text{or, } f = 245 \text{ N}.$$

Since there is no motion in the vertical direction.

$$\text{We can write, } \sum F_y = ma_y = 0$$

$$\text{or, } Y - (400 \text{ N}) \sin 50^\circ - mg = 0$$

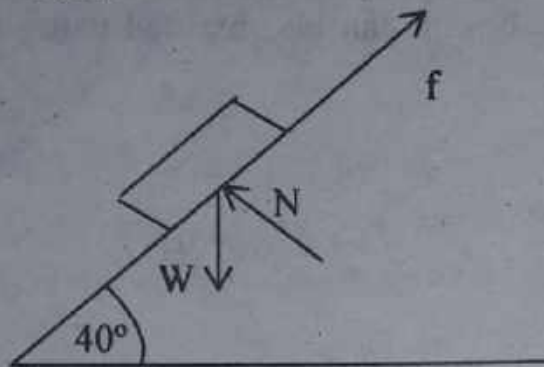
$$Y - 306 \text{ N} - (25)(9.8) \text{ N} = 0$$

$$\text{or, } Y = 551 \text{ N}.$$

from  $f = \mu \cdot Y$ , we get

$$\mu = \frac{f}{Y} = \frac{245}{551} = 0.44.$$

**Example 10.3.** A 12-kg box is released from the top of an incline that is 5.0 m long and makes an angle of  $40^\circ$  to the horizontal. A 60-N friction force impedes the motion of the box. (a) what will be the acceleration of the box and (b) how long will it take to reach the bottom of the incline? (c) what is the coefficient of friction between the box and the incline?

**Soln.**

The three forces acting on the block is shown in the figure. These are the frictional force  $f = 60\text{N}$ ; the normal force  $N$ , which is perpendicular to the incline; and the weight of the block,  $W = mg = (12\text{ kg})(9.8\text{ m/s}^2) = 118\text{ N}$ .

Let us choose the  $x$  axis along the incline with downward as positive. Using  $\sum F_x = ma_x$ , we get

$$w \sin 40^\circ - f = ma_x$$

$$(118\text{N})(0.642) - (60\text{ N}) = (12\text{ kg}) a_x$$

$$15.756\text{ N} = (12\text{ kg}) a_x$$

$$\text{or, } a_x = \frac{15.756\text{ N}}{12\text{ Kg}} = 1.313\text{ m/s}^2.$$

The time to reach the bottom can be obtained from the relation

$$x = v_{0x} t + \frac{1}{2} a_x t^2$$

$$\text{where } x = 5.0\text{ m} \quad \text{and } v_{0x} = 0$$

$$\text{or, } 5.0\text{ m} = 0 + \frac{1}{2} (1.313\text{ m/s}^2) t^2$$

$$\text{or, } t = (7.63\text{ s}^2)^{1/2} = 2.76\text{s}.$$

Again referring to the figure we find  $\sum F_y = 0$ ;

$$\text{or, } N - w \cos 40^\circ = 0$$

$$\text{or, } N - (118\text{ N})(0.766) = 0$$

$$\text{or, } N - 90.4 = 0$$

$$\therefore N = 90.4\text{ N}.$$



The coefficient of kinetic friction between box and incline is given by

$$m_k = f/N = \frac{60}{90.4} = 0.6637.$$

### 10.5 Rotation of rigid bodies

We shall discuss here the rotation of a rigid body about a fixed axis. The terms rigid body and fixed axis impose some restrictions. The first of these restrictions means that we shall not discuss the rotation of objects which are not rigid bodies – for example, the sun, a ball of gas – is not a rigid body. The second restriction rules out objects like a bowling ball rolling down a bowling lane because the ball is rotating about a moving axis-not a fixed axis.

#### Some definitions

Suppose we have a rigid body of arbitrary shape, with a fixed axis, within or without it. If a force is now applied to it, it cannot move bodily as a whole relatively to the axis, *i.e.*, no motion of translation is possible. The body simply moves round or rotates about the axis such that every particle of it undergoes the same angular displacement. A body, so rotating about a fixed axis, is said to perform rotatory or circular motion.

The fixed axis about which rotation is produced is called the *axis of rotation*. The force producing the rotation is said to have a moment about the axis of rotation which is given by the product of the force and the perpendicular distance between its line of action and the axis of rotation.

If the rotation is anti-clockwise, the moment of the force is said to be positive, and if it be in the clockwise direction, the moment is said to be negative. Since the moment of a force is a vector quantity, it follows that if a number of forces act simultaneously on a body, the algebraic sum of their individual moments about the given axis of rotation will be equal to the moment of their resultant about it.

## Angular position

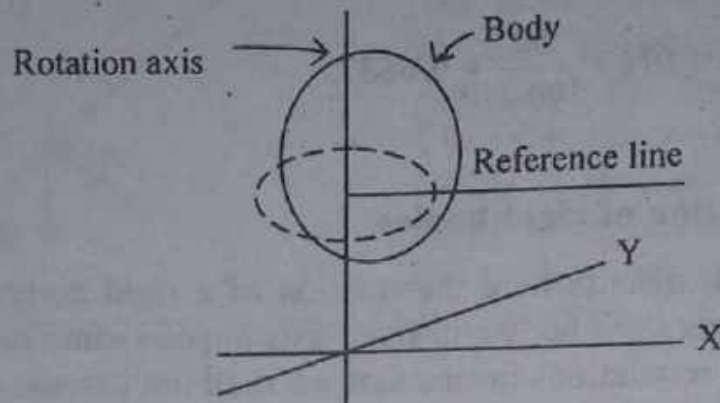


Fig. 10.2

Let us consider a reference line, fixed in the body, perpendicular to the rotation axis, and rotating with the body (Fig. 10.2). The motion of the rotating body can then be described by specifying the angular position of this line, *i.e.*, the angle of the line relative to a fixed axis (Fig. 10.3). In Fig. 10.3, the angular position  $\theta$  is measured relative to the x-axis, and is given by

$$\theta = \frac{s}{r} \quad (10.4)$$

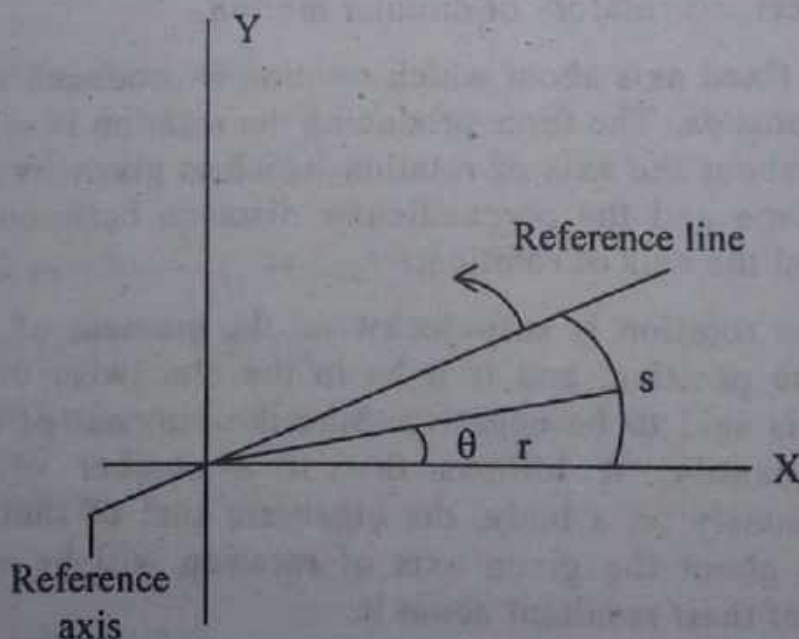


Fig. 10.3

Here  $s$  is the length of the arc of any circle cut off by the  $x$ -axis and the reference line, and  $r$  is the radius of that circle. An angle defined in this way is measured in radians (rad) rather than in revolutions (rev) or degrees. Being the ratio of two lengths, the radian is a pure number and thus has no dimension. Because the circumference of a circle of radius  $r$  is  $2\pi r$ , there are  $2\pi (= 2\pi r/r)$  radians in a complete circle.

$$1 \text{ rev} = 360^\circ = \frac{2\pi r}{r} = 2\pi \text{ rad} \quad (10.5)$$

$$\text{or, } 1 \text{ rad} = 57.3^\circ = 0.159 \text{ rev.} \quad (10.6)$$

If we know  $\theta(t)$ , the angular position of the body's reference line as a function of time, we know all there is to know about the rotation of the body.

### Angular displacement

If the body rotating about the rotation axis changes the angular position of the reference line from  $\theta_1$  to  $\theta_2$ , the body undergoes an angular displacement  $\Delta\theta$  given by

$$\Delta\theta = \theta_2 - \theta_1 \quad (10.7)$$

This definition of angular displacement holds not only for the rigid body as a whole but also for every particle of the body. The angular displacement can be either positive or negative, depending on whether the body is rotating in the direction of increasing  $\theta$  (counter clockwise) or decreasing  $\theta$  (clockwise).

### Angular velocity

Suppose the rotating body is at angular position  $\theta_1$  at time  $t_1$  and at angular position  $\theta_2$  at time  $t_2$ . Then the average angular velocity of the body may be defined as

$$\bar{\omega} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t} \quad (10.8)$$

where  $\Delta\theta$  is the angular displacement that occurs during the time interval.

The instantaneous angular velocity  $\omega$ , with which we shall be concerned most of the time, is the limit of the ratio in eqn. (10.8) as  $\Delta t$  is made to approach zero.



$$\omega = \lim_{\Delta t \rightarrow 0} \frac{\Delta \theta}{\Delta t} = \frac{d\theta}{dt} \quad (10.9)$$

If  $\theta(t)$ , is known, the angular velocity  $\omega$  can be obtained by differentiation. If the rate of rotation of the body be uniform, *i.e.*, if its angular velocity be constant and it describes an angle  $\theta$  (radians) in time  $t$  (seconds), then we have

$$\text{angular velocity of the body, } \omega = \frac{\theta}{t}.$$

If the body makes  $n$  revolutions in time  $t$ , the angle described by the body is equal to  $2\pi n$  radians. Or,  $\theta = 2\pi n$ .

$$\text{Therefore, its angular velocity, } \omega = \frac{2\pi n}{t}.$$

The unit of angular velocity is commonly *radian per second* (rad/s) or *revolution per second* (rev/s).

The definition of angular velocity as given above holds not only for the rotating rigid body as a whole but also for every particle of that body. Fig. 3 shows a body rotating about a fixed axis through  $O$ . Then the particles composing it, at any distance from  $O$ , such as at

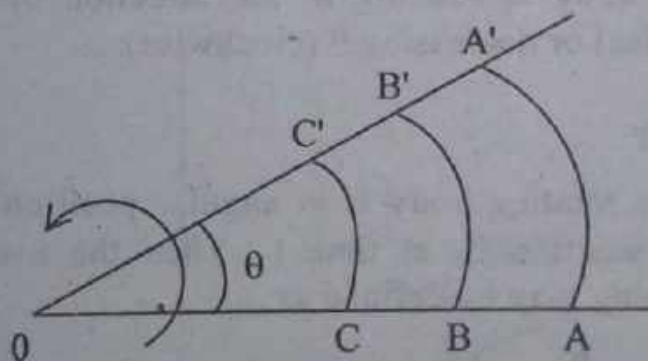


Fig. 10.4

A, B, C, etc. complete one rotation at the same time *i.e.*, they describe the same angle in the same time. In other words, they all have the same angular velocity. Now, although the angle  $\theta$  described by all particles of the body in a given time  $t$  is the same, the linear distances travelled by them are different. Thus the particles at A, B

and C cover the distances AA', BB' and CC' respectively (which are arcs of radii OA, OB and OC) depending upon their respective distances from the axis of rotation through O.

Let  $OA = r_1$ ,  $OB = r_2$  and  $OC = r_3$ ; then  
the arc  $AA' = r_1\theta$ ;  $BB' = r_2\theta$  and  $CC' = r_3\theta$

(arc = radius  $\times$  angle subtended by it)

$\therefore$  linear velocity of A =  $r_1\theta/t$ , that of B =  $r_2\theta/t$  and  
that of C =  $r_3\theta/t$ .

or, in general, linear velocity  $v$  of a particle at a distance  $r$  from the axis of rotation is  $r\theta/t$ .

or,  $v = r\omega$

$\therefore$  **linear velocity = distance from the axis of rotation  $\times$  angular velocity.**

Angular velocity of a body can be either positive or negative depending on whether the body is rotating in the direction of increasing  $\theta$  or decreasing  $\theta$ .

### Angular acceleration

If the angular velocity of a rotating body is not constant, then the body has an angular acceleration. Let  $\omega_1$  and  $\omega_2$  be the angular velocities at time  $t_1$  and  $t_2$  respectively. Then the average angular acceleration of the body is defined as

$$\bar{\alpha} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t} \quad (10.10)$$

where  $\Delta\omega$  is the change of angular velocity during a time interval  $\Delta t$ . The instantaneous angular acceleration, with which we are most concerned, is the limit of the ratio given by eqn. 7 as  $\Delta t$  approaches zero. Or,

$$\bar{\omega} = \lim_{\Delta t \rightarrow 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} \quad (10.11)$$

Thus angular acceleration may be defined as rate of change of angular velocity ( $d\omega/dt$ ). The unit of angular acceleration is



commonly given by *radian per second per second* ( $\text{rad/sec}^2$ ) or *revolutions per second per second* ( $\text{rev/sec}^2$ ).

The definition of angular acceleration as given above holds not only for the rotating rigid body as a whole but also for every particle of that body. Now, if the distance of a particle from the axis of rotation be  $r$ , its linear velocity changes from  $r\omega_1$  to  $r\omega_2$  in time interval  $\Delta t$ . Therefore, the rate of change of its linear velocity *i.e.*, its linear acceleration is given by

$$a = \frac{r\omega_2 - r\omega_1}{\Delta t} = \frac{r(\omega_2 - \omega_1)}{\Delta t} = \frac{r\Delta\omega}{\Delta t}$$

or,  $a = \lim_{\Delta t \rightarrow 0} \frac{r\Delta\omega}{\Delta t} = r \cdot \frac{d\omega}{dt}$  (10.12)

Thus,

**linear acceleration = distance from axis of rotation  $\times$  angular acceleration.**

When the particle rotates about an axis with uniform angular velocity,

$$\frac{d\omega}{dt} = 0 \quad \therefore a = r \cdot \frac{d\omega}{dt} = 0$$

It means both the linear and angular accelerations are zero.

## 10.6 Rotational kinetic energy – moment of inertia

According to Newton's first law of motion, a body must continue in its state of rest or of uniform motion in a straight line unless compelled by some external agency called Force. The *inertness* or inability of a body to change its state of motion by itself is called *inertia* and is a fundamental property of matter. For a given force, the greater the mass, the higher will be opposition to motion, or the larger will be the inertia. Thus for translatory motion, the *mass of a body measures the co-efficient of inertia*.

Similarly in rotatory motion, a body which is free to rotate about a given axis, opposes any change desired to be produced in its state. The measure of this opposition depends on (i) the mass of the body and (ii) the distribution of mass about the axis of rotation. The



co-efficient of inertia in rotatory motion is called the *moment of inertia* of the body about the given axis. The moment of inertia plays the same role in rotatory motion as that of mass in translatory motion. Also, as force has to be applied to bring about a change of state in case of translatory motion, a *torque* or *couple* has to be applied to bring about a change in the state of rotatory motion.

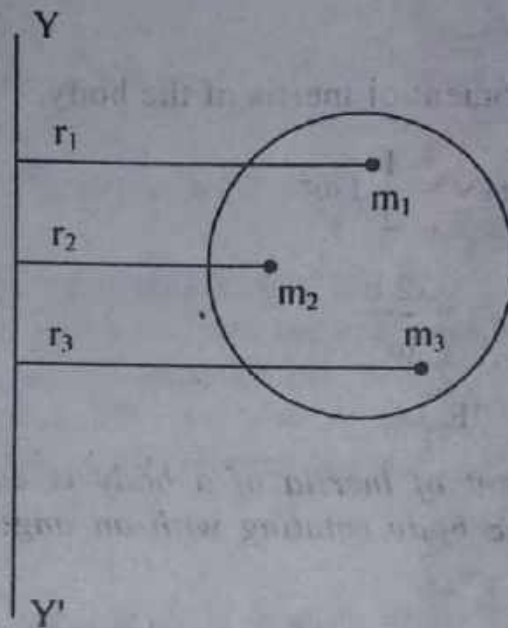


Fig. 10.4

Let a body of mass  $M$  rotate about the axis  $YY'$  with an angular velocity  $\omega$ . Let the body be made up of large number of particles  $m_1, m_2, m_3, \dots$ . Let their distances from the axis of rotation be  $r_1, r_2, r_3, \dots$  etc. All the particles move with the same angular velocity but with different linear velocities. If  $v_1, v_2, v_3, \dots$  etc. be the linear velocities of the particles, then the total kinetic energy of the body is given by

$$\begin{aligned}
 E &= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \frac{1}{2} m_3 v_3^2 + \dots \\
 &= \frac{1}{2} m_1 r_1^2 \omega^2 + \frac{1}{2} m_2 r_2^2 \omega^2 + \frac{1}{2} m_3 r_3^2 \omega^2 + \dots \quad [\because v = r\omega] \\
 &= \frac{1}{2} \omega^2 [m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \dots]
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2} \omega^2 \left[ \sum mr^2 \right] \\
 &= \frac{1}{2} \omega^2 I \quad (10.13)
 \end{aligned}$$

where

$$I = \sum mr^2 \quad (10.14)$$

$I$  is called the moment of inertia of the body.

$$\therefore \text{Kinetic energy} = \frac{1}{2} I \omega^2$$

$$\text{or,} \quad I = \frac{2E}{\omega^2}$$

$$\text{If } \omega = 1, \quad I = 2E.$$

(10.15)

Thus, the moment of inertia of a body is equal to twice the Kinetic energy of the body rotating with an angular velocity of 1 radian per second.

Suppose the body consists of  $n$  particles each of mass  $m$ . Then

$$I = \sum mr^2 = m \left[ r_1^2 + r_2^2 + \dots + r_n^2 \right]$$

$$= mn \left[ \frac{r_1^2 + r_2^2 + \dots + r_n^2}{n} \right]$$

$$= M \cdot K^2$$

where  $M = mn$  is the total mass of the body

$$\text{and } K^2 = \frac{r_1^2 + r_2^2 + \dots + r_n^2}{n}$$

$$\therefore K = \sqrt{\frac{r_1^2 + r_2^2 + \dots + r_n^2}{n}} \quad (10.16)$$

$K$  is called the *radius of gyration* and is equal to the root mean square distance of the particles from the axis of rotation.

If the whole mass of the body is supposed to be concentrated at a point P whose distance K from the axis of rotation is such that the body rotates about the axis with the same kinetic energy as before, then

$$\frac{1}{2} \omega^2 \sum mr^2 = \frac{1}{2} \omega^2 \sum MK^2; \text{ or, } \sum mr^2 = \sum MK^2 = I.$$

$$\text{or, } K = \sqrt{\frac{I}{M}} \quad (10.17)$$

$$\text{Hence the radius of gyration} = \sqrt{\frac{\text{moment of inertia}}{\text{total mass of the body}}}$$

*So the radius of gyration of a body with respect to an axis is defined to be the distance from the axis of a point inside the body such that if the whole mass of the body were concentrated in a particle at that point, the moment of inertia of the particle would have been the same as the moment of inertia of the original body.*

### 10.7 Torque

Let us consider a particle of mass  $m$  moving about an axis in a circular path of radius  $r$ . Let an external force  $F$  act on the particle along the tangent to the circular path. Now the moment of a force about a point is given by the product of the magnitude of the force and the perpendicular distance from its line of action to the point where its effect is felt. Hence the moment of the force  $F$  in the present case is given by  $F.r$  where  $r$  is the perpendicular distance of the line of action of the force from the axis of rotation. The moment of the force is also called *torque* represented by the symbol  $\tau$ .

$$\tau = Fr$$

$$\text{But } F = ma = mr\alpha$$

$$\therefore \tau = (mr\alpha) r = mr^2\alpha$$

$$\text{But } mr^2 = I$$

$$\therefore \tau = I\alpha \quad (10.18)$$

Hence torque is equal to the product of the moment of inertia and angular acceleration.



**Torque = Moment of Inertia  $\times$  Angular acceleration.**

Torque plays the same part in rotatory motion as force in translatory motion. It is a vector quantity. Its dimensions are  $ML^2T^{-2}$ . Its unit is  $kg\cdot m^2/s^2$ .

### 10.8 Work done by a torque

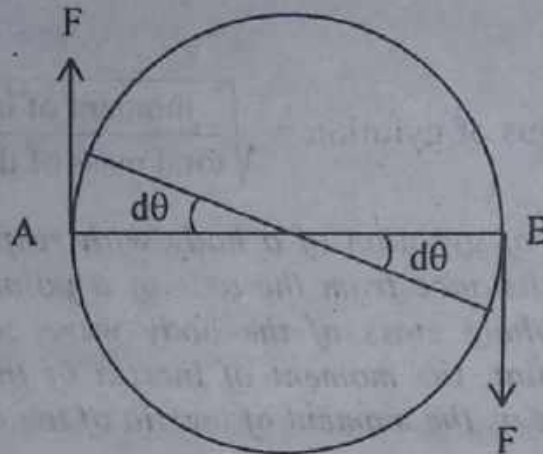


Fig. 10.5

A and B are two particles on the circumference of a disc at diametrically opposite points. Let  $F$  and  $-F$  be the two forces acting on the particles A and B respectively. If the particles are moved through an angle  $d\theta$ , then the displacement  $ds$  along the circumference of a circle is  $ds = r d\theta$  in each case where  $r$  is the radius of the circle.

Work done by each force,

$$= F \cdot ds = Fr d\theta$$

$$\text{Total work done} = F (2r) d\theta$$

But  $2r$  is the perpendicular distance between the lines of action of the two forces. Hence

$$F (2r) = \text{moment of the force} = \text{torque, } \tau$$

$$\therefore \text{work done} = \tau \cdot d\theta$$

If the particles get displaced through an angle  $\theta$ , under the action of an external torque  $\tau$ , the total work done

$$= \tau \cdot \theta. \quad (10.19)$$

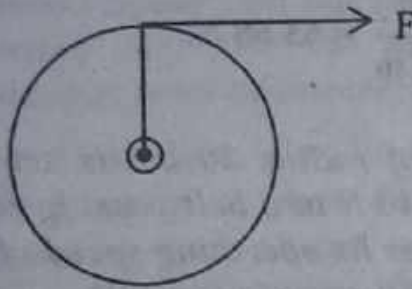
When a torque acting on a particle for a time  $dt$  displaces the particle through  $d\theta$ ,

$$\text{work done} = \tau \cdot d\theta.$$

$$\text{rate of doing work} = \tau \cdot \frac{d\theta}{dt} = \tau \cdot \omega$$

$$= \text{torque} \times \text{angular velocity}.$$

Torque is an important consideration for machine design. A force  $F$  is applied to a belt pulley at a distance  $r$  from the shaft. The torque acting on the pulley is



$$\tau = F \cdot r$$

If the pulley makes  $n$  revolutions, then the total angular displacement

$$= 2\pi \cdot n \text{ radians}$$

Fig. 10.6

$$\text{Hence the work done} = \tau \times 2\pi n = Fr (2\pi n)$$

$$\text{Hence, work done} = \text{torque} \times \text{angular movement in radians}.$$

**Example 10.4.** What average torque is required at the driving shaft of a loom if 400 W are needed to drive it at 240 rev/min? If the driving pulley has a radius of 25 cm, what is the average effective tension in the driving belt?

**Soln.**

$$400 \text{ W} = 400 \text{ J/sec.}$$

$$\text{Hence the work done per second } 400 \text{ J.}$$

$$\text{Total angular movement per second}$$

$$= \frac{240 \times 2\pi}{60} \text{ rad} = 25.13 \text{ rad.}$$

Now

work done = torque  $\times$  angular movement

$$\therefore \text{Torque} = \frac{400}{25.13} = 15.92 \text{ J.}$$

Therefore the torque required to drive the shaft  
= 15.9 J.

Now

Torque = average effective belt tension (force)  
 $\times$  pulley radius.

Hence

$$\text{Tension} = \frac{\text{Torque}}{\text{pulley radius}} = \frac{15.92 \text{ J}}{0.25 \text{ m}} = 63.68 \text{ N.}$$

**Example 10.5.** A heavy fly-wheel of radius 20 cm is set into motion by exerting a driving tension of 40 N in a belt passing round its circumference. If the fly-wheel reaches its operating speed of 140 rev/min in 10 sec. from rest, calculate its moment of inertia.

**Soln.**

We have

$$\text{Torque} = \text{Force} \times \text{radius} = 40 \times 0.2 = 8 \text{ J.}$$

For the uniform acceleration during start-up :

initial angular velocity,  $\omega_0 = 0$

final angular velocity,  $\omega = 150 \text{ rev/min} = 5\pi \text{ rad/s.}$

time,  $t = 10 \text{ s.}$

angular acceleration =  $\alpha$

Then, from the eqn.  $\omega = \omega_0 + \alpha t$ , we get

$$5\pi = 0 + \alpha \cdot 10$$

$$\text{or, } \alpha = \frac{\pi}{2} \text{ rad/s}^2.$$

But torque = moment of inertia  $\times$  angular acceleration



$$\therefore 8 = I \times \frac{\pi}{2}$$

$$\text{or, } I = \frac{16}{\pi}$$

Thus the moment of inertia =  $5.09 \text{ m}^2 \text{ kg}$ .

## 10.9 Theorem of perpendicular axes

i) For a plane lamina body

The theorem of perpendicular axes states that, *the moment of inertia of a plane lamina about an axis, perpendicular to the plane of lamina, is equal to the sum of the moments of inertia of the lamina about two mutually perpendicular axes in the plane of the lamina such that the three mutually perpendicular axes have a common point of intersection.*

Consider a plane lamina having the perpendicular axes OX and OY lying in its plane and intersecting at the point O. The axis OZ passes through O and is perpendicular to the plane of the lamina (Fig. 10.7). Let the lamina be divided into a large number of particles, each of mass  $m$ . Let one such particle of mass  $m$  be located at the point P which is at a distance  $r$  from O, the point of intersection of the axes.

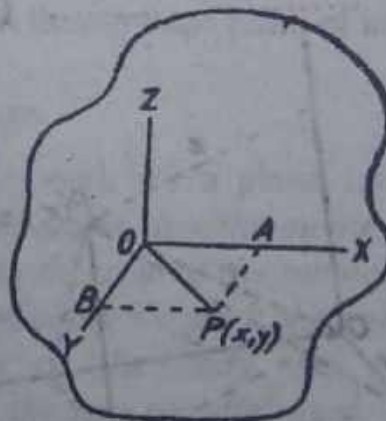


Fig. 10.7

The moment of inertia of the particle P about the axis OZ =  $mr^2$ .

The moment of inertia of the whole lamina about the axis OZ is then given by,

$$I_z = \sum mr^2 \dots \dots \dots (i)$$

$$\text{Now, } r^2 = x^2 + y^2 \dots \dots \dots (ii)$$

$$I_z = \sum mx^2 + \sum my^2 \dots \dots \dots (iii)$$

But  $\sum my^2$  is the moment of inertia of the whole lamina about the axis OX, or

$$I_x = \sum my^2 \dots \dots \dots (iv)$$

Similarly the moment of inertia of the whole lamina about the axis OY is

$$I_y = \sum mx^2 \dots \dots \dots (iv)$$

Hence from eqn. (iii), we have

$$I_z = \sum mx^2 + \sum my^2 = I_y + I_x \dots \dots (v)$$

ii) For a three dimensional body

Let us consider a three dimensional body shown by dotted lines in Fig. 10.8. The three mutually perpendicular axes meet at the point O inside the body. P is a particle of mass  $m$  situated at the point  $x, y, z$  which is at a distance  $r$  from O.

$$\therefore r^2 = x^2 + y^2 + z^2 \dots \dots \dots (i)$$

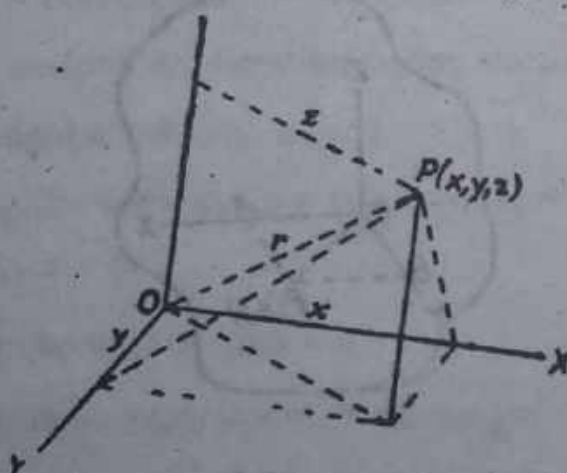


Fig. 10.8

The moment of inertia of the particle P about Z-axis =  $m(x^2 + y^2)$

Then the moment of inertia of the whole body about the Z-axis

$$I_z = \sum m(x^2 + y^2) \dots\dots\dots (ii)$$

Similarly,

$$I_x = \sum m(y^2 + z^2) \dots\dots\dots (iii)$$

$$\text{and, } I_y = \sum m(x^2 + z^2) \dots\dots\dots (iv)$$

The moment of inertia of the body about any axis passing through O,

$$I = \sum mr^2 = \sum m(x^2 + y^2 + z^2) \dots\dots\dots (v)$$

Adding (ii), (iii) and (iv), we get

$$\begin{aligned} I_x + I_y + I_z &= \sum m(y^2 + z^2) + \sum m(x^2 + z^2) + \sum m(x^2 + y^2) \\ &= 2 \sum m(x^2 + y^2 + z^2) \\ &= 2I \end{aligned}$$

$$\text{or, } I = \frac{1}{2}(I_x + I_y + I_z)$$

Thus, the moment of inertia of a three dimensional body about any axis is equal to half the sum of the moments of inertia about three mutually perpendicular axes about the common point of intersection.

### 10.10 Theorem of parallel axes

This theorem which is true both for a plane laminar body as well as a three dimensional body states that *the moment of inertia of a body about any axis is equal to the sum of the moment of inertia of the body about a parallel axis passing through the center of gravity and the product of the mass of the body and the square of perpendicular distance between the two parallel axes.*

Let G be the center of gravity of the plane laminar body (Fig. 10.9). The axis XX' passes through G and is perpendicular to the plane of the lamina. The axis X<sub>1</sub>X<sub>1</sub>', which is parallel to XX', passes through the point O, the distance between the two axes being x.



Let the lamina be divided into a large number of particles, each of mass  $m$ . The moment of inertia of one such particle  $P$ , located at a distance  $r$  from  $X_1X_1'$  is given by  $mr^2$ .

Then the moment of inertia of the whole lamina about the axis  $X_1X_1'$  is given by

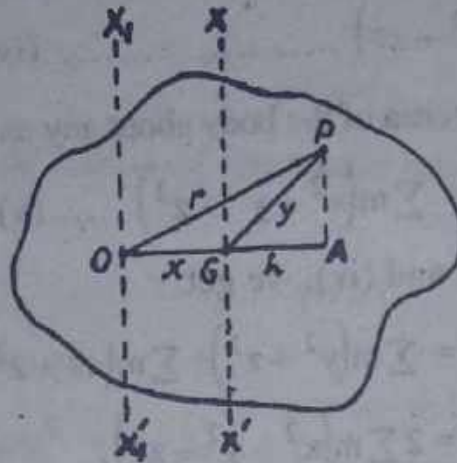


Fig. 10.9

$$I_o = \sum mr^2$$

In the triangle OPA,

$$OP^2 = (OA)^2 + (AP)^2$$

$$r^2 = (x + h)^2 + (AP)^2$$

$$= x^2 + 2xh + h^2 + (AP)^2$$

$$= x^2 + y^2 + 2xh$$

$$I_o = \sum mx^2 + \sum my^2 + \sum m(2xh)$$

$$= Mx^2 + I_G + 2x \sum mh \quad (i)$$

$$\text{Here } \sum my^2 = I_G$$

Since a body always balances about an axis passing through its center of mass, it is obvious that the algebraic sum of the moments of the weights of its individual particles about the center of mass must be zero. Hence  $\sum mg.GA$  (the algebraic sum of such moments about  $G$ ) must be zero. Since  $g$  is constant at a given place, the expression  $\sum m.GA = \sum m.h$  is equal to zero. Consequently

$$2x \sum mh = 0$$

Therefore, eqn. (i) becomes

$$I_o = Mx^2 + I_G$$

Thus, the theorem of parallel axes states that the moment of inertia of a body about any axis is equal to the sum of the moment of inertia of the body about a parallel axis passing through the center of gravity and the product of the mass of the body and the square of the perpendicular distance between the two parallel axes.

### 10.11 Moment of inertia of standard geometric objects

One frequently encounters standard geometric shapes in machine parts and a knowledge of their moment of inertia is very useful. Calculations of moment of inertia for a number of fixed geometric shapes will be carried out below. This will not only provide the students with a few standard values, but will also demonstrate the method of procedure for the calculation of moment of inertia so that it can be applied in other cases.

#### 1. Moment of inertia of a thin uniform bar (rod)

Perhaps the simplest type of geometric object is the long, thin rod. In engineering, one comes across this rod in the form of machine shafts, levers and so on.

It will be assumed that the length of the rod is very much greater than its diameter. We will calculate the moment of inertia of such a rod about an axis PQ, perpendicular to the rod AB at its centre C. The moment of inertia of a small element  $\delta x$ , at a distance

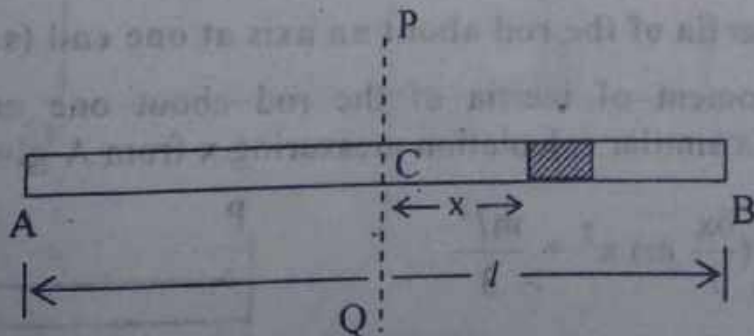


Fig. 10.10

$x$  from C about PQ is given by

$$(\delta I)_x = \left( \frac{m}{l} \cdot \delta x \right) x^2 \quad (i)$$

where  $l$  and  $m$  are the length and mass of the rod respectively.

$$[(\delta I)_x = \delta m \cdot x^2; \quad \text{but} \quad \delta m = \frac{m}{l} \cdot dx]$$

Since the rod is uniform we can integrate expression (i) over the limits  $x = 0$  to  $x = l/2$  for the total possible range of  $x$  and then double the value to include the other half of the rod.

Thus,

$$I = 2 \int_0^{l/2} (\delta I)_x = 2 \int_0^{l/2} \left( \frac{dx}{l} \times m \right) x^2$$

$$= \frac{2m}{l} \int_0^{l/2} x^2 dx = \frac{2m}{l} \left[ \frac{x^3}{3} \right]_0^{l/2}$$

$$= \frac{2m}{l} \left[ \frac{l^3}{24} \right]$$

$$= \frac{ml^2}{12} \quad (ii)$$

Also  $I = mK^2$

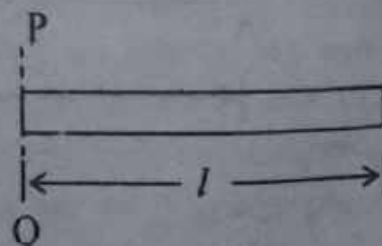
$$\therefore mK^2 = \frac{ml^2}{12}; \quad \text{or,} \quad K = \frac{l}{2\sqrt{3}} \quad (iii)$$

### Moment of inertia of the rod about an axis at one end (say A)

If the moment of inertia of the rod about one end, A, is required, then a similar calculation measuring  $x$  from A gives,

$$I = \int_0^l \left( \frac{\delta x}{l} m \right) x^2 = \frac{ml^2}{3}$$

But  $I = mK^2$

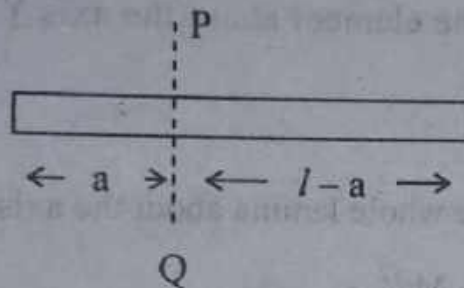




$$\therefore mK^2 = \frac{ml^2}{3}; \quad \text{or, } K = \frac{l}{\sqrt{3}} \quad (\text{iv})$$

**Moment of inertia of the rod about an axis at a distance 'a' from one end**

$$\begin{aligned} \text{Here } I &= \int_{-a}^{l-a} \left( \frac{\delta x}{l} \cdot m \right) x^2 \\ &= \frac{m}{l} \left[ \frac{x^3}{3} \right]_{-a}^{l-a} \\ &= m \left[ \frac{2}{3} - la + a^2 \right] \quad (\text{v}) \end{aligned}$$



## 2. Moment of inertia of a rectangular beam

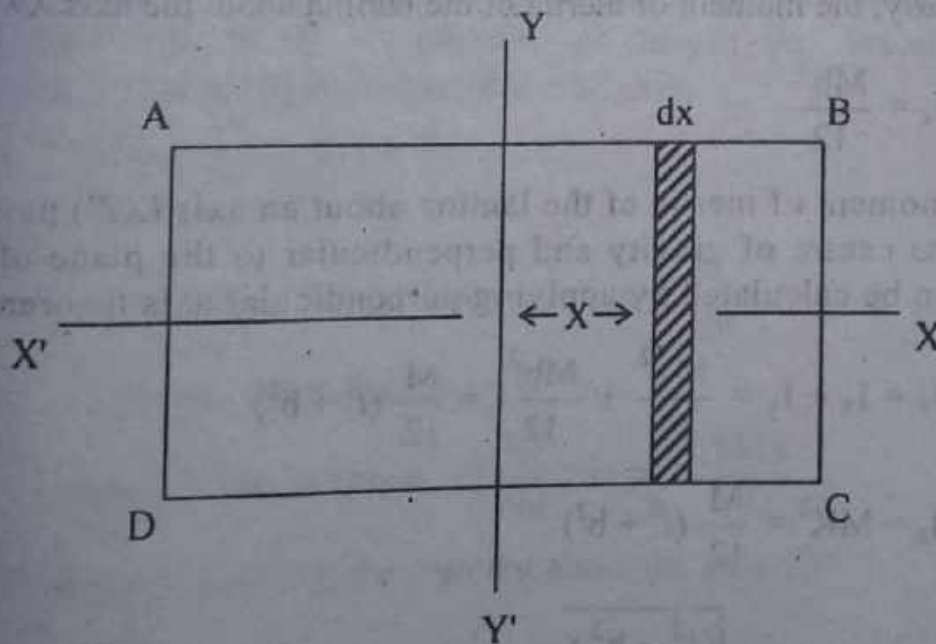


Fig. 10.11

ABCD is a rectangular lamina of uniform thickness, length  $l$  and breadth  $b$ . Let an element of length  $dx$  be at a distance  $x$  from the axis  $YY'$ .

Mass of the lamina =  $M$

Area of the lamina =  $l \times b$

$$\therefore \text{mass per unit area} = \frac{M}{l \times b}$$

area of the element =  $dx \times b$

$$\therefore \text{mass of the element} = \left( \frac{M}{l \times b} \right) b \times dx = \frac{M}{l} \times dx$$

Moment of inertia of the element about the axis  $YY'$

$$= \left( \frac{M}{l} dx \right) x^2$$

Moment of inertia of the whole lamina about the axis  $YY'$

$$I_y = \int_{-l/2}^{+l/2} \frac{M}{l} x^2 dx = \frac{M l^2}{12}$$

Similarly, the moment of inertia of the lamina about the axis  $XX'$ .

$$I_x = \frac{M b^2}{12}$$

The moment of inertia of the lamina about an axis ( $ZZ'$ ) passing through the centre of gravity and perpendicular to the plane of the lamina, can be calculated by applying perpendicular axis theorem.

$$I_z = I_x + I_y = \frac{M l^2}{12} + \frac{M b^2}{12} = \frac{M}{12} (l^2 + b^2)$$

$$I_z = M K^2 = \frac{M}{12} (l^2 + b^2)$$

$$\therefore K = \sqrt{\frac{(l^2 + b^2)}{12}}$$

### 3. Moment of inertia of a solid uniform bar of rectangular cross-section

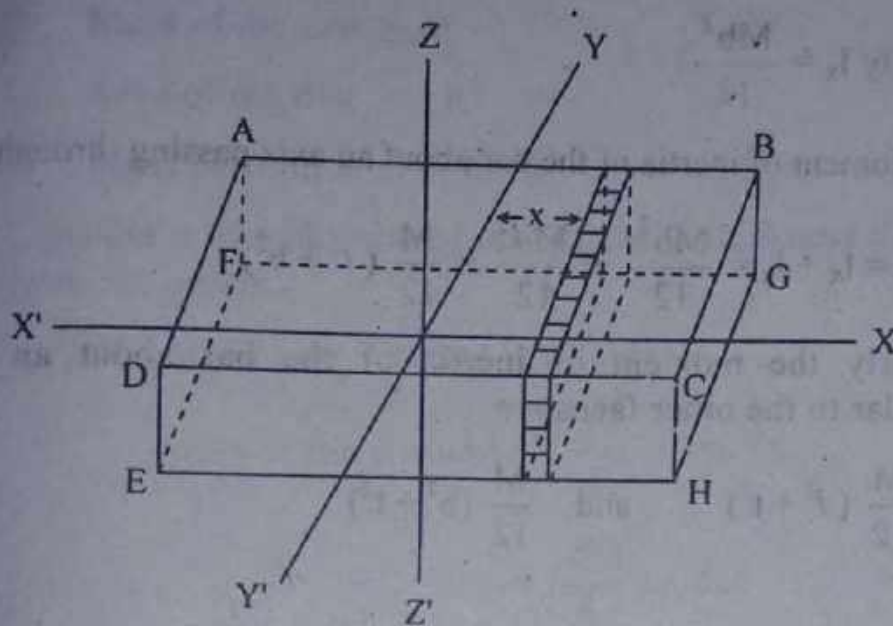


Fig. 10.12

ABCDEFGH is a uniform solid bar of length  $l$ , breadth  $b$  and thickness  $t$ . Consider an element of length  $dx$ , breadth  $b$  and thickness  $t$  at a distance  $x$  from the  $YY'$  axis.

Mass of the bar =  $M$

Volume of the bar =  $l b t$

$$\therefore \text{mass of the bar per unit volume} = \frac{M}{l b t}$$

Volume of the element =  $dx \cdot b \cdot t$

$$\text{mass of the element} = \left( \frac{M}{l b t} \right) dx b t = \frac{M dx}{l}$$

Moment of inertia of the element about the axis  $YY'$

$$= \frac{M}{l} dx \cdot x^2$$

$\therefore$  moment of inertia of the whole bar about  $YY'$



$$I_y = \int_{-l/2}^{+l/2} \frac{M}{l} x^2 dx = \frac{M l^2}{12}$$

$$\text{Similarly } I_x = \frac{M b^2}{12}$$

∴ Moment of inertia of the bar about an axis passing through O.

$$I_z = I_x + I_y = \frac{M b^2}{12} + \frac{M l^2}{12} = \frac{M}{12} (l^2 + b^2)$$

Similarly the moment of inertia of the bar about an axis perpendicular to the other faces are

$$\frac{M}{12} (l^2 + t^2) \quad \text{and} \quad \frac{M}{12} (b^2 + t^2)$$

#### 4. Moment of inertia of a uniform circular disc

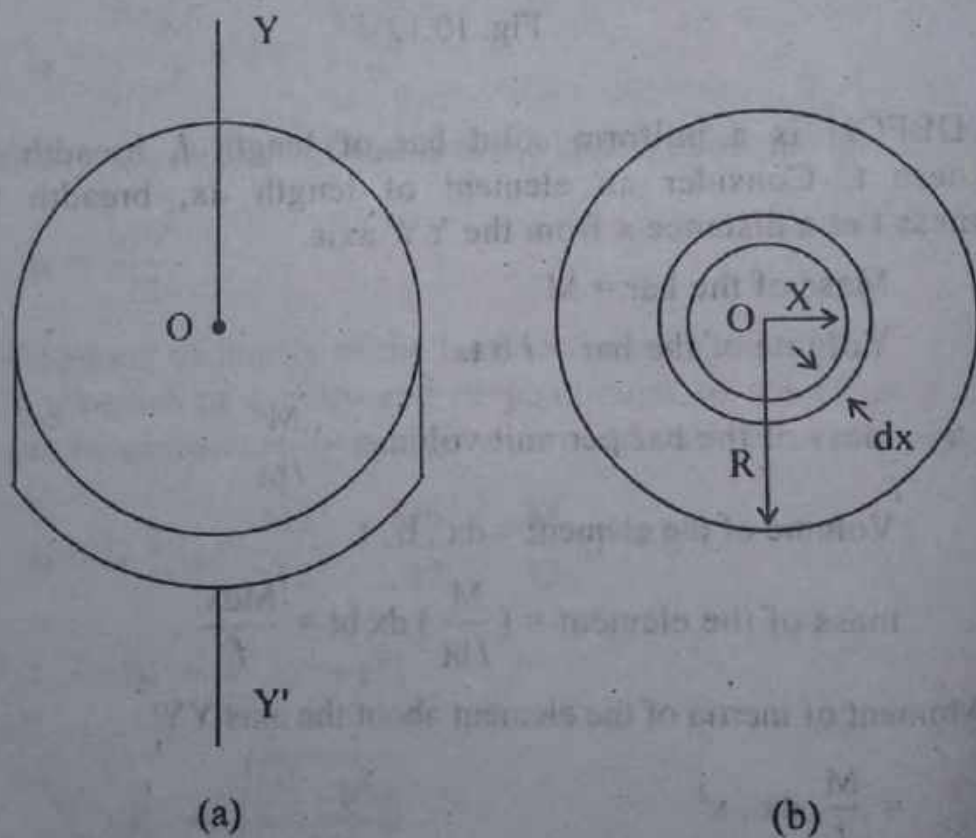


Fig. 10.13

Consider a uniform circular disc of mass  $M$  and radius  $R$  rotating about an axis passing through its centre and perpendicular to its plane.

$$\text{Mass of the disc} = M$$

$$\text{Area of the disc} = \pi R^2$$

$$\text{Mass per unit area} = M/\pi R^2 \quad (i)$$

Consider a thin element of disc of radial thickness  $dx$  at a distance  $x$  from the centre.

$$\text{The area of the element} = 2\pi x \cdot dx$$

$$\begin{aligned} \text{Mass of the element} &= \left( \frac{M}{\pi R^2} \right) 2\pi x \, dx \\ &= \left( \frac{2M}{R^2} \right) x \cdot dx. \end{aligned}$$

moment of inertia of the element about the axis of rotation

$$= \text{mass} \times (\text{distance})^2$$

$$= \left[ \left( \frac{2M}{R^2} \right) x \, dx \right] x^2 = \left( \frac{2M}{R^2} \right) x^3 \, dx$$

Moment of inertia of the whole disc about the axis of rotation,

$$\begin{aligned} I &= \int_0^R \left( \frac{2M}{R^2} \right) x^3 \, dx \\ &= \frac{2M}{R^2} \int_0^R x^3 \, dx = \frac{2M}{R^2} \left[ \frac{x^4}{4} \right]_0^R \\ &= \frac{2M}{R^2} \left[ \frac{R^4}{4} \right] = \frac{MR^2}{2} \quad (ii) \end{aligned}$$

### 5. Moment of inertia of an annular disc

Consider a uniform circular (annular) disc of inner radius  $R_1$  and outer radius  $R_2$ . Let the mass of the disc =  $M$

$$\text{Area of the disc} = \pi (R_2^2 - R_1^2)$$

$$\therefore \text{Mass per unit area} = \frac{M}{\pi (R_2^2 - R_1^2)}$$

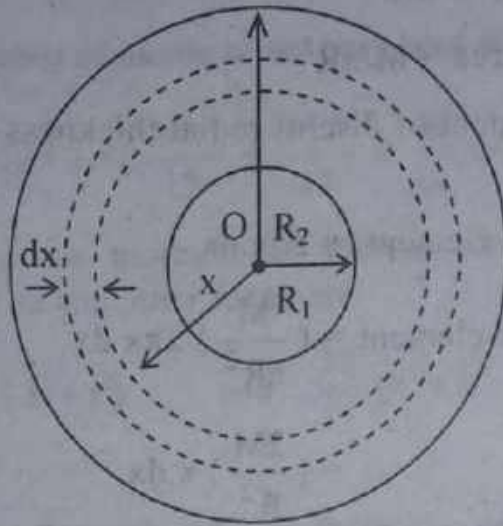


Fig. 10.14

Consider an element of disc of radial thickness  $dx$  at a distance  $x$  from the centre (O).

$$\text{Area of the element} = 2\pi x \cdot dx$$

$$\begin{aligned} \therefore \text{mass of the element} &= \left[ \frac{M}{\pi (R_2^2 - R_1^2)} \right] 2\pi x dx \\ &= \frac{2 M x dx}{(R_2^2 - R_1^2)} \end{aligned}$$

Moment of inertia of the element about an axis passing through the centre and perpendicular to the plane of the disc.

$$= \frac{2 M x dx}{(R_2^2 - R_1^2)} \cdot x^2$$

Moment of inertia of the whole disc

$$I = \int_{R_1}^{R_2} \frac{2 M x dx}{(R_2^2 - R_1^2)} x^2 = \int_{R_1}^{R_2} \frac{2 M}{(R_2^2 - R_1^2)} x^3 dx$$



$$\begin{aligned}
 &= \frac{2M}{(R_2^2 - R_1^2)} \left[ \frac{x^2}{4} \right]_{R_1}^{R_2} = \frac{2M}{(R_2^2 - R_1^2)} \frac{(R_2^2 - R_1^2)}{4} \\
 &= \frac{M}{2} (R_2^2 - R_1^2)
 \end{aligned}$$

**Note :** A wheel rim, a bushing, a large bearing or any similar annulus exemplify the case of an annular circular ring.

## 6. Moment of inertia of a flywheel

A flywheel is a heavy metal disc with its mass concentrated mostly in its rim. The wheel is capable of rotation about a horizontal or a vertical axis. A thick rod which is called the axle passes through the center of gravity of the wheel which rotates about the rod as axis. The rod and the wheel are rigidly connected. The wheel is on a supported horizontal axis (Fig. 10.15). One end of a flexible cord is fixed to a small peg on the axle. The other end of the cord, which is wrapped round the axle, carries a mass  $M$ . The length of the cord is

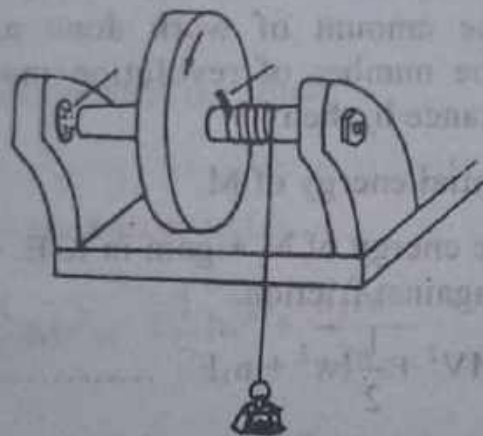


Fig. 10.15

such that it becomes detached from the axle when the mass strikes the ground.

Fig. 10.16, shows a mass  $M$ , attached by means of the cord to the axle of flywheel of radius  $r$ . Let  $I$  be the moment of inertia of the wheel about its axis of rotation. As already mentioned, the length of the cord is such that it becomes detached from the axle when the

mass strikes the floor. Suppose in falling through this distance, the potential energy of the mass has been converted into rotational and translational kinetic energy. If  $w$  be the maximum angular velocity of

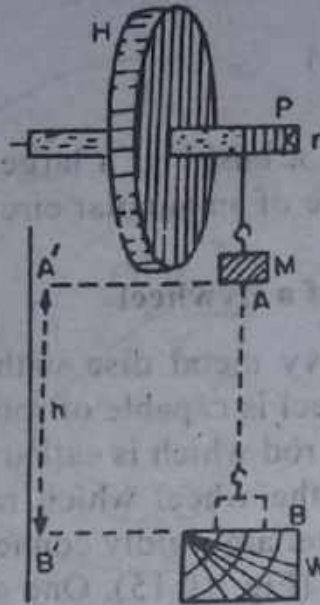


Fig. 10.16

the wheel,  $F$  be the amount of work done against friction per revolution and  $n_1$  the number of revolution made while the mass falls through the distance  $h$ , then

the loss in potential energy of  $M$   
 = gain in kinetic energy of  $M$  + gain in K.E. of flywheel  
 + work done against friction.

$$\text{or, } Mgh = \frac{1}{2}MV^2 + \frac{1}{2}Iw^2 + n_1F \quad \dots\dots\dots (i)$$

After the mass strikes the ground the wheel executes a further  $n_2$  revolution and the angular velocity gradually decreases to zero. It means that the rotational kinetic energy  $\frac{1}{2}Iw^2$  has been used up in overcoming frictional forces. Hence

$$\frac{1}{2}Iw^2 = F.n_2 \quad \dots\dots\dots (ii)$$

If  $n_2$  revolution takes a time  $t$ , then the average angular velocity  $w_a$  is given by  $w_a = \frac{2\pi n_2}{t}$ .

Since the angular velocity decreases uniformly from a maximum  $w$  to a minimum zero, the average angular velocity  $w_a$  is also given by  $w_a = \frac{w+0}{2} = \frac{w}{2}$ .

Also, the motion is uniform, hence  $\frac{w}{2} = \frac{2\pi n_2}{t}$

$$\text{i.e., } w = \frac{4\pi n_2}{t} \quad \text{..... (iii)}$$

from (ii) we get

$$\frac{1}{2} I w^2 = n_2 F$$

$$\text{or, } F = \frac{I w^2}{2 n_2} \quad \text{..... (iv)}$$

If the radius of the axle is  $r$ , then

$$v = r w \quad \text{..... (v)}$$

substituting these values in (i),

$$Mgh = \frac{1}{2} M r^2 w^2 + \frac{1}{2} I w^2 + \frac{n_1 I w^2}{2 n_2}$$

$$\text{or, } 2Mgh - M r^2 w^2 = I w^2 \left(1 + \frac{n_1}{n_2}\right)$$

$$\text{or, } I = \frac{2Mgh - M r^2 w^2}{w^2 \left(1 + \frac{n_1}{n_2}\right)} \quad \text{..... (vi)}$$

Substituting the value of  $w = \frac{4\pi n_2}{t}$  in eqn. (vi)



$$I = \frac{\left[ 2Mgh - Mr^2 \left( \frac{4\pi n_2}{t} \right)^2 \right]}{\left( \frac{4\pi n_2}{t} \right)^2 \left( 1 + \frac{n_1}{n_2} \right)}$$

$$= \frac{M \left[ \left( \frac{ght^2}{8\pi^2 n_2^2} \right) - r^2 \right]}{\left( 1 + \frac{n_1}{n_2} \right)} \dots\dots\dots (vii)$$

$I$  can be calculated from eqn. (vii)

**Example 10.6.** A uniform rod of length 20 cm has a square cross-section of side 1 cm and is made from an alloy with a specific gravity of 3.0. Calculate its moment of inertia about an axis perpendicular to its length and (i) through the centre of gravity; (ii) through one end.

**Soln.**

$$\begin{aligned} \text{Volume of the rod} &= 20 \text{ cm} \times 1 \text{ cm}^2 \\ &= 20 \text{ cm}^3 \end{aligned}$$

$$\text{mass of the rod} = \text{vol} \times \text{S.G.}$$

$$= 20 \times 3 = 60 \text{ g.}$$

$$= 60 \times 10^{-3} \text{ kg.}$$

(i) Moment of inertia through the centre of gravity

$$\begin{aligned} &= \frac{M l^2}{12} = \frac{(60 \times 10^{-3} \text{ kg}) (0.2 \text{ m})^2}{12} \\ &= 2 \times 10^{-4} \text{ m}^2 \text{ Kg.} \end{aligned}$$

(ii) Moment of inertia through one end

$$\begin{aligned} &= \frac{M l^2}{3} = \frac{0.06 \times (0.2)^2}{3} \\ &= 8 \times 10^{-4} \text{ m}^2 \text{ kg.} \end{aligned}$$

**Example 10.7.** A flat gear wheel, made from a metal with a specific gravity of 5.6, has a radius of 25 cm and a uniform thickness of 2 cm. In order to reduce the mass of the gear train, it is replaced by an annular pinion of the same material and thickness, but with an annular width of 2 cm and with spokes of inertia of the component. If the first pinion required 15s to reach its operating speed of 200 rev./min, how much faster would the start-up time become as a result of the change?

**Soln.**

$$\text{Volume of first gear wheel} = \pi r^2 t$$

$$= \pi \times 25^2 \times 2 \text{ cm}^3$$

$$\text{mass of gear} = 2\pi \times 25^2 \times 5.6 \text{ g}$$

$$= 22.0 \text{ Kg.}$$

Hence

$$I = \frac{mr^2}{2} = \frac{22.0 \times 0.25^2}{2}$$

$$= 0.69 \text{ m}^2 \text{ Kg.}$$

Volume of the annular wheel

$$= \pi (r_1^2 - r_2^2) t$$

$$= \pi (25^2 - 23^2) \times 2 \text{ cm}^3$$

i.e., mass of second pinion

$$= 2\pi \times 5.6 \times 96 \text{ g}$$

$$= 3.4 \text{ Kg.}$$

$$\therefore \text{moment of inertia, } I = mr^2$$

$$= 3.4 \times 0.24^2$$

where 24 cm (0.24m) is the mean of the inside and outside radii

$$= 0.20 \text{ m}^2 \text{ Kg.}$$

Thus the moment of inertia decreases by

$$(0.69 - 0.20) \text{ m}^2 \text{ kg} = 0.49 \text{ m}^2 \text{ Kg.}$$

For starting the first pinion :

initial angular velocity,  $\omega_0 = 0$

gular velocity,  $w = 200 \text{ rev/min}$

$$= 20.94 \text{ rad/sec.}$$

time,  $t = 15 \text{ s.}$

From the equation,  $w = w_0 + \alpha t$ , we have

$$20.94 = 0 + \alpha \cdot 15$$

$$\text{or, } \alpha = 1.40 \text{ rad/s}^2.$$

Hence the torque acting on the pinion,

$$\tau = I\alpha = 0.69 \times 1.40 = 0.97 \text{ J.}$$

We assume that the same torque is still available for starting the second pinion. Then, if  $\alpha'$  is the new angular acceleration, from

$$t = I\alpha'$$

we have

$$0.97 = 0.20 \alpha'$$

$$\text{or, } \alpha' = 4.85 \text{ rad/s}^2.$$

For starting the second pinion,

$$w_0 = 0$$

$$w = 200 \text{ rev/min} = 20.94 \text{ rad/s}$$

$$\alpha = 4.85 \text{ rad/s}^2$$

$t$  is unknown

substituting these values in  $w = w_0 + \alpha't$ , we get

$$20.94 = 0 + 4.85 t$$

$$\text{or, } t = 4.3 \text{ s.}$$

hence the start-up time is reduced from

15 s to 4.3 s.

**Example 10.8.** A flywheel has a moment of inertia of  $1 \text{ kg-m}^2$ . It is rotating at a speed of 2 revolutions per second. Find the constant torque required to stop the wheel in 5 rotations. Calculate also the work done by the braking torque.

**Soln.**



Let the torque be  $C$ .

Then the work done,

$$W = C\theta = \frac{1}{2}I\omega^2$$

$$\text{or, } C = \frac{I\omega^2}{2\theta}$$

$$\text{Here } I = 1 \text{ kg-m}^2$$

$$\omega = 2\pi \times n \text{ rad.}$$

$$\therefore C = \frac{1 \times (4\pi)^2}{2 \times 10\pi}$$

$$= 4\pi \text{ rad/sec (n = 2)}$$

$$= 2.5136 \text{ N-m}$$

$$\theta = 5 \times 2\pi = 10\pi \text{ rad.}$$

$$\therefore \text{Work done} = C\theta = 2.5136 \times 10\pi$$

$$= 78.96 \text{ J.}$$

**Example 10.9.** A grinder is in the form of a circular disc of mass 10kg and diameter 0.4m. Calculate the constant torque that has to be applied so that the disc acquires an angular velocity of 4 revolutions per second in 5 seconds. Calculate also the rate at which work is done by the torque at the end of 5 seconds.

**Soln.**

Angular acceleration

$$\alpha = \frac{\omega_2 - \omega_1}{t}$$

$$\text{Here } \omega_2 - \omega_1 = 4 \times 2\pi \text{ rad/s}$$

$$t = 5 \text{ seconds}$$

$$= \frac{8\pi}{5} \text{ rad/sec}^2.$$

$$\text{Torque} = I\alpha = \left( \frac{MR^2}{2} \right) \alpha$$

$$M = 10 \text{ kg}$$

$$R = 0.2 \text{ m}$$

$$= \frac{10 \times (0.2)^2 \times 8\pi}{5 \times 2}$$

$$= 1.0048 \text{ N-m.}$$

Rate at which work is done

$$= \text{torque} \times \text{angular velocity}$$

$$= (1.0048) 8\pi$$

$$= 25.24 \text{ watts.}$$

**Example 10.10.** A flywheel of mass 100 kg and radius 0.5m makes 10 revolutions per second. Calculate (i) angular velocity, (ii) moment of inertia and (iii) the energy of the flywheel. Assume that the whole mass of the flywheel is concentrated at the rim.

**Soln.**

$$\text{Here } M = 100 \text{ kg, } R = 0.5 \text{ m, } n = 10$$

(i) Angular velocity

$$\begin{aligned} \omega &= 2\pi n = 2\pi \times 10 \\ &= 62.84 \text{ rad/s.} \end{aligned}$$

(ii) Moment of inertia

$$\begin{aligned} I &= MR^2 = 100 \times (0.5)^2 \\ &= 25 \text{ kg-m}^2. \end{aligned}$$

(iii) Energy

$$\begin{aligned} E &= \frac{1}{2} I \omega^2 \\ &= \frac{1}{2} \times 25 \times (62.84)^2 \\ &= 4.938 \times 10^4 \text{ J.} \end{aligned}$$

**Example 10.11.** The driving wheel of a belt attached to an electric motor has a radius of 0.25m. It makes 40 revolutions per second. The tension in the belt is 200N on slack side and 1000N on light side. Calculate the power transmitted by the belt.

**Soln.**

$$F = 1000 - 200 = 800 \text{ N}$$

$$r = 0.25 \text{ m}$$

Torque,

$$\tau = F \times r = 800 \times 0.25 = 200 \text{ N-m}$$

$$w = 40 \text{ revolutions per second.}$$

$$= 40 \times 2\pi = 80\pi \text{ radians/sec.}$$

$$\text{Power} = \tau \times w = 200 \times 80\pi \text{ watts}$$

$$= 50.24 \text{ KW.}$$

**Example 10.12.** A wheel and axle is made to rotate about a horizontal axis with the help of a body of mass 5 kg attached to a string wound around the axle. The radius of the axle is 10 cm. The body falls vertically through 5 metres in 10 seconds starting from rest. Calculate the moment of inertia of the wheel and axle.

**Soln.**

Linear acceleration of the body = a

$$h = \frac{1}{2}at^2; \quad \text{here } h = 5 \text{ m, } t = 10 \text{ s.}$$

$$\text{or, } a = \frac{2h}{t^2} = \frac{2 \times 5}{(10)^2} = 0.1 \text{ m/s}^2.$$

$$\text{Angular acceleration, } \alpha = \frac{a}{r} \quad a = 0.1 \text{ m/s}^2$$

$$\therefore \alpha = \frac{0.1}{0.1} = 1 \text{ rad/s}^2. \quad r = 10 \text{ cm} = 0.1 \text{ m}$$

$$\text{Also } Mg - T = Ma \quad M = 5 \text{ kg}$$

$$\text{or, } T = Mg - Ma = M(g - a)$$

$$\therefore T = 5(9.8 - 0.1) \\ = 48.5 \text{ N.}$$

$$\text{Torque} = T \times r \propto I \alpha$$

$$\therefore I = \frac{Tr}{\alpha} = \frac{48.5 \times 0.1}{1}$$

$$= 4.85 \text{ Kg-m}^2.$$



## EXERCISES

- [1] Distinguish between static and kinetic frictions. Discuss the properties of friction.
- [2] Explain the terms moment of inertia and radius of gyration. Discuss the physical significance of moment of inertia.
- [3] State and prove the theorem of perpendicular axis in moment of inertia.
- [4] State and prove the theorem of parallel axis in moment of inertia.
- [5] Define moment of inertia and radius of gyration. State and prove theorems of perpendicular and parallel axes.
- [6] Derive an expression for the moment of inertia of a thin uniform bar about an axis passing through (i) its centre and (ii) one end and perpendicular to its length.
- [7] Derive an expression for the moment of inertia of a rectangular lamina about an axis passing through its centre and perpendicular to its plane.
- [8] Show that the moment of inertia of a uniform circular disc about an axis passing through its centre and perpendicular to its plane is  $0.5 MR^2$ .
- [9] Derive an expression for the moment of inertia of an annular disc of inner radius  $r_1$  and outer radius  $r_2$  about an axis passing through its centre and perpendicular to the plane of the disc.
- [10] What is a flywheel? Discuss a method for determining the moment of inertia of a fly-wheel.
- [11] A 400-g block originally moving at 120 m/s coasts 70 cm along a tabletop before coming to rest. What is the coefficient of friction between the block and the table? [0.105]
- [12] An 8.0-kg box is released on a  $30^\circ$  incline and accelerates down the incline at  $0.30 \text{ m/s}^2$ . Find the frictional force impeding its motion. How large is the coefficient of friction in this situation? [36.8 N, 0.54]
- [13] A horizontal force  $P$  is exerted on a 20-kg box in order to slide it up a  $30^\circ$  incline. The frictional force retarding the motion is 80 N. How large must  $P$  be if the acceleration of the moving box is to be (a) zero and (b)  $0.75 \text{ m/s}^2$ ? [206 N, 223 N]
- [14] A circular disc of mass 100 grams and radius 10 cm is making 120 rpm about an axis passing through its centre and perpendicular to its plane. Calculate its kinetic energy. [ $3.949 \times 10^{-2} \text{ J}$ ]

- [15] A thin metal ring (hoop) of diameter 0.6m and mass 1 kg starts from rest and rolls down an inclined plane. Its linear velocity on reaching the foot of the plane is 5m/s. Calculate (i) the moment of inertia of the ring and (ii) the Kinetic energy of rotation at the instant. [0.09kg-m<sup>2</sup>, 12.5 J]
- [16] A circular disc of mass M and radius R is set rolling on a table with a velocity v. Show that its total kinetic energy is 0.75 MV<sup>2</sup>.
- [17] Calculate the power transmitted by a rope that is being wound on a wheel of circumference 5 metres when the wheel makes 2 revolutions per second. Tension in the rope is 50 N. [500 watts]
- [18] A uniform circular disc of mass 10 kg and radius 0.2 m is subjected to a torque consisting of 2 equal and opposite forces acting tangentially to the circumference of the disc at two opposite ends of a diameter. The disc can freely rotate about an axis passing through its centre and perpendicular to its plane. Calculate (i) the magnitude of each force required so that the disc can rotate with an angular acceleration of 20 radians/s<sup>2</sup> and (ii) the kinetic energy of the disc after 10 seconds from the start. [(i) 10 N, (ii) 4000 J]
- [19] A flywheel of an engine starts from rest and acquires an angular velocity of 54 radians/s in 5 seconds. Calculate (i) angular acceleration and (ii) angular displacement in 5 seconds. [(i) 10 radian/s<sup>2</sup> (ii) 125 radians]
- [20] A thin metal hoop (ring) of diameter 0.8m and mass 1 kg starts from rest and rolls down an inclined plane. Calculate (i) its linear velocity on reaching the foot of the ring and (ii) the kinetic energy of rotation at that instant. [(i) 0.16 kg-m<sup>2</sup> (ii) 50 J]
- [21] A grinder is in the form of a circular disc of mass 5 kg and diameter 0.5m. Calculate the constant torque that has to be applied so that the disc acquires an angular velocity of 5 revolutions per second in 10 seconds. Calculate also the rate at which work is done by the torque at the end of 10 seconds. [0.4742 N-m, 1.489 W]
- [22] A circular disc of radius 0.08m can rotate freely about a horizontal axis passing through its centre. A thread of negligible mass is wound over its circumference and a mass of 0.25 kg is tied to the free end of the thread and allowed to fall freely. In the first 4 seconds the mass falls through 2 m. Calculate (i) angular acceleration and (ii) moment of inertia of the disc. [(i) 3.125 rad/s<sup>2</sup> (ii) 0.0611 kg-m<sup>2</sup>]
- [23] A flywheel has a moment of inertia of 5 kg-m<sup>2</sup>. It is rotating at a speed of 5 revolutions/s. Calculate the constant torque required to stop the flywheel in 10 rotations. Calculate also the work done by the braking torque. [(i) 39.27 N-m (ii) 2467.41 J]



## CHAPTER XI

## HYDRODYNAMICS

*Introduction - Rate of flow of a liquid - Streamline and turbulent motion - Tubes of flow - Equation of continuity - Energy of a fluid in motion - Bernoulli's equation - speed of efflux: Torricelli's theorem - Vena contracta - Venturimeter - The pitot tube - Solved problems - Exercises.*

## 11.1 Introduction

The properties of fluids at rest can be described by the simple concepts of pressure and density, by Archimedes' principle of buoyancy and by Pascal's law of transmission of pressure. When fluids are in motion, new properties become apparent, however, careful considerations show that the phenomenon of fluids in motion can be described in terms of the familiar principles of mechanics. The harnessing of water power, the building of efficient steam turbines, the designing of streamlined cars, trains and airplanes require knowledge of the behavior of fluids in motion. The term fluid is applicable to both liquids and gases and is used in considering their common properties.

## 11.2 Rate of flow of a liquid

An ideal liquid is considered to be *perfectly mobile*, *practically incompressible* and *non-viscous* i.e., having no internal friction. The amount of such a liquid flowing across any section of a tube in a given time is, therefore, always same. The rate of flow of the liquid is defined as its volume (or mass) that flows across any section in unit time.

Let A and B be two sections of a tube of cross-sectional area  $\alpha$  (Fig. 11.1). Obviously the volume of liquid flowing through the section AB is equal to the volume of the cylindrical column

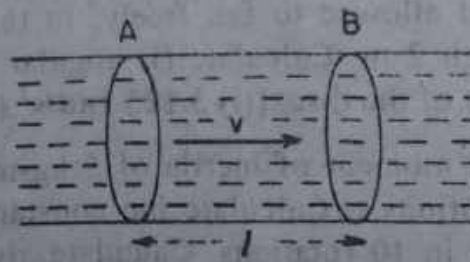


Fig. 11.1



$$= \alpha \times l = \alpha \times vt$$

where  $v$  is the velocity of the liquid in a direction perpendicular to the two sections A and B and  $t$  is the time taken by the liquid to flow from A to B. Thus  $\alpha \times vt$  is the volume of the liquid flowing across the section in time  $t$ .

$$\therefore \text{rate of flow of liquid} = \frac{\alpha \times vt}{t}$$

$$= v \times \alpha \text{ c.c./sec}$$

$$= \text{velocity of the liquid} \times \text{area of cross-section of the tube}$$

Sometimes, the rate of flow of the liquid is also expressed in terms of mass of the liquid flowing across any section in unit time. In this case, Rate of flow of liquid = mass of liquid flowing across any section per unit time

$$= \text{velocity of liquid} \times \text{area of cross-section} \times \text{density of liquid}$$

$$= v \times \alpha \times \rho \text{ gm / sec.}$$

### 11.3 Streamline and turbulent motion

The motion of a fluid may be either steady or unsteady. In steady or orderly motion, the velocity  $v$  at a given point is constant in time. Since velocity at a point does not change in time, every particle arriving at a point, say P, (Fig. 11.2) will pass on with the same speed in the same direction. The same is true for the points Q and R. Thus if we trace out the path of the particle, as indicated by the curve in the figure, then this curve will be the path of every particle arriving at P. This kind of motion is called orderly or *streamline* motion and the curve (i.e., the path traversed by the particle) is called a *streamline*. A streamline may, therefore, be defined as a curve, the tangent to which at any point gives the direction of flow of the liquid at the point. A streamline may be straight or curved, according as the lateral pressure on it is the same throughout or different. When the flow of liquid occurs steadily through a pipe or a tube of uniform cross-section all streamlines are parallel to the axis of the tube and, therefore, to each other.

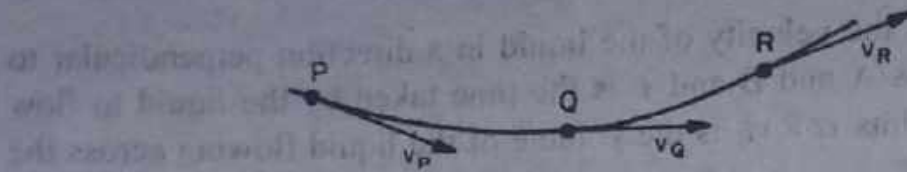


Fig. 11.2

No two streamlines can ever cross each other, for if they do, an oncoming particle arriving at the point of crossing can follow either of the two paths. Then the flow would not be steady. In steady flow, the pattern of streamlines is stationary with time.

The steady motion, however, holds only so long as the velocity of the liquid does not exceed a particular limiting value, called the *critical velocity*. Beyond this velocity, the flow of the liquid loses all its orderliness, and becomes *sinuous* or *zig-zag* and the motion is then called *turbulent motion*.

#### 11.4 Tubes of flow

In principle a streamline can be drawn through every point in the fluid. Therefore, if we consider an element of area  $A$  within the fluid and imagine streamlines to be passing through every point on the periphery of this area, then these lines will enclose a tubular region, called the *tube of flow*. The tube of flow may be thought of as made up of a bundle of streamlines. From the definition of streamlines, *no fluid can cross the side walls of the tube of flow* and hence the liquid in a tube of flow remains in the tube. *The tube behaves somewhat like a pipe of the same shape as its own.* It is convenient to imagine the whole region in which flow occurs to be divided into tubes. In the steady state there can be no mixing of liquids in different flow tubes. Therefore, *the liquid that enters at one end must leave at the other.*

#### 11.5 Equation of continuity

Fig. 11.3 represents a thin tube of flow. The velocity of the fluid inside, although parallel to the tube at any point, may have different magnitudes at different points. Let  $v_1$  and  $v_2$  be the velocities of the fluid



at sections P and Q of the tube respectively. If  $A_1$  and  $A_2$  are the respective areas of cross-sections of these sections then the amount (volume) of fluid flowing per second into the tube at the point P is  $A_1 v_1$  and that flowing out of the tube per second at the point Q is  $A_2 v_2$ . If  $\rho_1$  and  $\rho_2$  be the respective densities of the fluid at sections P and Q, then the mass of the fluid flowing in per second is  $A_1 v_1 \rho_1$  and that flowing out is  $A_2 v_2 \rho_2$ .



Fig. 11.3

Since no fluid can leave through the walls of the tube and there are no sources or sinks wherein the fluid can be created or destroyed, the mass crossing each section of the tube per unit time must be the same.

Hence

$$A_1 v_1 \rho_1 = A_2 v_2 \rho_2$$

If the fluid is incompressible, then  $\rho_1 = \rho_2$ . Thus the above equation takes on a simpler form

$$A_1 v_1 = A_2 v_2$$

or,  $Av = \text{constant}$

(11.1)

This equation is referred to as the *equation of continuity* and states that *in steady incompressible flow the volume flux or the flow rate given by the product  $Av$  across any section is constant*. The equation predicts that the speed of flow varies inversely with the cross-sectional area, being larger in narrower parts. The fact that  $Av$  remains constant also helps us to interpret the streamline picture somewhat. In the narrower section of the tube, the streamlines must crowd together than in the wider part. This means that the speed of the fluid also varies inversely as the distance between the streamlines; a high speed corresponding to regions of closely spaced streamlines and *vice versa*.



### 11.6 Energy of a fluid in motion

A fluid in steady or streamline motion may possess any or all of the following three types of energy, viz.

- (i) *kinetic energy* because of its inertia
  - (ii) *potential energy*, because of its position relative to the earth's surface
- and (iii) what is referred to as pressure energy because of its pressure, for if work is done on the fluid against its (hydrostatic) pressure, the fluid can do the same amount of work back for us and thus acquires energy.

Let us obtain the value of each type of energy per unit mass and per unit volume of the fluid.

- (i) Kinetic energy: If  $m$  be the mass of the fluid of density  $\rho$  flowing with a velocity  $v$ , its kinetic energy is given by  $\frac{1}{2}mv^2$ . Hence *kinetic energy per unit mass of the fluid*  $= \frac{1}{2}v^2$

Now the mass of unit volume of a fluid is equal to its density  $\rho$ . Therefore,

$$\text{kinetic energy per unit volume of the fluid} = \frac{1}{2}v^2 \times \rho = \frac{1}{2}\rho v^2$$

- (ii) Potential energy: If  $m$  be the mass of a fluid which is at a height  $h$  above the surface of the earth, then the potential energy acquired by the fluid  $= mgh$  where  $g$  is the acceleration due to gravity at the place.

Therefore,

$$\text{potential energy per unit mass of the fluid} = gh$$

and *potential energy per unit volume of the fluid*

$$= gh \times \rho = \rho gh$$

- (iii) Pressure energy: Suppose we have an incompressible, non-viscous fluid in a tube of uniform area of cross-section  $\alpha$  and let the hydrostatic pressure of the fluid be  $p$ . If now an additional mass of fluid  $\delta m$  be introduced into the tube against this pressure so gradually as not to impart any velocity and hence any kinetic energy to it and if this mass

occupies a length  $\delta l$  of the tube, then

$$\text{work done on the mass} = \text{force} \times \text{distance} = (p\alpha) \times (\delta l)$$

Since the work done on a mass  $\delta m$  of the fluid forms its pressure energy, we have pressure energy per unit mass of the fluid  $= \frac{p\alpha\delta l}{\delta m}$

$$\text{Now } \delta m = \text{volume} \times \text{density} = \alpha \cdot \delta l \times \rho$$

$$\therefore \text{Pressure energy per unit mass of the fluid} = \frac{p\alpha\delta l}{\alpha\delta l \times \rho}$$

$$= \frac{p}{\rho}$$

Hence pressure energy per unit volume of the fluid

$$= \frac{p}{\rho} \times \rho = p, \text{ its pressure.}$$

### 11.7 Bernoulli's equation

Let us consider an incompressible fluid flowing steadily through a pipe that is not level. The pressure will change from point to point along the pipe, and work must be done to make the fluid flow up. Again when the fluid flows through a horizontal tube of varying cross-section, its velocity changes *i.e.*, it accelerates or decelerates. This means that the resultant force or the pressure acting on the fluid must be different at different points along the flow tube even though the elevation does not change. *Thus the pressure at any point for an incompressible fluid in motion depends not only on its elevation from a reference surface, but also on the velocity of the fluid at that point.* The precise form of this variation is given by a relation known as Bernoulli's equation. The equation is a fundamental relation in fluid mechanics; this is not a new principle, but can be derived from the application of Newton's work-energy theorem.

Consider a steady, incompressible and non-viscous fluid flowing through a tube. Fig 11.4 represents a portion of the tube of flow. We are to follow a small element of the fluid, indicated by shading, as it moves from one point to another along the tube. Let



$y_1$ ,  $v_1$  and  $A_1$  be the elevation, velocity of the fluid and area of cross-section of the tube respectively at the first point P and  $y_2$ ,  $v_2$  and  $A_2$  be the corresponding values at the second point Q. Work must be done to make the fluid flow up from position P to Q. Let us calculate the work carried out on this element of fluid in a certain infinitesimal time interval. The amount of fluid flowing into the tube through the left side of the tube is the same as the amount of fluid flowing out of the tube through the right side of the tube during the time interval.

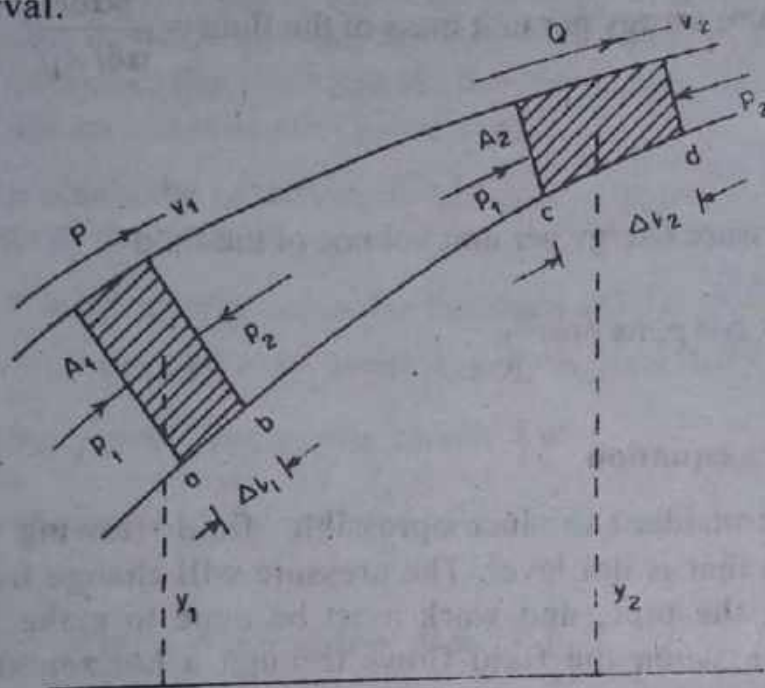


Fig. 11.4

The two forces that work on the shaded element of the liquid as it moves from one point to another are (i) *the force due to pressure* and (ii) *the force due to gravity*. As the fluid flows through the tube, the net effect of these forces is to raise an amount of liquid indicated by the shading from the first position P to the second position Q, *the amount of fluid remaining unchanged in the two positions*.

Since the fluid is under pressure at all points, inward forces, shown by the heavy arrows, are exerted against both faces of the element. As the element moves from the first point to the second point, positive work is done by the force acting on the left face, and negative work by the force acting on the right face (the displacement being against the direction of the force).



If  $A$  represents the cross-sectional area of the tube at any point and  $p$  represents the corresponding pressure, the force against a face of the element at the point is  $pA$ . The work done by the force acting on the left face of the element in motion in the diagram, is

$$\int_a^c F_s \cdot ds = \int_a^c pA \, ds$$

where  $ds$  is a short distance measured along the tube of flow. The limits of integration are from  $a$  to  $c$ , since these are the initial and final positions of the left face. This integral may be written as

$$\int_a^c pA \, ds = \int_a^b pA \, ds + \int_b^c pA \, ds$$

Similarly, the work done by the force acting on the right face of the element is

$$\int_b^d pA \, ds = \int_b^c pA \, ds + \int_c^d pA \, ds$$

The net work is, therefore,

$$\begin{aligned} & \int_a^b pA \, ds + \int_b^c pA \, ds - \int_b^c pA \, ds - \int_c^d pA \, ds \\ &= \int_a^b pA \, ds - \int_c^d pA \, ds \end{aligned}$$

The distances from  $a$  to  $b$  and from  $c$  to  $d$  are sufficiently small so that the pressures and areas may be considered constant along their extent.

Then

$$\int_a^b pA \, ds = p_1 A_1 \Delta l_1 = p_1 A_1 v_1 \Delta t$$

where  $\Delta t$  is the time during which the force  $p_1 A_1$  acts on the shaded element, producing a displacement  $\Delta l = v_1 \Delta t$

Similarly

$$\int_c^d pA \, ds = p_2 A_2 \Delta l_2 = p_2 A_2 v_2 \Delta t$$

Hence the net work done due to the pressure force, is

$$\begin{aligned} & \int_a^b pA \, ds - \int_c^d pA \, ds \\ &= p_1 A_1 v_1 \Delta t - p_2 A_2 v_2 \Delta t \end{aligned}$$

The work done in raising the element from an elevation of  $y_1$ , to an elevation of  $y_2$  by the force of gravity is  $-mg(y_2 - y_1)$ . Minus sign indicates that the work is done against gravity. Noting that during the same time interval  $\Delta t$ , the mass of fluid flowing through the tube is  $\rho A_1 v_1 \Delta t = \rho A_2 v_2 \Delta t$ , the work done against gravity is

$$-\rho g A_1 v_1 \Delta t (y_2 - y_1)$$

The total work carried out on the fluid is the sum of these terms. Hence

$$\begin{aligned} W &= p_1 A_1 v_1 \Delta t - p_2 A_2 v_2 \Delta t - \rho g A_1 v_1 \Delta t (y_2 - y_1) \\ &= A_1 v_1 \Delta t [(p_1 - p_2) - \rho g (y_2 - y_1)] \\ &\quad (\text{since } A_1 v_1 = A_2 v_2) \end{aligned}$$

According to work-energy theorem, the work done by the resultant force acting on a system is equal to the change in kinetic energy of the system. Now the change in kinetic energy of the fluid system is

$$\begin{aligned} \Delta K &= \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2 \\ &= \frac{1}{2} (\rho A_2 v_2 \Delta t) v_2^2 - \frac{1}{2} (\rho A_1 v_1 \Delta t) v_1^2 \\ &= \frac{1}{2} \rho A_1 v_1 \Delta t (v_2^2 - v_1^2) [\because A_1 v_1 = A_2 v_2] \end{aligned}$$

Hence,

$$A_1 v_1 \Delta t [(p_1 - p_2) - \rho g (y_2 - y_1)] = \frac{1}{2} \rho A_1 v_1 \Delta t (v_2^2 - v_1^2)$$

$$\text{or, } (p_1 - p_2) - \rho g (y_2 - y_1) = \frac{1}{2} \rho (v_2^2 - v_1^2)$$

On rearranging,

$$p_1 + \rho g y_1 + \frac{1}{2} \rho v_1^2 = p_2 + \rho g y_2 + \frac{1}{2} \rho v_2^2$$

Since the subscripts 1 and 2 simply refer to any two arbitrary locations along the tube, we can drop the subscripts and write

$$p + \frac{1}{2}\rho v^2 + \rho gy = \text{constant} \quad (11.2)$$

The above equation is referred to as *Bernoulli's theorem* (or *equation*) and may be stated as follows:

*For streamline motion of an incompressible non-viscous fluid, the sum of the pressure at any part plus the kinetic energy per unit volume plus the potential energy per unit volume there is always constant.*

For the special case when the flow of the fluid is horizontal, the height  $y$  is constant and Bernoulli's equation then reduces to

$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2 = \text{constant}$$

This means that the velocity along a horizontal flow-tube increases as the pressure decreases and *vice versa*. Thus *the points of maximum pressure correspond to those of minimum velocity and vice versa*.

A second special case of interest is that for which the fluid does not undergo any motion. Since  $v_1 = v_2 = 0$  in this case, Bernoulli's equation reduces to

$$p_1 + \rho gy_1 = p_2 + \rho gy_2$$

$$\text{or, } p_2 - p_1 = \rho g (y_1 - y_2)$$

$$= \rho gh$$

$$\text{where } y_1 - y_2 = h$$

The above relation expresses the law of pressure change with height in a fluid at rest and is the ordinary law of hydrostatics when the velocity is everywhere zero.

Dividing eqn. (11.2) by  $\rho g$ , we have

$$\frac{v^2}{2g} + \frac{p}{\rho g} + y = \text{constant} \quad (11.3)$$

Eqn. (11.3) is yet another form of Bernoulli's equation in terms of lengths or heads, as they are called. As can be seen, each term on the left



hand side of eqn.(11.3) has the dimensions of length or a height and hence called a head. Thus,  $\frac{1}{2} \frac{v^2}{g}$  is called the velocity head because the fluid must fall through this height to attain the velocity  $v$ . Similarly  $p / \rho g$  is called the pressure head while  $y$  is called the gravitational head.

Thus we have,

$$\begin{aligned} &\text{velocity head} + \text{pressure head} \\ &+ \text{gravitational head} = \text{total head} = \text{constant} \end{aligned}$$

Besides, all terms in the Bernoulli's equation have the dimension of pressure. The term  $(p + \rho gy)$  is the pressure which would be present even if there were no flow ( $v = 0$ ) and is called the static pressure. The term  $\frac{1}{2} \rho v^2$  is called the dynamic pressure.

### 11.8 Speed of efflux: Torricelli's theorem

Let us consider a tank  $T$  of cross-sectional area  $A_1$ , filled to a height  $H$  with a liquid of density  $\rho$  (Fig. 11.5). Suppose that a very small hole of cross-sectional area  $A_2$ , known as the orifice, is made in the side of the tank at a depth  $h$  below the surface of the liquid. We would like to calculate the velocity  $v$  of the liquid with which it issues out of the orifice. This velocity is referred to as the *speed of efflux*.

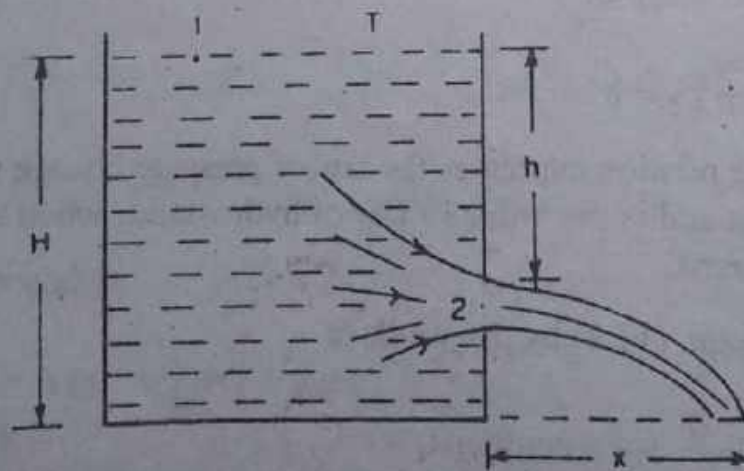


Fig. 11.5

Let us consider the entire volume of the moving fluid as a single tube of flow. Now consider a point 1 at the orifice. Applying Bernoulli's theorem to the points 1 and 2 and taking the orifice as our reference level, we have

$$p_1 + \rho gh + \frac{1}{2}\rho v_1^2 = p_2 + 0 + \frac{1}{2}\rho v_2^2 \quad (11.4)$$

where  $p_1$  and  $p_2$  are the respective pressures at the surface of the tank and at the opening *i.e.*, the orifice. Now both these pressures are the same *i.e.*, atmospheric since both these points are exposed to the atmosphere. Also since the cross-sectional area  $A_1$  is very large compared to area  $A_2$ , then from the relation

$$A_1 v_1 = A_2 v_2$$

$v_1$ , *i.e.*, the velocity at the surface of the tank is very much less than  $v_2$  and may be taken to be zero.

Hence eqn. (11.4) reduces to

$$\rho gh = \frac{1}{2}\rho v_2^2$$

$$\text{or } v_2^2 = 2gh$$

$$\text{or, } v_2 = v = \sqrt{2gh} \quad (11.5)$$

*The above equation was first obtained by Evangelista Torricelli in 1664. This has come to be known as Torricelli's theorem or law, or the law of efflux and may be stated as follows:*

*The speed of efflux of a liquid issuing out of an orifice is the same as that acquired by any body in falling freely through the vertical height between the liquid surface and the orifice.*

No liquid is ideal *i.e.*, completely free from viscosity or internal friction. Hence the velocity  $v = \sqrt{2gh}$  is never really attained in practice. The observed velocity is actually given by the relation  $v = C_v \sqrt{2gh}$  where  $C_v$  is a constant called the coefficient of velocity. In case of water its value lies between 0.95 to 0.99 depending upon the *head* of water *i.e.*, the depth of the orifice below

the water surface and the shape of the orifice. If the orifice is circular and sharp-edged then its value is usually taken to be 0.97.

### Range of a liquid issuing out through an orifice

The liquid jet issues out of the orifice in the form of a parabola and strikes the plane surface at a distance  $x$  from the tank called its range. If  $h_1$  is the height of the orifice from the plane surface, then the time  $t$  taken by the liquid, flowing out of the orifice, to fall through this height can be obtained from the relation

$$S = ut + \frac{1}{2} ft^2$$

Here  $u$  is the initial vertical component of the speed of efflux and is zero;  $s = h_1$  and  $f = g$ , the acceleration due to gravity. Then

$$h_1 = \frac{1}{2} gt^2; \text{ or, } t = \sqrt{\frac{2h_1}{g}}$$

$$\text{Hence } x = v \times t = \sqrt{2gh} \times \sqrt{\frac{2h_1}{g}}$$

$$= 2\sqrt{h \times h_1} \quad (11.6)$$

where  $h$  is the depth of the orifice from the water surface.

But  $h_1 = H - h$  where  $H$  is the height of the liquid column in the tank.

Hence

$$x = 2\sqrt{(H - h)h}$$

$$\text{or, } x^2 = 4(H - h)h$$

$$= 4Hh - 4h^2$$

Differentiating with respect to  $h$

$$2x \cdot \frac{dx}{dh} = 4H - 8h$$



For  $x$  to be maximum,  $\frac{dx}{dh} = 0$

$$\therefore 4H - 8h = 0$$

$$\text{or, } h = \frac{H}{2}; \quad \text{hence } h_1 = \frac{H}{2}$$

*Thus, for the range ( $x$ ) to be maximum, the orifice must be at half the height of the liquid column.*

### 11.9 Vena contracta

The whole of the liquid entering into the orifice does not come only from a direction perpendicular to its plane but from all sides with the streamlines getting curved near the edges as shown in Fig. 11.6. Due to inertia, the liquid entering into the orifice from the sides still possesses a lateral velocity and hence continues to move upwards or downwards, *i.e.*, to wards the centre of the issuing jet, until the outward pressure in it is just balanced by the atmospheric pressure on it. As a result, the

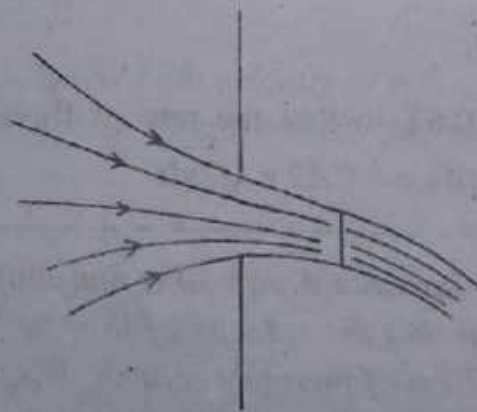


Fig. 11.6

cross-section of the jet goes on contracting after issuing out of the orifice, acquiring its minimum value at certain distance beyond the orifice. This distance is equal to about half the diameter of the orifice. This minimum cross-section or the *neck* is called the *vena contracta*. It is from here that the cross-section of jet remains uniform with the streamlines parallel to each other and perpendicular to the plane of the orifice. The velocity also remains constant after this point. Since the velocity varies inversely as the cross-section, it is obvious that the velocity has its maximum value at the vena contracta. It is this velocity which is given by Torricelli's expression  $v = \sqrt{2gh}$

The cross-section of the jet at the vena contracta is obviously smaller than that at the orifice. The ratio between the two is referred to as the *coefficient of contraction*, denoted by the symbol  $C_c$ . Theoretically its value is 0.611 but it may go up to 0.69, depending upon the *head* of the liquid column and the size and shape of the orifice. For sharp-edged circular orifice its value is nearly equal to 0.624.

Thus the actual rate of flow of the liquid issuing out of the orifice

$$= C_v \cdot v \times C_c \cdot a$$

where  $a$  is the area of cross-section of the orifice. The term  $C_v C_c = C_D$  is called the *coefficient of discharge*. Hence the rate of flow of liquid through the orifice

$$= C_D \cdot a \cdot \sqrt{2gh}$$

For water  $C_D = 0.62$  so that the rate of flow of water through a sharp-edged circular orifice  $= 0.62 a \sqrt{2gh}$

**Example 11.1.** A horizontal pipe of a non-uniform bore has water flowing through it such that the velocity of flow is 40 cm/sec at a point where the pressure is 2 cm of mercury column. What is the pressure at a point where the velocity of flow is 60 cm/sec?  $g = 980 \text{ cm/sec}^2$  and density of water  $= 1 \text{ gm/cc}$ .

**Soln.**

In accordance with Bernoulli's theorem we have,

$$p_1 + \rho gh + \frac{1}{2} \rho v_1^2 = p_2 + \rho gh + \frac{1}{2} \rho v_2^2$$

$$\text{or, } \frac{1}{2}\rho(v_2^2 - v_1^2) = p_1 - p_2$$

$$\text{or, } \frac{1}{2}(v_2^2 - v_1^2) = \frac{p_1 - p_2}{\rho} = p_1 - p_2$$

where  $p_1$  and  $p_2$  are the pressures at the first and the second point respectively,  $v_1$  and  $v_2$  are the corresponding velocities at the points and  $\rho = 1 \text{ gm/cc}$  is the density of water

$$\text{Now } p_1 = 2 \times 13.6 \times 980$$

$$= 26656 \text{ dynes/cm}^2$$

$$v_1 = 40 \text{ cm/sec and } v_2 = 60 \text{ cm/sec}$$

$$\therefore \frac{1}{2}(60^2 - 40^2) = 26656 - p_1$$

$$\text{or, } 1000 = 26656 - p_1$$

$$\text{or, } p_1 = 26656 - 1000$$

$$= 25656 \text{ dynes/cm}^2$$

$$= \frac{25652}{13.6 \times 980} = 1.925 \text{ cm of mercury.}$$

**Example 11.2.** Calculate the velocity of efflux of alcohol from a tall cylinder in which the total pressure is 2 atmospheres (sp. gr. of alcohol = 0.80).

**Soln.**

$$\text{Velocity of efflux, } v = \sqrt{2gh}$$

where  $h$  is the height of alcohol column above the orifice.

The pressure due to alcohol at the level of the axis of the orifice

$$= \text{total pressure} - \text{atmospheric pressure}$$

$$= (2-1) \text{ atmospheric pressure}$$

$$= 76 \times 13.6 \times 981 \text{ dynes/cm}^2$$



Again pressure =  $h \rho g$

$$\text{or, } 76 \times 13.6 \times 981 = h \times 0.80 \times 981$$

$$\text{or, } h = \frac{76 \times 13.6}{0.80} = 1292 \text{ cm}$$

Hence

$$\begin{aligned} v &= \sqrt{2gh} \\ &= \sqrt{2 \times 1292 \times 981} \\ &= 1592 \text{ cm/sec} \\ &= 15.92 \text{ m/sec} \end{aligned}$$

**Example 11.3** Water flows through a horizontal pipe line of varying cross-section. At a point where the pressure of water is 0.05 m of Hg the velocity of flow is 0.25 m/s. Calculate the pressure at another point where the velocity of flow is 0.4 m/s. Density of water =  $10^3 \text{ kg/m}^3$ ;  $g = 9.8 \text{ m/sec}^2$ .

**Soln.**

For a horizontal pipe line, we have

$$p_1 + \frac{1}{2} \rho v_1^2 = p_2 + \frac{1}{2} \rho v_2^2$$

$$\text{or, } p_2 = p_1 + \frac{1}{2} \rho (v_1^2 - v_2^2)$$

$$\begin{aligned} p_1 &= 0.05 \times 13.6 \times 10^3 \times 9.8 \text{ N/m}^2 \\ &= 6.664 \times 10^3 \text{ N/m}^2. \end{aligned}$$

$$v_1 = 0.25 \text{ m/sec}$$

$$v_2 = 0.4 \text{ m/sec}$$

$$\begin{aligned} \therefore p_1 &= 6.664 \times 10^3 + \frac{1}{2} \times 10^3 [(0.25)^2 - (0.4)^2] \\ &= 6.664 \times 10^3 - 48.75 \\ &= 6615.25 \text{ N/m}^2 \end{aligned}$$

$$= \frac{6615.25}{13.6 \times 10^3 \times 9.8} \text{ m of Hg}$$

$$= 0.04963 \text{ m of Hg}$$

**Example 11.4** In a horizontal pipe line of uniform area of cross-section, the pressure falls by  $5 \text{ N/m}^2$  between two points separated by a distance of  $1 \text{ km}$ . What is the change in kinetic energy per kg of the oil flowing at these points? Density of oil  $= 800 \text{ kg/m}^3$ .

**Soln.**

For a horizontal pipeline, we have in accordance with Bernoulli's principle

$$p_1 + \frac{1}{2} \rho v_1^2 = p_2 + \frac{1}{2} \rho v_2^2$$

$$\text{or, } p_1 - p_2 = \frac{1}{2} \rho (v_2^2 - v_1^2)$$

$$\text{or, } \frac{p_1 - p_2}{\rho} = \frac{1}{2} (v_2^2 - v_1^2)$$

= change in kinetic energy per kg of the oil

$$\text{Now } p_1 - p_2 = 5 \text{ N/m}^2$$

$$\rho = 800 \text{ kg/m}^3$$

$\therefore$  Change in kinetic energy per kg.

$$\frac{1}{2} (v_2^2 - v_1^2) = \frac{p_1 - p_2}{\rho}$$

$$= \frac{5}{800} = 6.25 \times 10^{-3} \text{ J/kg}$$

**Example 11.5** A tank containing water has an orifice on one vertical side. If the centre of the orifice is  $4.9 \text{ meters}$  below the surface level in the tank find the velocity of discharge, assuming that there is no wastage of energy.

**Soln.**

$$\begin{aligned} v &= \sqrt{2gh} \\ &= \sqrt{2 \times 9.8 \times 4.9} \\ &= 9.8 \text{ m/sec.} \end{aligned}$$

**Example. 11.6** Water is flowing smoothly through a closed - pipe system. At one point the speed of the water is 3.0 m/s, while at another point 1.0 m higher the speed is 4.0 m/s. (a) If the pressure at the lower point is 20  $\text{KP}_a$  ( $1\text{P}_a = 1\text{N/m}^2$ ), what is the pressure at the upper point? (b) At what pressure at the upper point the water would stop flowing, the pressure at the lower point being 18  $\text{KP}_a$ ? Density of water =  $10^3 \text{ kg/m}^3$

**Soln.**

(a) From Bernoulli's theorem we have

$$p_1 + \rho gh_1 + \frac{1}{2} \rho v_1^2 = p_2 + \rho gh_2 + \frac{1}{2} \rho v_2^2$$

$$\text{or, } p_1 - p_2 = \rho g(h_2 - h_1) + \frac{1}{2} \rho (v_2^2 - v_1^2)$$

here  $h_2 - h_1 = 1\text{m}$ ,  $v_2 = 4\text{m/s}$  and  $v_1 = 3\text{m/s}$

$$\begin{aligned} \therefore p_1 - p_2 &= (10^3)(9.8)(1) + (0.5)(10^3)(4^2 - 3^2) \\ &= 9800 + 3500 \\ &= 12300 \text{ P}_a \\ &= 12.3 \text{ KP}_a \end{aligned}$$

$$\begin{aligned} \therefore p_2 &= p_1 - 12.3 \text{ KP}_a = 20 \text{ KP}_a - 12.3 \text{ KP}_a \\ &= 6.7 \text{ KP}_a \end{aligned}$$

(b) When water stops flowing,  $v_1 = v_2 = 0$

$$\therefore p_1 - p_2 = \rho g(h_2 - h_1)$$

$$\begin{aligned} \text{or, } p_2 &= p_1 - \rho g(h_2 - h_1) \\ &= p_1 - (10^3)(9.8)(1) \end{aligned}$$



$$= p_1 - 9800 P_a$$

$$= 18 \text{ KP}_a - 9.8 \text{ KP}_a = 8.2 \text{ KP}_a$$

**Example. 11.7** Water flows out of a pipe at the rate of  $3.0 \text{ cm}^3/\text{s}$ . Find the velocity of the water at a point in the pipe where the diameter is (a)  $0.50 \text{ cm}$  and (b)  $0.80 \text{ cm}$

**Soln.**

From the equation of continuity, we have

$$Av = \text{constant.}$$

$$(a) \quad \pi r^2 v = 3.0$$

$$(b) \quad \pi r^2 v = 3.0$$

$$(3.14) \left( \frac{0.50}{2} \right)^2 v = 3.0$$

$$(3.14) \left( \frac{0.80}{2} \right)^2 v = 3.0$$

$$\therefore v = 15.28 \text{ cm/s}$$

$$v = 5.97 \text{ cm/s}$$

**Example 11.8** A water barrel stands on a table of height  $h$ . If a small hole is punched in the side of the barrel at its base, it is found that the resultant stream of water strikes the ground at a horizontal distance  $R$  from the barrel. What is the depth of water in the barrel?

**Soln.**

From Torricelli's theorem, the horizontal velocity of efflux from the barrel is  $v = \sqrt{2gd}$  where  $d$  is the depth of the water in the barrel. The time  $t$  taken by the water to hit the ground is given by (Art. 11.8)

$$h = \frac{1}{2}gt^2; \text{ or } t = \sqrt{2h/g}$$

So the horizontal distance  $R$  is given by

$$R = v \times t = \sqrt{2gd} \cdot \sqrt{\frac{2h}{g}} = 2\sqrt{dh}$$

$$\text{or, } R^2 = 4.dh \quad \therefore d = \frac{R^2}{4h}$$

**Example 11.9** The cross-sectional area of the aorta (the major blood vessel emerging from the heart) of a normal resting person is  $3 \text{ cm}^2$  and the speed of blood in the aorta is  $30 \text{ cm/sec}$ . A typical capillary (diameter  $\cong 6 \mu\text{m}$ ) has a cross-sectional area of  $3 \times 10^{-7} \text{ cm}^2$  and blood flows through the capillary with a speed of  $0.05 \text{ cm/sec}$ . how many capillaries does such a person have?

**Soln.**

Suppose the number of capillaries =  $n$ . All the blood that passes through the capillaries must have passed through the aorta. Let the area of cross-section of aorta be  $A$  and the flow speed of blood through it be  $v$  and let the corresponding values for capillary be  $A_0$  and the flow speed of blood through it be  $v_0$

respectively, then from eqn. of continuity, we have

$$Av = nA_0v_0 \quad ; \quad \text{or, } n = \frac{Av}{A_0v_0} = \frac{(3\text{cm}^2)(30\text{cm})/\text{sec}}{(3 \times 10^{-7}\text{cm}^2)(0.05\text{cm}/\text{sec})}$$

$$= 6 \times 10^9 \quad \text{or } 6 \text{ billion.}$$

**Example 11.10** A hole of area  $1 \text{ mm}^2$  opens in the pipe near the lower end of a large water-storage tank, and a stream of water shoots from it. If the top of the water in the tank is  $20 \text{ m}$  above the point of the leak, how much water escapes in 1 second?

**Soln.**

The flow of water in 1 second =  $vA$  where  $A$  is the area of the hole and  $v$  is the speed of water escaping through the hole

According to Torricelli's theorem

$$\therefore v = \sqrt{2gh} = \sqrt{(2)(9.8)(20)} = 19.8 \text{ m/s}$$

$$\therefore vA = (19.8)(1 \times 10^{-6})$$

$$= 19.8 \times 10^{-6} \text{ m}^3/\text{sec}$$

$$= 19.8 \text{ mL/s}$$

**Example 11.14** A vessel is filled with water to a height 20m. At the bottom of the vessel, there is a sharp-edged circular orifice of cross-section  $0.75 \text{ cm}^2$ . What amount of water flows through the orifice in 1 second?

**Soln.**

For a sharp-edged circular orifice, the rate of flow of water is (Art.9.9)

$$\begin{aligned}
 &= 0.62A\sqrt{2gh} \\
 &= (0.62)(0.75 \times 10^{-4} \text{ m}^2)(\sqrt{(2)(9.8)(20)}) \text{ m/sec} \\
 &= 9.20 \times 10^{-4} \text{ m}^3/\text{s} \\
 &= 920 \text{ mL/s}
 \end{aligned}$$

### 11.10 Venturimeter

*Venturimeter* is an arrangement to measure the rate of flow of a liquid in a pipe. It is usually used to measure the rate of flow of water in the city water supply main; it is then called *venturi water meter*. The underlying principle of a venturimeter is based on Bernoulli's theorem that when a liquid flows through a tube of varying bore or cross-section, the velocity and pressure vary along the tube, the velocity being maximum where the pressure is least and vice versa.

As shown in Fig. 11.7 a venturimeter consists of three distinct parts (i) a convergent cone AB, (ii) a short horizontal part BC and (iii) a divergent cone CD. The diameter of the outer end of the convergent cone is  $d_1$  at A, tapering down to a much smaller diameter  $d_2$  at its inner end B. The slopes of its sides are in the ratio 1:4 or 1:5, the angle of convergence being between  $20^\circ$  to  $30^\circ$ . The short horizontal part BC has a uniform diameter  $d_2$ , and is known as the *throat*. The diameter of the inner end C of the diverging cone CD has the same diameter  $d_2$  of the throat while its diameter at the outer end D is the same as the diameter  $d_1$  at A of the convergent cone AB. But the slope of its side is very much smaller than that of the convergent cone, being in the ratio of 1:10 or even 1:20, the angle of divergence lying between  $5^\circ$  and  $14^\circ$ .



the rate of flow of water is to be determined. The ratio of  $d_2/d_1$  is called the *throat ratio*. Its usual value is  $\frac{1}{2}$  although it may vary between  $\frac{1}{4}$  to  $\frac{3}{4}$ . As already mentioned, the function of the convergent cone is to increase the velocity of the stream at the throat by decreasing the pressure there. This is achieved by reducing the cross-sectional area of the stream at the throat, thereby, producing an appropriate pressure difference between the inlet A of the cone and the throat. However, the ratio  $d_2/d_1$  should not be such that the pressure at the throat becomes negative i.e. below the atmospheric pressure. Otherwise, the water would start getting vaporized by giving out its dissolved air, thereby, breaking the continuity of the water stream. Further, the venturimeter must always run full of water or else any air present in it may disrupt the continuity of water flow.

To measure the pressure difference between the inlet A and the throat, two narrow vertical glass tubes (piezometer tubes)  $P_1$  and  $P_2$  are fixed in the water main, close to the end A and the mid-point of the throat respectively as shown in Fig. 11.7.

### Calculation

Since the velocity at the throat is much higher than the velocity in the main (or inlet A), the pressure at the throat will be correspondingly lower than that at the main. The height of the water column in the tube  $P_2$  will be lower than that in the tube  $P_1$ , the heights being measured from a common reference level- the axis of the tube. If the pressure at the main is  $P_1$  and that in the throat is  $P_2$  then we have

$$P_1 - P_2 = h \rho g$$

where  $h$  is the difference in height of the water levels in the two tubes,  $\rho$  the density of water, regarded as constant throughout the tube and  $g$  the acceleration due to gravity.

Let  $a_1$  and  $a_2$  are the respective cross-sectional areas of the main (or the inlet A) and the throat. If  $v_1$  and  $v_2$  be the velocities of the water stream in the main and the throat respectively, then we have, from the *equation of continuity*

the rate of water flow,

$$Q = a_1 v_1 = a_2 v_2$$

$$\text{or, } \frac{v_2}{v_1} = \frac{a_1}{a_2} \quad (11.8)$$

Applying Bernoulli's theorem to a point in the main and a point in the throat,

$$p_1 + y_1 \rho g + \frac{1}{2} \rho v_1^2 = p_2 + y_2 \rho g + \frac{1}{2} \rho v_2^2$$

Since the venturimeter is horizontal,

$$y_1 = y_2$$

$$\therefore p_1 + \frac{1}{2} \rho v_1^2 = p_2 + \frac{1}{2} \rho v_2^2$$

$$\text{or, } p_1 - p_2 = \frac{\rho}{2} (v_2^2 - v_1^2)$$

$$\text{or, } h \rho g = \frac{\rho}{2} (v_2^2 - v_1^2)$$

Rearranging the right hand side of the equation we have

$$h \rho g = \frac{\rho v_1^2}{2} \left( \frac{v_2^2}{v_1^2} - 1 \right)$$

Substituting the value of  $\frac{v_2}{v_1}$  from the relation (12.8), we have

$$p_1 - p_2 = \frac{\rho v_1^2}{2} \left[ \frac{a_1^2}{a_2^2} - 1 \right]$$

$$= \frac{\rho v_1^2}{2} \left[ \frac{a_1^2 - a_2^2}{a_2^2} \right]$$

$$\text{or, } v_1^2 = \frac{2(p_1 - p_2)a_2^2}{\rho[a_1^2 - a_2^2]}$$

$$\text{or, } v_1 = a_2 \sqrt{\frac{2(p_1 - p_2)}{\rho(a_1^2 - a_2^2)}}$$

$$\begin{aligned}
 &= a_2 \sqrt{\frac{2h\rho g}{\rho(a_1^2 - a_2^2)}} \\
 &= a_2 \sqrt{\frac{2hg}{a_1^2 - a_2^2}} \quad (11.10)
 \end{aligned}$$

$$(\because p_1 - p_2 = h\rho g)$$

Hence the *rate of flow (or rate of discharge) of water* through the main is

$$Q = a_1 v_1 = a_1 a_2 \sqrt{\frac{2(p_1 - p_2)}{\rho(a_1^2 - a_2^2)}} \quad (11.11)$$

$$= a_1 a_2 \sqrt{\frac{2hg}{a_1^2 - a_2^2}} \quad (11.12)$$

$a_1$  and  $a_2$  depend upon the dimensions of the venturimeter,  $h$  is the difference in the water levels in the two tubes  $p_1$  and  $p_2$ . These quantities are known for a given venturimeter. Thus from the knowledge of  $g$  at the given place, the rate of flow or rate of discharge of water through the main can be determined.

The factor  $a_1 a_2 \sqrt{2g/(a_1^2 - a_2^2)}$  for a given venturimeter at a given place is a constant and is called the *constant of the venturimeter*, denoted by the letter  $C$ .

Then the rate of flow or rate of discharge,

$$Q = C\sqrt{h} \quad (11.13)$$

Eqn. (9.13) gives the *theoretical rate of discharge of water*. When the slight fall in the pressure head between  $A$  and  $B$  is taken into account, the rate of discharge is given by

$$Q = kC\sqrt{h}$$

where  $k$  is called the *coefficient of discharge*. The usual value of  $k$  is 0.97 so that the rate of discharge is given by

$$Q = 0.97 C\sqrt{h} \quad (11.14)$$



Venturimeter was actually devised by an American engineer, *Clemens Herschel*, in 1886. But he gave it the name venturimeter in honour of *G. B. Venturi*, an Italian philosopher and engineer who had earlier studied the phenomenon of liquid flow through the pipes in 1791.

### 11.11 The Pitot tube

The *Pitot tube*, also based on Bernoulli's theorem, is a simple device for measuring the velocity of water flow and hence to determine the rate of flow of water.

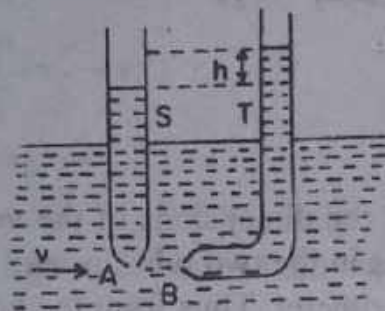


Fig. 11.8

A pitot tube consists of two tubes. One of the tubes S is straight with a small aperture at one end, the other end being wide open (Fig 11.8). The other tube T is bent in L- form, with an aperture at the smaller bent end, whereas the other end is again wide open. The combination of this two tubes can be placed in the liquid separately or joined together at the upper ends. Let us study the working of the Pitot tube.

Let the Pitot tube be immersed in the liquid, with the aperture B at the nose of the bent tube facing the liquid flow, *i.e.*, with the plane of the aperture normal to the direction of the liquid flow. The aperture A at the end of the straight tube is on the same streamlines with B but lying some distance away from it with its aperture parallel to the direction of flow of water. The liquid flow will be brought to stop at the nose B, which is, therefore, known as

*stagnation point* of the tube. The velocity of water, therefore, becomes zero at this point and its kinetic energy is reduced from  $\frac{1}{2}\rho v^2$  to zero where  $v$  is the velocity of water in the concerned streamline. The whole of this kinetic energy is converted into potential energy and consequently the water rises to a higher level in the tube  $T$  than in  $S$ . This liquid column in the tube thus measures the total *stagnation* pressure at  $B$ . Let it be  $p_B$ . The height of the water column in  $S$  measures the static pressure in  $S$ . Let it be  $p_A$ .

Let us now apply *Bernoulli's equation* to the points  $A$  and  $B$ . We get

$$p_A + \frac{1}{2}\rho v^2 = p_B + 0$$

the velocity of the liquid being zero at  $B$ .  $\rho$  is the density of the liquid, supposed to remain constant.

$$\therefore \frac{1}{2}\rho v^2 = p_B - p_A = h\rho g \quad (11.15)$$

where  $h$  is the difference in heights of the liquid columns in  $T$  and  $S$ .

$$\therefore v^2 = 2hg$$

$$\text{or, } v = \sqrt{2hg} \quad (11.16)$$

If  $a$  is the area of cross-section of the pipe or the channel at the place where the Pitot tube is immersed, then the *rate of liquid flow*

$$Q = av = a\sqrt{2gh} \quad (11.17)$$

**Example 11.15** A horizontal tube has different areas of cross-section at two points  $P$  and  $Q$ . The diameter at  $P$  is  $0.04\text{m}$  and the diameter at  $Q$  is  $0.02\text{m}$ . When a liquid of density  $800\text{ kg/m}^3$  flows through the tube, the pressure difference between the points  $P$  and  $Q$ , as recorded by two manometer limbs fixed at these points, is  $0.08\text{ m}$  column of the liquid.

**Soln.**

According to eqn. (11.11)

$$\text{Rate of flow} = a_1 v_1$$

$$= a_1 a_2 \sqrt{\frac{2(p_1 - p_2)}{\rho(a_1^2 - a_2^2)}}$$

$$a_1 = \pi r^2 = \pi \times (0.02)^2 = 4\pi \times 10^{-4} \text{ m}^2$$

$$a_2 = \pi r^2 = \pi \times (0.01)^2 = \pi \times 10^{-4} \text{ m}^2$$

$$p_1 - p_2 = h\rho g = 0.08 \times 800 \times 9.8$$

$$a_1^2 = 16\pi^2 \times 10^{-8}$$

$$a_2^2 = \pi^2 \times 10^{-8}$$

$$h = 0.08 \text{ m}$$

$$\therefore \text{rate of flow} = 4\pi \times 10^{-4} \times \pi \times 10^{-4} \times \sqrt{\frac{2 \times 0.08 \times 800 \times 9.8}{800(16\pi^2 \times 10^{-8} - \pi^2 \times 10^{-8})}}$$

$$= \frac{4\pi^2 \times 10^{-8}}{\pi} \sqrt{\frac{1.568}{15 \times 10^{-8}}}$$

$$= 12.56 \times 10^{-8} \times 32.32 \times 10^2$$

$$= 405.94 \times 10^{-6}$$

$$= 4.0595 \times 10^{-4} \text{ m}^3/\text{sec}$$

**Example 11.16** Calculate the speed at which the velocity head of a stream of water is equal to 0.50 m of Hg.

**Soln.**

$$\text{Velocity head} = \frac{v^2}{2g} \text{ meters of water.}$$

But velocity head = 0.5 m of Hg

$$= 0.5 \times 13.6 \text{ m of water}$$

$$\therefore \frac{v^2}{2g} = 0.5 \times 13.6$$



$$v^2 = 2 \times 9.8 \times 0.5 \times 13.6$$

$$v = \sqrt{9.8 \times 13.6}$$

$$= 11.54 \text{ m/sec}$$

**Example 11.17** Water is flowing through a horizontal pipeline. At two points A and B, the diameters are 0.6 m and 0.2 m. The pressure difference between the points A and B is 1 meter column of water. Calculate the volume of water flowing per second

**Soln.**

Rate of flow

$$= a_1 a_2 \sqrt{\frac{2(p_1 - p_2)}{\rho(a_1^2 - a_2^2)}}$$

$$= a_1 a_2 \sqrt{\frac{2hg}{a_1^2 - a_2^2}}$$

$$a_1 = \pi r^2 = \pi \times \left(\frac{0.6}{2}\right)^2 = 0.09\pi \text{ m}^2$$

$$a_2 = \pi r^2 = \pi \times \left(\frac{0.2}{2}\right)^2 = 0.01\pi \text{ m}^2$$

$$a_1^2 = 0.0081\pi^2$$

$$a_2^2 = 0.0001\pi^2$$

$$a_1^2 - a_2^2 = 0.0080\pi^2$$

$$= 8 \times 10^{-3} \pi^2$$

$$h = 1 \text{ m}, g = 9.8 \text{ m/sec}^2$$

$\therefore$  Rate of flow

$$= (0.09\pi \times 0.01\pi) \sqrt{\frac{2 \times 1 \times 9.8}{8 \times 10^{-3} \pi^2}}$$

$$= \frac{9 \times 10^{-4} \times \pi^2}{\pi} \sqrt{2.45 \times 10^3}$$

$$= 28.26 \times 4.949 \times 10 \times 10^{-4}$$

$$= 139.85 \times 10^{-3}$$

$$= 0.13985 \text{ m}^3/\text{sec}$$

**Example 11.18** Air flows through the horizontal main tube of a venturimeter. The radii of the wide and narrow parts of the main tube are 1.0 cm and 0.50 cm, respectively. If the speed of the air entering the main tube be 15.0 m/s, find the difference in mercury levels in the two limbs of the manometer tube.

**Soln.**

According to Bernoulli's theorem, we have

$$p_1 + \rho gh_1 + \frac{1}{2} \rho v_1^2 = p_2 + \rho gh_2 + \frac{1}{2} \rho v_2^2$$

$$\text{But } h_1 = h_2$$

$$\therefore p_1 - p_2 = \frac{1}{2} \rho_{\text{air}} (v_2^2 - v_1^2) \quad (\text{i})$$

from eqn. of continuity

$$A_1 v_1 = A_2 v_2 \quad \therefore v_2 = v_1 \frac{A_1}{A_2} = v_1 \frac{r_1^2}{r_2^2}$$

$$\therefore v_2^2 - v_1^2 = v_1^2 \left( \frac{r_1^4}{r_2^4} - 1 \right)$$

Inserting this value in eqn (i), we get

$$p_1 - p_2 = \frac{1}{2} \rho_{\text{air}} v_1^2 \left( \frac{r_1^4}{r_2^4} - 1 \right) \quad (\text{ii})$$

$$\text{But } p_1 - p_2 = h \rho_{\text{mer}} g$$

$$\text{or, } h = \frac{p_1 - p_2}{\rho_{\text{mer}} g}$$

$$= \frac{\rho_{\text{air}} v_1^2}{2 \rho_{\text{mer}} g} \left( \frac{r_1^4}{r_2^4} - 1 \right)$$

$$\rho_{\text{air}} = 1.3 \text{ kg/m}^3$$

$$\rho_{\text{mer}} = 13.6 \times 10^3 \text{ kg/m}^3$$

$$v_1 = 15.0 \text{ m/s}$$

$$r_1 = 1.0 \text{ cm}$$

$$r_2 = 0.5 \text{ cm}$$

$$g = 9.8 \text{ m/sec}^2$$

$$h = \frac{(13 \text{ kg/m}^3)(15.0 \text{ m/s})^2}{2(13.6 \times 10^3 \text{ kg/m}^3)(9.8 \text{ m/sec}^2)} \left[ \left( \frac{1.0 \text{ cm}}{0.5 \text{ cm}} \right)^4 - 1 \right]$$

**Example. 11.19.** A water main of 20 cm diameter has a Pitot tube fixed into it and the pressure difference indicated by the gauge is 5 cm of water column. Calculate the rate of flow of water through the main. ( $g = 980 \text{ cm/sec}^2$  and  $\rho$  for water =  $1 \text{ gm/cc}$ ).

**Soln.**

$$\text{Rate of flow} = av = a\sqrt{2gh}$$

$$a = \pi r^2 = \pi \times (10)^2 = 100\pi$$

$$\therefore \text{rate of flow} = 100 \times \pi \sqrt{2 \times 980 \times 5}$$

$$= 314\sqrt{9800}$$

$$= 314 \times 98.99$$

$$= 31084.4 \text{ cc/sec}$$

$$= 31.08 \text{ liters/sec}$$

**Example 11.20** A Pitot tube is fixed on the wing of an aeroplane to measure its speed. The tube contains a liquid of density  $800 \text{ kg/m}^3$ . The difference in level between the two limbs is 0.5 m. Density of air =  $1.293 \text{ Kg/m}^3$ . Calculate the speed of the aero plane.

**Soln.**

From eqn. (12.15), we have

$$\frac{1}{2} \rho v^2 = p_A - p_B = h d g$$



$$\text{or, } v^2 = \frac{2hdg}{\rho}$$

where  $\rho$  is the density of air and  $d$  is the density of the liquid in the pitot tube

$$\therefore v^2 = \frac{2hdg}{\rho}$$

$$\rho = 1.293 \text{ kg/m}^3$$

$$d = 800 \text{ kg/m}^3$$

$$h = 0.5 \text{ m}$$

$$g = 9.8 \text{ m/sec}^2$$

$$\text{or, } v = \sqrt{\frac{2hdg}{\rho}}$$

$$= \sqrt{\frac{2 \times 0.5 \times 800 \times 9.8}{1.293}}$$

$$= 77.85 \text{ m/sec}$$

### EXERCISES

- [1] Explain the terms stream line motion and rate of flow of a fluid.
- [2] Distinguish between streamline and turbulent flow of a liquid.
- [3] What is the equation of continuity? Derive an expression for the equation of continuity.
- [4] State Bernoulli's principle for fluid motion.
- [5] Derive Bernoulli's equation for a fluid in streamline motion.
- [6] Discuss the various forms of energy possessed by liquid in motion.
- [7] Show that for a liquid in streamline motion  $\frac{p}{\rho} + gh + \frac{v^2}{2} = \text{constant}$ .
- [8] Enunciate Torricelli's theorem and obtain expression for the velocity of efflux of a liquid from a tank.

- [9] Describe a venturimeter and explain its working principle. Derive an expression for the rate of flow of a liquid through a pipe line.
- [10] Describe a Pitot tube and explain how it can be used to measure the rate of flow of a liquid.
- [11] State and prove Torricelli's theorem. What is vena contracta?
- [12] State and prove Bernoulli's theorem for fluid motion and give two of its applications.
- [13] Write short notes on:
- (i) Equation of continuity
  - (ii) Torricelli's theorem
  - (iii) Vena contracta
  - (iv) Pitot tube
- [14] Water flows through a horizontal pipe having a tapering bore. The velocity of water is 2 m/sec at the broader end and the pressure is 1 kN/m<sup>2</sup> less at the narrow end. What is the velocity of water at the latter end? [6m/sec].
- [15] Water is flowing through a horizontal main having a non-uniform bore. The velocity of water is 100cm/sec at a point in it where the pressure is 7/19th of that of the atmosphere. Calculate the velocity of water at the point where the pressure is half this value. [251.7 cm/sec.]
- [16] A fire engine pumps water from a hydrant at the rate of 10<sup>3</sup> liters/sec. It ejects it from a nozzle 5 m above the surface of water in the hydrant with a velocity of 10 m/s. Calculate (i) the pressure difference between the water at the pump and the nozzle and (ii) power of the engine. [(i) 99 x 10<sup>3</sup> N/m<sup>2</sup>, (ii) 99 kilowatts].
- [17] A venturimeter has a pipe diameter of 0.2 m and a throat diameter of 0.15. The levels of water column in the two limbs differ by 0.1 m. Calculate the amount of water discharged through the pipe in one hour. Density of water = 10<sup>3</sup> kg/m<sup>3</sup> [107.6 m<sup>3</sup>/hr].
- [18] Water flows through a horizontal pipe-line of varying cross-section at the rate of 0.2 m<sup>3</sup>/s. Calculate the velocity of water at a point where the area of cross-section of the pipe is 0.02 m [10m/sec]
- [19] A venturimeter is connected to a horizontal main of radius 20 cm. If the radius of the throat of the venturimeter be 15 cm and the difference of water level in the piezometer tubes be 10 cm, calculate the rate of flow of water per hour through the main [430.65 x 10<sup>3</sup> liters / hr]



## CHAPTER XII

## VISCOSITY

*Laminar or viscous flow-Viscosity-Newton's law of viscous flow-Coefficient of viscosity-Units and dimensions of viscosity-Fugitive elasticity-Critical velocity-Flow of a liquid through a narrow tube-Poiseuille's equation—Determination of coefficient of viscosity of a liquid by the capillary flow method-Motion in a viscous medium :Stokes' law-Stokes' method for the determination of coefficient of viscosity of a liquid-The falling sphere viscometer-Rotation viscometer-Solved problems-Exercises.*

## 12.1 Laminar or viscous flow

Let us consider a certain amount of liquid between two parallel plates as in Fig. 12.1 (a). The bottom plate is stationary. If now a shearing stress is applied to the top surface of the liquid, say by moving the top plate forward, then the top surface (or layer) of the liquid in contact with this plate will begin to flow with the same velocity as the top plate since by its very definition, a liquid cannot permanently withstand a shearing stress. The layer of liquid in contact

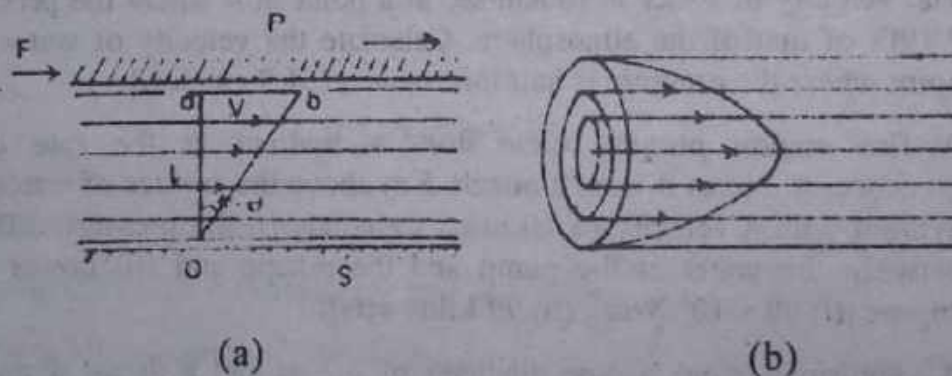


Fig. 12.1

with the fixed bottom plate, however, remains at rest. In between these two plates, *i.e.*, the boundary surfaces, velocities of the intermediate layers increase uniformly from the bottom surface to the top surface. The greater the distance of the layer from the fixed surface, the greater is its velocity; so that the *velocity-distribution curve* or the *profile* of the flowing liquid is a *straight line* Ob [Fig 12.1 (a)] inclined to the vertical. The magnitudes of the velocities of the different layers are indicated by the lengths of the arrows in the figure.



Thus, if the flow of the liquid be slow or steady, if it does not exceed a certain velocity called the *critical velocity*, then it behaves as though it consists of a number of *layers of laminae*, one above the other, with the uppermost layer moving with the greatest velocity and each succeeding layer slower than the one above it, with the lowermost layer in contact with the bottom fixed surface remaining at rest. *Such a liquid-flow, in which the different layers or laminae glide over one another at a slow and steady velocity without intermixing, is called a laminar, streamline or viscous flow.*

If the liquid flows through a tube or pipe, then the layers are cylindrical in form, one inside the other [Fig.12.1 (b)]. Again the cylindrical layer which is in contact with the walls of the tube or pipe remains at rest whereas the one *farthest from this stationary layer i.e.*, the layer along its axis moves with the greatest velocity. Thus in case of a liquid flow through a tube, the velocity of the layers increases from *zero* at the walls of the tube (or pipe) to a *maximum* along its axis. The *velocity-distribution curve* or the *profile* of the flowing liquid here is parabolic in form as shown in Fig. 12.1 (b).

## 12.2 Viscosity

As discussed in Art. 12.1, if we consider any particular layer of a liquid which is in streamline or viscous flow, we have a layer immediately *below* it moving slower than it and a layer immediately *above* it moving faster than it. Thus the layer below it tends to retard its motion while that above it tends to *accelerate* it. The two layers taken together tend to destroy their relative motion as though a tangential, backward *dragging force*, frictional in nature, comes into play between the layers tending to oppose their relative motion.

This *tangential backward dragging force* which comes into play in between two adjacent layers of a liquid, tending to oppose their relative motion, is called the *viscous force* or the *viscous drag*.

And, the property of a liquid which gives rise to such viscous force by virtue of which it tends to oppose the relative motion between the different layers of the liquid is known as *viscosity* or *internal friction* of the liquid.

It is, therefore, clear that if a relative motion between the layers

of a liquid is to be maintained, an external force must be applied to it. In the absence of any such external force, the relative motion between the layers is destroyed and the liquid ceases to flow.

### 12.3 Newton's law of viscous flow-coefficient of viscosity

As mentioned in the previous article, an external force must be applied to maintain the relative motion between the layers of the liquid. Referring back to Fig. 12.1(a), let this external force be  $F$ , and  $A$  the surface area of the layer of the liquid over which this force is applied. Then the ratio  $F/A$  will be the *tangential or shearing stress*, exerted on the liquid. If as a result of this stress, a point  $a$  on the topmost layer moves to a point  $b$ , then the ratio

$\theta = \frac{ab}{l}$  is known as the *angle of shear or shearing strain*. Newton

found that this shear strain  $\theta$  is a function of the *shearing stress*  $F/A$  and is a measure of the viscosity of the liquid. He termed the ratio between  $F/A$  and  $\theta$  as the *coefficient of viscosity* of the liquid. Thus *coefficient of viscosity of the liquid*

$$\eta = \frac{F/A}{\theta}$$

Now in the case of a solid, when a shearing stress is applied, the effect is to produce a certain displacement, say  $ab$ , of the solid. If  $l$  be the transverse dimension of the solid, then the angle of shear is  $\theta = ab/l$  and within elastic limit, the shearing stress is proportional to shearing strain. Further, *the displacement once produced stays constant so long as the shearing stress remains operative*. But as pointed out by Maxwell, when a shearing stress is applied to a liquid, unlike solid, the liquid breaks down under the shearing stress only for the shear to be formed again. This is because the point  $a$  on the liquid surface gets displaced through a distance  $ab$ , say after every one second. Thus a fresh shear  $\theta$  is formed at intervals of one second. In the case of liquid  $\theta$ , therefore, does not stand for shear strain, but for the *rate of formation of shear*. Since the transverse distance  $l$  is constant, the rate of formation of shear or the rate of change of shear strain is equal to  $1/l$  times the rate of change of  $ab$ . But the rate of change of  $ab$  is the velocity of the point  $a$ , or the velocity  $v$  of the moving wall. Hence, the rate of change of shearing



strain =  $\frac{v}{l}$ . Thus, the *coefficient of viscosity* of a liquid,

$$\eta = \frac{\text{shearing stress}}{\text{rate of change of shearing strain}}$$

$$= \frac{F/A}{v/l}$$

$$\text{or, } F = \eta A \frac{v}{l}$$

Assuming the velocity of the layers of the liquid to vary linearly with distance from the stationary layer, the quantity  $v/l$  represents the *rate of change of velocity with distance* (or the *space rate of change of velocity*) in a direction perpendicular to that of the liquid flow. It is called the *velocity gradient* in that direction. Thus, we have

$$\eta = \frac{F/A}{\theta} = \frac{F/A}{v/l} = \frac{\text{shearing stress}}{\text{velocity gradient}}$$

If the velocity-distribution curve is *not a linear one* as shown in (Fig 12.2), the velocity gradient is given by  $dv/dl$ . We then have

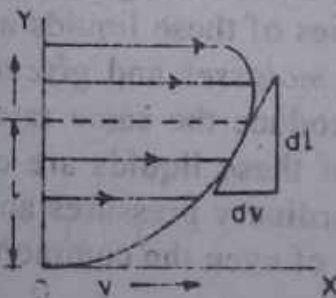


Fig. 12.02

$$\eta = \frac{F/A}{dv/dl}$$

$$\text{or, } F = \eta A \frac{dv}{dl}$$



This proportionality was first shown by Newton and eqn.(12.2) is referred to as *Newton's law of viscous flow of a liquid*. The law is applicable only when the flow is *laminar or streamline*.

Now, if  $A = 1$  and  $dv / dl = 1$  in eqn. (12.2), then  $F = \eta$ .

Thus *the coefficient of viscosity* of a fluid may be defined as the tangential force required per unit area to maintain a unit velocity gradient, *i.e.*, to maintain unit relative velocity between two layers separated by unit distance.

$\eta$  is often referred to as the *absolute viscosity* or *dynamic viscosity* or simply *viscosity* to distinguish it from what is called *kinematic viscosity* which is the *ratio between dynamic viscosity and the density* of the liquid. Kinematic viscosity is usually denoted by the letter  $\nu$ . Thus

$$\begin{aligned} \text{Kinematic viscosity, } \nu &= \frac{\text{dynamic viscosity}}{\text{density}} \\ &= \frac{\eta}{\rho} \end{aligned}$$

The reciprocal of viscosity is called *fluidity*.

Liquids, such as *water and kerosene* which flow readily, have relatively small shearing stress for a given rate of change of shearing strain. Thus the viscosities of these liquids are also relatively small. But for liquids such as *molasses and glycerine*, a greater shearing stress is necessary to produce the same rate of change of shearing strain. The viscosities of these liquids are correspondingly greater. Viscosities of gases at ordinary pressures and temperatures are very much smaller than those of even the common liquids.

Viscosity of all fluids depends markedly on temperature, increasing for gases and decreasing for liquids as the temperature is increased. It has been observed by Thorpe and Rodger that the coefficients of viscosity of a large number of liquids agree very closely with the empirical formula given by  $\eta = c/(1+bt)^n$  where  $\eta$  is the coefficient of viscosity at the temperature  $t$ , and  $c$ ,  $b$  and  $n$  are constants depending on the nature of the liquid. An important consideration for the design of lubricating oils for engines is to minimize this temperature variation of viscosity.

### 12.4 Units and dimensions of viscosity

From eqn. 12.2, the unit of viscosity is

$(\text{force} \times \text{distance}) / (\text{area} \times \text{velocity})$ .

In the M.K.S. system, the unit is

$$1 \text{ N m m}^{-2} (\text{ms}^{-1})^{-1}$$

$$= 1 \text{ N s m}^{-2}$$

$$= 1 \text{ Newton-second per metre}^2$$

$$= 1 \text{ Pascal second (Pa s)}$$

$$= 1 \text{ Poiseuille (Pl)}$$

Thus

$$1 \text{ poise} = 1 \text{ dyne-s cm}^{-2}$$

$$= 10^{-1} \text{ N-s m}^{-2}$$

Sometimes smaller units of viscosities are used. These are centipoises ( $1 \text{ cp} = 10^{-2} \text{ poise}$ ) or micro poises ( $1 \mu \text{ p} = 10^{-6} \text{ poise}$ ).

On substituting the dimension of force in mass, length and time, the dimensions of the coefficient of viscosity are

$$\left( \frac{\text{force} \times \text{distance}}{\text{area} \times \text{velocity}} \right)$$

$$= \frac{\text{MLT}^{-2} \times \text{L}}{\text{L}^2 \times \text{LT}^{-1}}$$

$$= \text{MLT}^{-2} \cdot \text{L} \cdot \text{L}^{-2} \cdot \text{L}^{-1} \cdot \text{T}$$

$$= \text{ML}^{-1} \text{T}^{-1}$$

The C.G.S. unit of kinematic viscosity is  $\text{cm}^2 \text{sec}^{-1}$ . It is called *stokes* after Sir G.G. Stokes, an English scientist

Since

Kinematic viscosity

$$= \frac{\text{dynamic viscosity}}{\text{density}}$$

the dimensions of kinematic viscosity are

$$\left( \frac{\text{dynamic viscosity}}{\text{density}} \right)$$

$$= \frac{ML^{-1}T^{-1}}{ML^{-3}}$$

$$= ML^{-1}T^{-1}M^{-1}L^3$$

$$= L^2T^{-1}$$

### 12.5 Fugitive elasticity

The relation for coefficient of viscosity may be rearranged as

$$\eta = \frac{F/A}{dv/dl} = \frac{\text{tangential stress}}{\text{velocity gradient}}$$

Now the coefficient of rigidity of a solid is given by

$$\eta = \frac{F/A}{\theta} = \frac{F/A}{ab/l} = \frac{F/A}{dy/dx}$$

$$= \frac{\text{tangential stress}}{\text{displacement gradient}}$$

Comparing the above two relations, it can be seen that the viscosity of liquid is strikingly similar to the limiting case of the rigidity of a solid when it breaks down under the shearing stress applied. Maxwell, therefore, surmised very fittingly that a *liquid too possesses some rigidity but that it quickly breaks down under the shearing strain (or shear) produced by the shearing stress to which it is subjected*. However, the shear is formed again only to be broken down again and this process of breaking down and reformation of shear goes on continually so long as the shearing stress continues to be applied. *A liquid thus offers only a momentary or a fleeting resistance to a shearing stress*. The liquid is, therefore, said to possess a *fleeing or fugitive rigidity or elasticity*.

### 12.6 Critical velocity

It has already been pointed out in Art. 12.3 that the flow of a liquid remains steady or orderly only so long as its velocity does not exceed a



certain limiting value for it, called *critical velocity*. Beyond the critical velocity, the liquid flow loses all orderliness and becomes turbulent. *Most of the energy needed to drive the liquid is then dissipated in setting up eddies and whirlpools in it.*

Osborne Reynolds showed experimentally that the critical velocity  $v_c$  for a liquid is given by  $v_c = k\eta/\rho r$  where  $\rho$  and  $\eta$  are the density and coefficient of viscosity of the liquid respectively,  $r$  is the radius of the tube and  $k$  is a number, called *Reynolds number*. The relation has come to be known as *Reynolds's formula*. The value of  $k$  is very high and is, therefore, usually represented on a logarithmic scale.

For narrow circular tubes the motion will be streamline when  $k < 1000$ . The motion becomes turbulent when  $k$  exceeds 2000, the motion will be generally streamline if the entrance point of the liquid remains clear.

It should, however, be emphasized that Reynolds's formula applies only to narrow tubes. For tubes of wide bore, the value of  $v_c$  is very much greater, and may even be thousand times greater than that given by the above relation.

The expression for  $v_c$  may be easily deduced by the method of dimensions. Since  $v_c$  is found to depend on (i)  $\eta$  (ii)  $\rho$  (iii)  $r$ , let us say

$$v_c = k\eta^a \rho^b r^c$$

where  $k$  is a constant and  $a, b, c$  are respectively the dimensions of  $\eta, \rho$  and  $r$ .

Putting the dimensions of the quantities involved in the above relation, we have

$$[LT^{-1}] = [ML^{-1}T^{-1}]^a [ML^{-3}]^b [L]^c$$

$$\text{or, } LT^{-1} = M^{a+b} L^{-a-3b+c} T^{-a}$$

$k$  being a constant has no dimension.

Since the dimensions on the two sides of the equation must be same, we have, from the principle of homogeneity of dimensions,

$$a + b = 0 \quad (i)$$

$$-a - 3b + c = 1 \quad (ii)$$

$$-a = -1 \quad (iii)$$

From (iii) we have  $a = 1$

Adding (i) and (iii) we get  $b = -1$ . Substituting the values of  $a$  and  $b$  in (ii) we have  $c = -1$ .

$$\text{Hence } v_c = \frac{k\eta}{\rho r}$$

Thus the critical velocity of a liquid is

(i) directly proportional to its viscosity ( $v_c \propto \eta$ )

(ii) inversely proportional to its density ( $v_c \propto \frac{1}{\rho}$ )

and (iii) inversely proportional to the radius of the tube through which it flows ( $v_c \propto \frac{1}{r}$ ).

It is, therefore, obvious that *narrow tubes, and liquids of low density and high viscosity promote orderly motion whereas tubes of wide bore and liquids of high density and low viscosity lead to turbulence.*

Again, if we have a perfectly mobile liquid for which  $\eta = 0$ , then, obviously  $v_c = 0$ . Thus even for the smallest velocity and narrowest of tubes, the flow would be turbulent.

*Thus it is the viscosity of a liquid alone which determines if the flow of a liquid will be orderly and thus approximate to that of a perfect liquid.*

### 12.7 Flow of a liquid through a narrow tube: Poiseuille's equation

Poiseuille's equation is an expression for the rate of flow of a liquid through a narrow tube due to pressure difference between the two ends of the tube. While calculating this rate of flow, it will be assumed that

- (i) *the tube is perfectly horizontal*
- (ii) *the liquid flows through the tube in laminar or streamline fashion with the streamlines parallel to the axis of the tube.*

(iii) *There is no radial flow of the liquid, the pressure, therefore, in accordance with Bernoulli's theorem, is constant over any given cross-section of the tube.*

(iv) *The liquid in contact with the walls of the tube is at rest.*

All the above assumptions are found to be quite valid if the tube be narrow and the velocity of the liquid flow really small.

Let a liquid of coefficient of viscosity  $\eta$  be flowing through a narrow horizontal tube of length  $L$  and inner radius  $R$  under a difference of pressure  $P$  between the ends. Let us imagine a cylindrical layer or shell of radius  $r$  and co-axial with the tube (Fig. 12.3). Then *the velocity of flow at all points on this cylindrical shell will be same.*

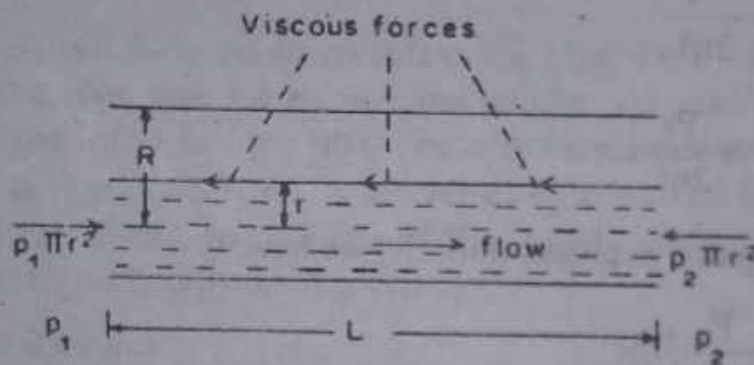


Fig. 12.3

Let it be  $v$ . Since the layer at the centre (axis) of the tube moves with the maximum velocity, decreasing to zero for layers in contact with the walls, it is obvious that the liquid inside the imaginary cylinder is moving faster than that outside it. The *backward tangential force* exerted by the outer slower moving layer on the inner faster moving layer is given by

$$F = \eta A \frac{dv}{dr}$$

where

$\eta$  = coefficient of viscosity of the liquid

$A = 2\pi r.L$  is the lateral surface area of the cylindrical shell of radius  $r$  and length  $L$  and  $\frac{dv}{dr}$  = velocity gradient at the layer concerned.



Let  $P = (p_1 - p_2)$  be the difference in pressure at the two ends of the tube. Then the net forward force on the cylindrical shell in the direction of flow is clearly  $p \times \pi r^2$ . This is the force which tends to accelerate the motion of the liquid.

Therefore, if the motion of the liquid is steady, we have

$$-\eta \cdot 2\pi rL \cdot \frac{dv}{dr} = +P \cdot \pi r^2 \quad (i)$$

The minus sign of the dragging force indicates that it acts in a direction opposite to that of the driving force.

Rearranging eqn. (i), we have

$$\frac{dv}{dr} = -\frac{Pr}{2\eta L}$$

$$\text{or, } dv = -\frac{Pr}{2\eta L} \cdot dr$$

Integrating this expression for  $dv$ , we have

$$v = -\frac{P}{2\eta L} \int r dr$$

$$\text{or, } v = -\frac{P}{2\eta L} \cdot \frac{r^2}{2} + C_1$$

where  $C_1$  is a constant of integration.

Now  $v = 0$  at  $r = R$

$$\therefore 0 = -\frac{PR^2}{4\eta L} + C_1$$

$$\text{or, } C_1 = +\frac{PR^2}{4\eta L}$$

$$\therefore v = -\frac{Pr^2}{4\eta L} + \frac{PR^2}{4\eta L}$$

$$= \frac{P}{4\eta L}(R^2 - r^2) \quad (12.3)$$

Eqn. (12.3), therefore, is the expression for the velocity of flow of the liquid at a distance  $r$  from the axis of the tube. A mere glance at the expression shows that the velocity decreases from a maximum value  $\frac{PR^2}{4\eta L}$  at the centre (axis) ( $r = 0$ ) to zero at the walls ( $r = R$ ). Thus the

*maximum velocity* is proportional to the square of the tube radius and is also proportional to the pressure change per unit length  $P/L$ , called the *pressure gradient*. The *profile* or the *velocity distribution curve* as given by this equation is shown in Fig. (12.1b). As can be seen the curve is *parabolic*.

Eqn. (12.3) can be used to calculate the *total* flow of the liquid through the tube. For this, let us imagine another *co-axial* cylindrical shell of the liquid, of radius  $(r + dr)$ . The *cross-sectional* area between the two shells is  $2\pi r.dr$ . Since  $v$  is the velocity of flow of the liquid in between these two shells, the volume of the liquid flowing per second through this cross-sectional area is given by

$$dV = 2\pi r.dr.v$$

Imagining the entire tube to be made up of such concentric cylindrical shells, the volume  $V$  of the liquid flowing per second through the tube is obtained by integrating the expression for  $dV$  between the limits  $r = 0$  and  $r = R$

$$\begin{aligned} \therefore V &= \int_0^R dV = \int_0^R 2\pi r dr.v \\ &= \int_0^R 2\pi r \cdot \frac{P}{4\eta L}(R^2 - r^2) dr \\ &= \frac{\pi P}{2\eta L} \int_0^R (rR^2 - r^3) dr \\ &= \frac{\pi P}{2\eta L} \left[ \frac{R^2 r^2}{2} - \frac{r^4}{4} \right]_0^R \\ &= \frac{\pi P}{2\eta L} \left[ \frac{R^4}{2} - \frac{R^4}{4} \right] \end{aligned}$$

$$= \frac{\pi P R^4}{8 \eta L} \quad (12.4)$$

Eqn. (12.4) gives an expression for volume  $V$  of the liquid that flows out of the tube per second in terms of the coefficient of viscosity  $\eta$  of the tube, the radius of the tube, the length of the tube and the pressure difference between the two ends of the tube.

Rearranging eqn. (12.4). We have

$$\eta = \frac{\pi P R^4}{8 V L} \quad (12.5)$$

Eqn. (12.4) or (12.5) is referred to as *Poiseuille's equation* and can be used to determine the value of  $\eta$ .  $P$ ,  $R$ ,  $V$  and  $L$  can be experimentally determined.

The assumptions made in the derivation of Poiseuille's equation are more or less valid for tubes of narrow bore (small radius) and for flow of liquids at low velocities. For tubes of narrow bore, the critical velocity is rather large ( $v_c \propto \frac{1}{r}$ ) and the velocity of the liquid flowing through such a tube is usually well below its critical velocity. The rate of flow of the liquid then, depends mainly on viscosity ( $\eta$ ), quite in accordance with Poiseuille's formula.

### Corrections

*Poiseuille's equation, however, needs two important corrections.*

(i) In deriving the equation it was assumed that the entire pressure difference  $P$  was utilized in overcoming the viscous force. However, a part of the thrust due to the pressure difference  $P$  between the two ends of the tube is used up to impart some kinetic energy to the liquid. This means that the whole of the pressure  $P$  is not available for overcoming the *viscous resistance* of the liquid. If the portion of the pressure used for imparting kinetic energy be  $P'$ , then the effective pressure available for overcoming the viscous resistance is  $P_1 = P - P'$ . This should be substituted for  $P$  in the relation for  $\eta$ .



Now the kinetic energy imparted to the liquid per second

$$E = \int_0^R \frac{1}{2} mv^2$$

mass = volume  $\times$  density

$$= (2\pi r \cdot dr \cdot v) \times \rho$$

where  $\rho$  is the density of the liquid.

Therefore,

$$E = \int_0^R \frac{1}{2} [(2\pi r dr v) \rho] v^2$$

$$= \pi \rho \int_0^R v^3 r dr$$

$$\text{but } v = \frac{P(R^2 - r^2)}{4\eta L}$$

$$\therefore E = \pi \rho \int_0^R \left( \frac{P}{4\eta L} \right)^3 (R^2 - r^2)^3 r dr$$

$$= \pi \rho \left( \frac{P}{4\eta L} \right)^3 \int_0^R [R^6 r - 3R^4 r^3 + 3R^2 r^5 - r^7] dr$$

$$= \pi \rho \left( \frac{P}{4\eta L} \right)^3 \left( \frac{R^8}{8} \right)$$

$$= \left( \frac{\pi P R^4}{8\eta L} \right)^3 \cdot \frac{\rho}{\pi^2 R^4}$$

$$= \frac{V^3 \rho}{\pi^2 R^4}$$

Now if  $P'$  is the part of the pressure difference responsible for imparting kinetic energy to a volume  $V$  flowing out of the tube per second, then the pressure energy for this is  $P'V$ .

Hence

$$P'V = E$$

$$P' = \frac{E}{V} = \frac{V^2 \rho}{\pi^2 R^4}$$

$$\begin{aligned} & [\text{pressure} \times \text{volume}] \\ & = (\text{Force} / \text{Area}) \times \text{Volume} \end{aligned}$$

$$= \left( \frac{M \cdot L \cdot T^{-2}}{L^2} \right) \times L^3$$

$$= M \cdot \frac{L^2}{T^2} = M \cdot \left( \frac{L}{T} \right)^2$$

$$= \text{mass} \times (\text{velocity})^2]$$

Then

$$P_1 = P - \left[ \frac{V^2 \rho}{\pi^2 R^4} \right]$$

If  $V$  is negligibly small,  $P_1 = P$

(ii) The liquid-flow all along the tube is not uniform or streamline as assumed.

Upon entering the flow-tube the liquid is clearly accelerated and does not become uniform or streamline until after covering a good length of the tube. This error can, however, be easily corrected by considering the effective length of the flow-tube to be  $(L + 1.64R)$  instead of  $L$ . This correction is thus quite independent of the length of the flow-tube and depends only on its radius.

The corrected relation for  $\eta$ , therefore, becomes

$$\begin{aligned} \eta &= \frac{\pi P R^4}{8V(L + 1.64R)} - \left( \frac{V^2 \rho}{\pi^2 R^4} \right) \frac{\pi R^4}{8V(L + 1.64R)} \\ &= \frac{\pi P R^4}{8V(L + 1.64R)} - \frac{V \cdot \rho}{8\pi(L + 1.64R)} \quad (12.6) \end{aligned}$$

### 12.8 Determination of coefficient of viscosity of a liquid by the capillary flow method

The experimental arrangement consists of a constant level water tank N with a rubber cork R fitted at the bottom (Fig. 12.4). There are two holes in the rubber cork. Through one hole the outlet pipe O passes. This pipe extends about  $\frac{2}{3}$ rd up the tank and serves as the over-flow pipe. A bent tube passes through the other hole in the cork. The bent tube projects slightly above the cork R. A capillary tube C about 40 to 50 cm long, is attached to the tube with a rubber tube K and is supported in its position by a clamp (not shown). The supply of water through the inlet pipe I is slightly in excess of that flowing out through the capillary tube C and the excess

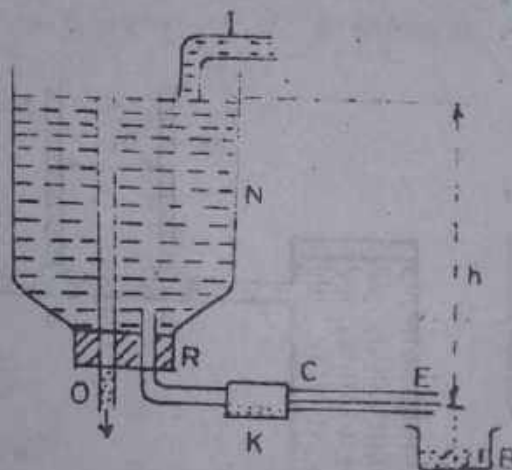


Fig. 12.4

liquid flows out through the tube O, thus keeping the liquid level constant at the upper end of it. The height  $h$  of the free surface of the liquid in the tank above the end E of the capillary tube can be measured with a meter scale or cathetometer. The difference of pressure  $P$  is then given by  $P = h\rho g$  where  $\rho$  is the density of the liquid and  $g$  the acceleration due to gravity. The liquid flowing out through the capillary tube in a slow trickle is collected in a weighed beaker B during a known interval of time. The volume of this liquid is determined by dividing the mass of this liquid by its density at the temperature of the experiment. From the time of collection, the



volume  $V$  flowing out per second is determined. Substituting the value of  $V$ , so obtained, in eqn (12.5), the coefficient of viscosity ( $\eta$ ) of the liquid can be easily calculated from the measured values of  $R$  and  $L$ .

A better experimental arrangement is shown in Fig. 12.5. The flow tube  $T$  is fairly long and of uniform circular cross-section. The pressure difference for a length  $AB$  of the tube is given by the manometer formed by the narrow vertical tubes  $M_1$  and  $M_2$  admitted into two fine water-tight holes at  $A$  and  $B$ .  $A$  and  $B$  lie at a distance of at least 10 cms from the two end of the flow tube respectively so that the velocity of the out flowing liquid becomes uniform near about them. This very much minimizes the two sources of error mentioned in Art. 12.7- the second one almost completely.

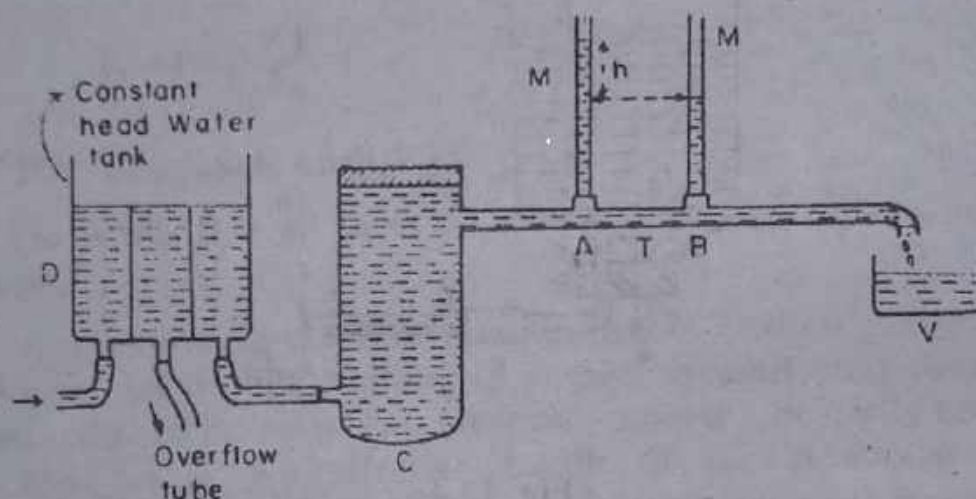


Fig. 12.5

Thus with a slow rate of flow of the liquid and a fairly small size of the holes at  $A$  and  $B$ , no further corrections are necessary.

**Example 12.1.** A flat plate of area 20 sq. cm is placed on a horizontal surface coated with a layer of glycerine 1 mm thick. What force must be applied to the plate to keep it moving with a speed of 1 cm/sec over the horizontal surface? Coefficient of viscosity of glycerine =  $20 \text{ gm cm}^{-1} \text{ sec}^{-1}$ .



$$P = h\rho g = 20 \times 1 \times 980$$

$$r = \frac{0.1}{2} \text{ cm} = 0.05 \text{ cm}$$

$$\eta = 0.0089 \text{ CGS unit}$$

$$l = 40 \text{ cm}$$

$$\therefore V = \frac{3.14 \times 20 \times 1 \times 980 \times (0.05)^4}{8 \times 0.0089 \times 40}$$

$$= 0.1353 \text{ c.c./sec}$$

$$\therefore \text{Volume of water flowing out in 10 minutes}$$

$$= 0.1353 \times 60 \times 10 \text{ c.c.}$$

$$= 81.18 \text{ c.c.}$$

Hence, mass of water flowing out in 10 minutes

$$V \times \rho = 81.18 \times 1 = 81.18 \text{ gm}$$

*Example 12.4 In an experiment with Poiseuille's apparatus the following figures were obtained:*

*Volume of water issuing per minute = 7.08 c.c.*

*Head of water = 34.1 cm.*

*Length of tube = 56.45 cm.*

*Radius of the tube = 0.0514 cm*

*Find the Coefficient of viscosity.*

**Soln.**

rate of flow of water,

$$V = \frac{\pi P r^4}{8 \eta l}$$

$$\text{or, } \eta = \frac{\pi P r^4}{8 V l}$$

$$P = h\rho g = 34.1 \times 1 \times 980$$



$$r = 0.0514 \text{ cm}$$

$$V = \frac{7.08}{60} \text{ c.c.}$$

$$l = 56.45 \text{ cm}$$

$$\eta = ?$$

$$\therefore \eta = \frac{3.14 \times 34.1 \times 1 \times 980 \times (0.0514)^4 \times 60}{8 \times 7.08 \times 56.45}$$

$$= 0.013745 \text{ dyne cm}^{-2} \text{ sec (or poise).}$$

**Example 12.5** A capillary tube  $10^{-3} \text{ m}$  in diameter and  $0.2 \text{ m}$  in length is fitted horizontally to a vessel kept full of alcohol of density  $0.8 \times 10^3 \text{ kg/m}^3$ . The depth of the centre of the capillary tube below the surface of alcohol is  $0.3 \text{ m}$ .

Calculate the volume of alcohol that flows in 5 minutes. Viscosity of alcohol is  $0.0012 \text{ N-s/m}^2$ .

**Soln.**

$$\text{Rate of flow} = \frac{\pi P r^4}{8 \eta l}$$

$$P = h \rho g = 0.3 \times 0.8 \times 10^3 \times 9.8$$

$$r = 5 \times 10^{-4} \text{ m}$$

$$l = 0.2 \text{ m}$$

$$\eta = 0.0012 \text{ N-s/m}^2$$

$\therefore$  rate of flow

$$= \frac{3.14 \times 0.3 \times 0.8 \times 10^3 \times 9.8 \times (5 \times 10^{-4})^4}{8 \times 0.0012 \times 0.2}$$

$$= 2.404 \times 10^{-7} \text{ m}^3 / \text{sec}$$

Hence total volume flowing out in 5 minutes

$$\begin{aligned}
 &= V \times t \\
 &= 2.404 \times 10^{-7} \times 5 \times 60 \\
 &= 7.212 \times 10^{-5} \text{ m}^3.
 \end{aligned}$$

**Example 12.6** Water is conveyed through a horizontal tube 0.08 m in diameter and 4km in length at the rate of 20 liters per second. Assuming only viscous resistance, calculate the pressure difference required to maintain the flow. Viscosity of water =  $10^{-3}$  N-s/m<sup>2</sup>.

**Soln.**

According to Poiseuille's formula

$$V = \frac{\pi P r^4}{8 \eta l}$$

$$\text{or, } P = \frac{8 V \eta l}{\pi r^4}$$

$$V = 20 \text{ litres/sec}$$

$$= 20 \times 10^{-3} \text{ m}^3/\text{sec}$$

$$\eta = 10^{-3} \text{ N-s/m}^2$$

$$l = 4 \text{ km} = 4 \times 10^3 \text{ m}$$

$$r = 0.04 \text{ m}$$

$$\therefore P = \frac{8 \times 20 \times 10^{-3} \times 10^{-3} \times 4 \times 10^3}{\pi \times (0.04)^4}$$

$$= 7.96 \times 10^4 \text{ N/m}^2$$

**Example 12.7.** A wide vertical tube has a short capillary tube fixed horizontally at its lower end. When it is filled with glycerine it takes 45 seconds for the level to fall between two fixed marks. When the tube is filled with oil it takes 67.5 seconds for the level to fall between the same two fixed marks. Calculate the coefficient of viscosity of oil. Given, density of glycerine =  $1.2 \times 10^3 \text{ kg/m}^3$ , density of oil =  $0.8 \times 10^3 \text{ kg/m}^3$ . Viscosity of glycerine =  $0.85 \text{ N-s/m}^2$ .

**Soln.**

From Poiseuille's eqn. we have

$$V_1 = \frac{\pi P_1 r^4}{8\eta_1 l} \text{ and } V_2 = \frac{\pi P_2 r^4}{8\eta l}$$

Since the total volume of fluid flowing out between the two fixed marks is the same in both cases.

$$V_1 t_1 = V_2 t_2$$

$$\text{or, } \frac{\pi P_1 r^4}{8\eta_1 l} \times t_1 = \frac{\pi P_2 r^4}{8\eta l} \times t_2$$

$$\text{or, } \frac{\eta_2}{\eta_1} = \left(\frac{P_1}{P_2}\right) \times \left(\frac{t_2}{t_1}\right)$$

$$= \left(\frac{h\rho_2 g}{h\rho_1 g}\right) \times \left(\frac{t_2}{t_1}\right) = \left(\frac{\rho_2}{\rho_1}\right) \times \left(\frac{t_2}{t_1}\right)$$

$$\text{or, } \eta_2 = \eta_1 \times \left(\frac{\rho_2}{\rho_1}\right) \times \left(\frac{t_2}{t_1}\right)$$

$$\eta_1 = 0.8 \times \text{N} - \text{s/m}^2$$

$$\rho_1 = 1.2 \times 10^3 \text{ kg/m}^3$$

$$\rho_2 = 0.8 \times 10^3 \text{ kg/m}^3$$

$$t_1 = 45 \text{ sec. } t_2 = 67.5 \text{ sec.}$$

$$\begin{aligned} \therefore \eta_2 &= 0.85 \times \frac{0.8 \times 10^3}{1.2 \times 10^3} \times \frac{67.8}{45} \\ &= 0.85 \text{ N} - \text{s/m}^2. \end{aligned}$$

**Example 12.8** A capillary tube 1mm in diameter and 30 cms in length is attached horizontally to a vessel kept full of a liquid of density 1.5 gms per c.c. The height of the liquid surface above the centre of the capillary tube is 25 cms. If the viscosity of the liquid is 0.008 C. G. S unit, find the rate of flow of the liquid.



**Soln.**

$$\text{Rate of flow} = \frac{\pi P r^4}{8\eta l}$$

$$\text{Here, } P = h\rho g = 25 \times 1.5 \times 980$$

$$h = 0.008$$

$$r = 0.5 \text{ mm} = 0.05 \text{ cm}$$

$$l = 30 \text{ cm}$$

$\therefore$  rate of flow

$$= \frac{3.14 \times 25 \times 1.5 \times 980 \times (0.05)^4}{8 \times 0.008 \times 30}$$

$$= 0.39 \text{ c.c. /sec}$$

**Example 12.9** Write down Poiseuille's formula for the rate of flow of a liquid through a capillary tube. From this show that if two capillaries of radii  $a_1$  and  $a_2$ , having lengths  $l_1$  and  $l_2$  respectively, are set in series, the rate of flow  $V$  is given by

$$V = \frac{\pi P}{8\eta} \left( \frac{l_1}{a_1^4} + \frac{l_2}{a_2^4} \right)^{-1}$$

where  $P$  is the pressure across the arrangement and  $\eta$ , the coefficient of viscosity of the liquid.

**Soln.**

Let  $P_1$  be the pressure across the first capillary and  $P_2$  that across the second capillary. So that  $P = P_1 + P_2$ .

$$\text{or, } P_2 = P - P_1$$

In accordance with the equation of continuity, it is obvious that the rate of flow through either capillary will be the same, say  $V$ . Therefore, according to Poiseuille's equation,

$$V = \frac{\pi P_1 a_1^4}{8\eta l_1} = \frac{\pi P_2 a_2^4}{8\eta l_2} = \frac{\pi (P - P_1) a_2^4}{8\eta l_2}$$

$$\text{or, } \frac{P_1 a_1^4}{l_1} = \frac{P a_2^4}{l_2} - \frac{P_1 a_2^4}{l_2}$$

$$\text{or, } P_1 \left( \frac{a_1^4}{l_1} + \frac{a_2^4}{l_2} \right) = \frac{P a_2^4}{l_2}$$

$$\therefore P_1 = \frac{P a_2^4 / l_2}{a_1^4 / l_1 + a_2^4 / l_2} = \frac{P}{\left( \frac{a_1^4}{l_1} \times \frac{l_2}{a_2^4} + 1 \right)}$$

$$= \frac{P}{\frac{a_1^4}{l_1} \left( \frac{l_2}{a_2^4} + \frac{l_1}{a_1^4} \right)} = \frac{P l_1}{a_1^4 \left( \frac{l_2}{a_2^4} + \frac{l_1}{a_1^4} \right)}$$

$$= \frac{P l_1}{a_1^4} \left( \frac{l_2}{a_2^4} + \frac{l_1}{a_1^4} \right)^{-1}$$

Substituting this value of  $P_1$  in the expression

$$V = \frac{\pi P a_1^4}{8 \eta l_1}, \text{ the rate of flow of the liquid}$$

$$V = \frac{\pi P}{8 \eta} \left( \frac{l_1}{a_1^4} + \frac{l_2}{a_2^4} \right)^{-1}.$$

**Example 12.10** Three capillaries of lengths  $8L$ ,  $0.2L$  and  $2L$  with their radii  $r$ ,  $0.2r$  and  $0.5r$  respectively, are connected in series. If the total pressure across the system is  $P$ , deduce the pressure across the shortest capillary.

**Soln.**

According to equation of continuity, the rate of flow of liquid across each capillary is the same. So, if  $P_1$ ,  $P_2$  and  $P_3$  be the pressures across the three capillaries respectively, then according to Poiseuille's equation we have

$$V = \frac{\pi P_1 r^4}{8\eta(8L)} = \frac{\pi P_2 (0.2r)^4}{8\eta(0.2L)} = \frac{\pi P_3 (0.5r)^4}{8\eta(2L)}$$

$$\text{or, } \frac{P_1}{64} = \frac{P_2}{1000} = \frac{P_3}{256}$$

which gives

$$P_1 = \frac{P_2}{1000} \times 64 \quad \text{and} \quad P_3 = \frac{P_2}{1000} \times 256$$

$$= \frac{8}{125} P_2$$

$$= \frac{32}{125} P_2$$

$$\text{Now } P = P_1 + P_2 + P_3$$

$$= \frac{8}{125} P_2 + P_2 + \frac{32}{125} P_2$$

$$= \frac{165}{125} P_2$$

Therefore, the pressure across the shortest capillary, i.e.,

$$P_2 = \frac{125}{165} P$$

$$= 0.7575 P.$$

**Example 12.11** Calculate the mass of water flowing in 10 seconds through a horizontal capillary tube of circular cross-section of radius  $10^{-3}$  m. The tube is fitted at the bottom of a constant level tank at a depth of 1m. Length of the tube is 0.3142m. coefficient of viscosity of water =  $10^{-3}$  N-s/m<sup>2</sup>, density of water =  $10^3$  kg/m<sup>3</sup>.

**Soln.**

$$\text{Rate of flow, } V = \frac{\pi P r^4}{8\eta L} \text{ m}^3$$

$$P = h\rho g = 1 \times 10^3 \times 9.8$$



$$r = 10^{-3} \text{ m.}$$

$$l = 0.3142 \text{ m.}$$

$$\begin{aligned} \therefore V &= \frac{3.142 \times 10^3 \times 9.8 \times (10^{-3})^4}{8 \times 10^{-3} \times 0.3142} \\ &= 1.225 \times 10^{-5} \text{ m}^3 \end{aligned}$$

So volume of water flowing out in 10 seconds

$$= 1.225 \times 10^{-5} \times 10 = 1.225 \times 10^{-4} \text{ m}^3.$$

Hence mass of water flowing out in 10 seconds

$$= \text{volume} \times \text{density}$$

$$= 1.225 \times 10^{-4} \times 10^3$$

$$= 0.1225 \text{ kg.}$$

**Example 12.12** A vessel of cross-section 20sq. cm has at the bottom a horizontal capillary tube of length 10 cm and internal radius 0.5 mm. It is initially filled with water to a height of 20 cm above the capillary tube. Find the time taken by the vessel to empty one half of its contents, given that the viscosity of water is 0.01 CGS units.

**Soln.**

Let  $h$  be the height of the water column in the vessel at any given instant and  $dh$  the fall in its height in a small interval of time  $dt$ . Then, if  $A$  be the area of cross-section of the vessel, we have

Rate of flow of water through the capillary tube,

$$\text{i.e., } V = -A \frac{dh}{dt}$$

minus sign indicates that as  $t$  increases,  $h$  decreases. But according to Poiseuille's eqn., the rate of flow of water through the capillary tube,

$$V = \frac{\pi P r^4}{8 \eta l}$$

$$\text{where } P = h \rho g = h g \quad (\rho = 1 \text{ for water})$$

We, therefore, have

$$-A \frac{dh}{dt} = \frac{\pi h g r^4}{8\eta l}$$

$$\text{or, } dt = - \frac{8\eta l/A}{\pi g r^4} \cdot \frac{dh}{h}$$

Integrating, we get

$$\int_0^t dt = \int_{h_1}^{h_2} - \frac{8\eta l/A}{\pi g r^4} \cdot \frac{dh}{h}$$

$$\text{whence } t = \frac{8\eta l A}{\pi g r^4} \log_e \frac{h_1}{h_2}$$

Here  $A = 20 \text{ sq.cm}$

$$r = 0.5 \text{ mm} = 5 \times 10^{-2} \text{ cm}$$

$$l = 10 \text{ cm}$$

$$h = 0.01 \text{ C.G.S. unit}$$

$$g = 980 \text{ cm/sec}^2$$

$$\frac{h_1}{h_2} = 2 \quad (\because h_2 = \frac{h_1}{2})$$

$$\therefore t = \frac{8 \times 0.01 \times 1 \times 20}{3.142 \times 980 \times (5 \times 10^{-2})^2} \log_e 2$$

$$= \frac{16}{0.019245} \times 0.693$$

$$\approx 576 \text{ secs}$$

$$\approx 9 \text{ min } 36 \text{ secs.}$$

**Example 12.13.** Castor oil is forced through a pipe of circular cross-section by a pump that maintains a gauge pressure of 950 Pa. The pipe has a diameter of 2.6 cm and a length of 65 cm. The castor oil emerging from the free end of the pipe at atmospheric pressure is

collected. An amount of 1.23 Kg is collected in 90s. If the density of castor oil at room temperature is  $0.96 \times 10^3 \text{ Kg/m}^3$ , what is the coefficient of viscosity of castor oil at this temperature?

**Soln.**

From eqn. (12.5), we have

$$\eta = \frac{\pi P R^4}{8 V L}$$

Here

$P$  = pressure difference between two ends of the tube = 950 Pa

$V$  = volume of castor oil flowing out per second

=  $\frac{\text{mass flowing out per second}}{\text{density}}$

$$= \frac{m}{\rho} = \frac{(1.23/90)}{0.96 \times 10^3} = \frac{0.0137}{0.96 \times 10^3}$$

$$R = \frac{2.6}{2} \text{ cm} = 0.013 \text{ m}$$

$$L = 65 \text{ cm} = 0.65 \text{ m}$$

$$\begin{aligned} \therefore \eta &= \frac{(3.14) (0.96 \times 10^3 \text{ Kg/m}^3) (0.013 \text{ m}) (950 \text{ Pa})}{(8) (0.0137 \text{ Kg/sec}) (0.65 \text{ m})} \\ &= 1.15 \text{ N.s/m}^2 = 1.15 \text{ Pa.s} = 1.15 \text{ Pl.} \end{aligned}$$

## 12.9 Motion in a viscous medium : Stokes's law

When a small body, say a steel ball, is left gently on the surface of a long vertical column of a viscous liquid, the body moves vertically downward. As the body falls, it is accelerated initially due to gravity. However, its motion is opposed by the viscous drag of the liquid and the upward force of buoyancy of the liquid until the resultant force acting on the body is zero. Then the body is in equilibrium and moves with a constant velocity, called the *terminal velocity* ( $v_t$ ).



Stokes showed that the opposing force,  $F$ , acting on the body due to viscosity of the liquid is given by

$$F = 6\pi v_l r \eta$$

where  $r$  is the radius of the body and  $\eta$  the coefficient of viscosity of the medium.

The above relation is known as *Stokes' law* and was obtained on the assumption that

(i) *the body is perfectly rigid and smooth.*

(ii) *while moving through the viscous liquid the body does not set up any waves and eddies i.e., the body moves through the liquid with a very small velocity not exceeding a certain critical velocity  $v$  ( $= 0.6\eta/\sigma r$ ) where  $r$  is the radius of the body and  $\sigma$  the density of the liquid medium.*

(iii) *the medium is infinite in depth and extent and homogeneous in so far as the body is concerned. (homogeneous means that the intramolecular space in the medium is very much smaller than the diameter of the body)*

and (iv) *there is no slip between the body and the medium.*

### **Deduction of Stoke's law by method of dimensions**

Stokes' law may also be easily deduced by the method of dimensions as discussed below:

Let us suppose that for a slow-moving spherical body the viscous drag  $F$  varies in certain manner with

(i) its terminal velocity ( $v_l$ ); (ii) radius  $r$  of the moving body and (iii) coefficient of viscosity  $\eta$  of the medium through which it is moving. Then we may write

$$\text{or, } F = k r^a v_l^b \eta^c \quad (12.8)$$

where  $k$  is a constant and  $a$ ,  $b$  and  $c$  are the dimensional coefficients of  $r$ ,  $v_l$ , and  $\eta$  respectively.

Putting the dimensions of the physical quantities involved in the two sides of the above equation, we have

$$[MLT^{-2}] = [L^a][LT^{-1}]^b[ML^{-1}T^{-1}]^c$$

$$= M^c L^{a+b-c} T^{-b-c}$$

(k being a number, has no dimension)

whence  $c = 1$ ,  $a + b - c = 1$  and  $-b - c = -2$

Solving the above equations we get

$$a = 1, b = 1 \text{ and } c = 1,$$

$\therefore$  the relation (12.8) becomes

$$F = k v_t r \eta$$

The value of k was shown by Stokes' to be equal to  $6\pi$ .

Hence,

$$F = 6\pi v_t r \eta \quad (12.9)$$

### Expression for terminal velocity

If the density of the material of the spherical body is  $\rho$ , then

its weight = mass  $\times$  g

$$= \text{volume} \times \rho \times g$$

$$= \frac{4}{3} \pi r^3 \cdot \rho \cdot g.$$

The force due to buoyancy of the liquid or the upthrust on the body is the weight of the liquid displaced by the body. If  $\sigma$  be the

density of the liquid, then this upthrust on the body =  $\frac{4}{3} \pi r^3 \sigma \cdot g$ .

Then the sum of the forces acting on the body-considering upward forces to be positive, is

$$6\pi\eta v_t r + \frac{4}{3} \pi r^3 \sigma g - \frac{4}{3} \pi r^3 \rho g = 0$$

(viscous drag)      (buoyancy)      (weight of the body)

$$\text{or, } 6\pi\eta v_t r = \frac{4}{3} \pi r^3 \rho g - \frac{4}{3} \pi r^3 \sigma g$$

$$= \frac{4}{3} \pi r^3 (\rho - \sigma)g$$

$$\text{or, } v_t = \frac{\frac{4}{3} \pi r^3 (\rho - \sigma)g}{6\pi\eta r}$$

$$= \frac{2}{9} \frac{r^2 g (\rho - \sigma)}{\eta} \quad (12.10)$$

$$= \frac{2}{9} \frac{r^2 g \rho (1 - \frac{\sigma}{\rho})}{\eta} \quad (12.11)$$

Thus the terminal velocity of a small body falling through a viscous liquid is

- (i) *directly proportional to the square of its radius ( $r^2$ )*
- (ii) *directly proportional to the difference in the densities of the body and the medium ( $\rho - \sigma$ )*
- and (iii) *inversely proportional to the coefficient of viscosity ( $\eta$ ) of the medium.*

Eqn. (12.10) can be rearranged as

$$\eta = \frac{2}{9} \frac{r^2 g (\rho - \sigma)}{v_t} \quad (12.12)$$

$$= \frac{2}{9} \frac{r^2 g \rho (1 - \frac{\sigma}{\rho})}{v_t} \quad (12.13)$$

Thus by measuring the terminal velocity of a small sphere of known radius and density through a viscous fluid of known density, the coefficient of viscosity of the fluid can be determined. Alternately, by measuring the terminal velocity of a sphere of known density while it is moving through a viscous fluid whose density and coefficient of viscosity are known, the radius of the sphere can be determined.



### Corrections

Ladenburg pointed out that Stokes' derivation of the formula for  $\eta$  is valid only if the liquid is *unbound* i.e., *infinite both in extent and depth*. Since in the laboratory measurements of  $\eta$ , the liquid is contained in a jar, it is bound in extent as well as depth by the wall and bottom of the vessel, no matter how wide or tall the vessel is. This calls for corrections to the so-called *wall effect* and *end effect*.

(i) *wall-effect* : If  $v_t$  be the observed terminal velocity of the sphere of radius  $r$  in a liquid contained in a jar of radius  $R$ , then its value when *corrected for the wall effect* is

$$v_{\infty} = v_t \left( 1 + 2.4 \frac{r}{R} \right),$$

$v_{\infty}$  being the terminal velocity in the liquid of infinite extent.

(ii) *end-effect* : If  $\mu$  be the depth of the liquid column in the jar or vessel, and  $v_t$ , the observed terminal velocity of the sphere in it, then its value when *corrected for the end effect* is

$$v_{\infty} = v_t \left( 1 + 3.3 \frac{r}{\mu} \right),$$

$v_{\infty}$  being the terminal velocity in a vessel of infinite depth.

With both these corrections made, the expression for the coefficient of viscosity becomes

$$\eta = \frac{2}{9} \frac{(\rho - \sigma) r^2 g}{v_t \left( 1 + 2.4 \frac{r}{R} \right) \left( 1 + 3.3 \frac{r}{\mu} \right)} \quad (12.14)$$

### 12.10 Stokes' method for the determination of coefficient of viscosity of a liquid – the falling sphere viscometer.

A tiny sphere, say a small steel ball bearing, is dropped centrally through a narrow glass tube so that it falls centrally into the liquid, contained in a tall and wide jar (Fig. 12.6). After the sphere has attained terminal velocity, the time taken by it to cover the distance between the two gummed papers is noted carefully. The first and the second gummed papers should be about 10 – 12 cm

below the top surface and above the bottom surface of the liquid respectively. In view of its small size the sphere is assumed to have attained the terminal velocity before crossing the first gummed paper. If  $t$  is the time taken by the sphere to cover the distance  $d$  between the gummed papers, then  $v_t = d/t$ . Substituting this value for  $v_t$  in eqn. (12.12), we have

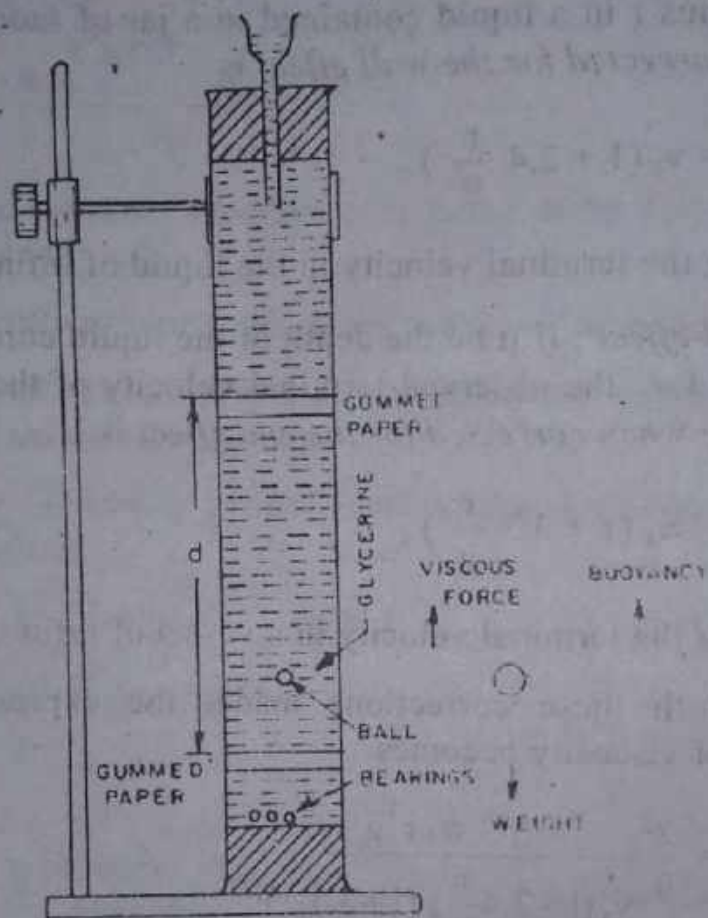


Fig. 12.6

$$\eta = \frac{2}{9} \frac{(\rho - \sigma) r^2 g t}{d}$$

where  $r$  is the radius of the sphere determined accurately with the help of a micrometer screw gauge.

### 12.11 Rotation viscometer

A rotation viscometer consists of two co-axial cylinders A and B. The cylinder A of radius  $a$ , is suspended co-axially inside the cylinder B with the help of a long and thin suspension wire attached to a torsion head H. The cylinder B, of radius  $b$ , is clamped on to a table which can be rotated by means of a small electric motor. Let the space between the two cylinders be filled up to a height, say  $l$ , with the liquid whose coefficient of viscosity is to be determined (Fig. 12.7).

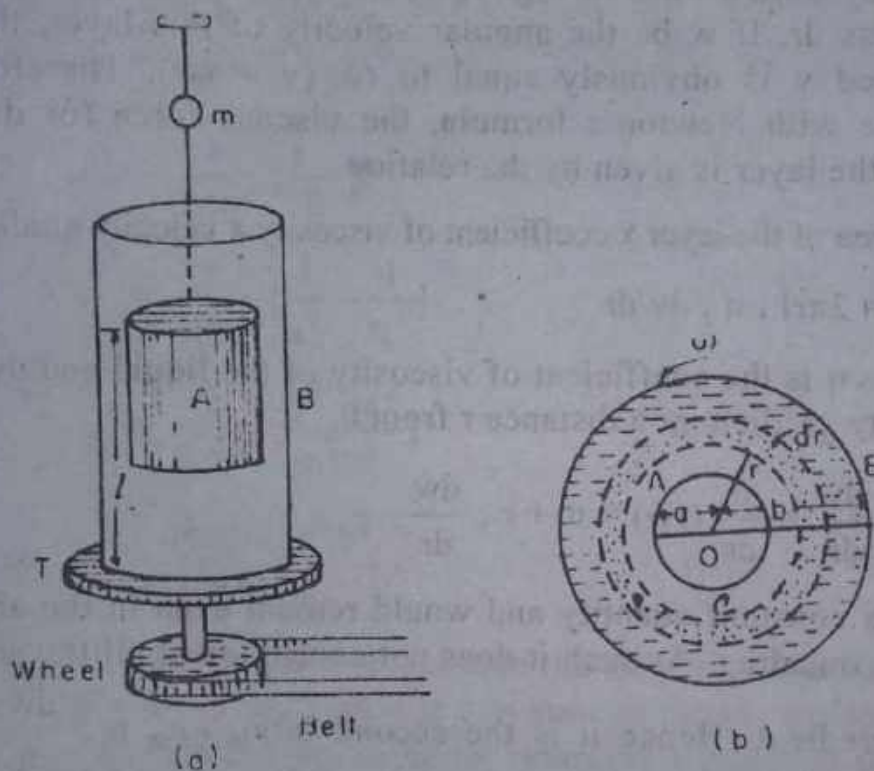


Fig. 12.7

When the outer cylinder B is rotated with a suitable angular velocity  $\omega_0$  (so as not to cause turbulence in the liquid), the layer of the liquid in contact with the inner surface of B will have the same angular velocity  $\omega_0$  as that of the cylinder. The angular velocity of the layers decreases radically inwards due to viscous drag. Due to rotation of B, the liquid in contact with A rotates the cylinder A through an angle. Let this angle of rotation be  $\theta$  when the cylinder A



finally attains equilibrium position. Then the angular velocity of the layers in contact with the outer surface A is zero. There is thus brought about a relative motion between the different layers of the liquid. And the rotation of B communicates a torque to the stationary cylinder A, the magnitude of the torque depending upon the coefficient of viscosity of the liquid.

Let the two cylinders be represented in Fig. 12.7(b) by the full circles A and B of radii  $a$  and  $b$  respectively, their common axis passing through O being perpendicular to the plane of the paper.

Let us consider a co-axial cylindrical layer C of the liquid, indicated by dotted lines in Fig. 12.7(b) at a distance  $r$  from O, and of thickness  $dr$ . If  $w$  be the angular velocity of this layer, then its linear speed  $v$  is obviously equal to  $rw$  ( $v = rw$ ). Therefore, in accordance with Newton's formula, the viscous force (or drag)  $F$  acting on the layer is given by the relation

$F = \text{area of the layer} \times \text{coefficient of viscosity} \times \text{velocity gradient.}$

$$= 2\pi r l \cdot \eta \cdot dv/dr$$

where  $\eta$  is the coefficient of viscosity of the liquid and  $dv/dr$  is the velocity gradient at a distance  $r$  from O.

$$\text{Now } \frac{dv}{dr} = \frac{d}{dr} (rw) = w + r \cdot \frac{dw}{dr}$$

$w$  is a constant quantity and would remain even in the absence of any viscous drag. As such it does not contribute at all towards the velocity gradient. Hence it is the second term, i.e.,  $r \cdot \frac{dw}{dr}$  which contributes towards the velocity gradient, responsible for the viscous drag on the imaginary cylindrical layer C.

Thus the viscous force  $F$  acting on the layer is given by

$$F = 2\pi r l \cdot \eta \cdot r \frac{dw}{dr} \quad (i)$$

Hence, moment of this force or the torque acting on the layer C is given by

$$\tau = F \times r.$$

$$= 2\pi r^2 l \eta \frac{dw}{dr} \times r$$

$$= 2\pi r^3 l \eta \frac{dw}{dr}$$

$$\text{or, } dw = \left( \frac{\tau}{2\pi\eta l} \right) \left( \frac{dr}{r^3} \right) \quad (\text{ii})$$

Integrating eqn. (ii) for the limits  $w = 0$  and  $w = w_0$  and  $r = b$  and  $r = a$ , we have

$$\int_0^{w_0} dw = \left( \frac{\tau}{2\pi\eta l} \right) \cdot \int_b^a \frac{dr}{r^3}$$

$$\text{or, } w_0 = \frac{\tau}{2\pi\eta l} \left[ \frac{r^{-2}}{2} \right]_b^a$$

$$= \frac{\tau}{4\pi\eta l} \left[ \frac{1}{a^2} - \frac{1}{b^2} \right]$$

$$= \frac{\tau}{4\pi\eta l} \left[ \frac{b^2 - a^2}{a^2 b^2} \right]$$

$$\text{or, } \tau = \frac{4\pi\eta/w_0 a^2 b^2}{b^2 - a^2} \quad (\text{iii})$$

The torque  $\tau$  tends to accelerate the motion of the fluid between C and A. But since the liquid is in a state of steady motion, the inner cylinder must also exert an equal retarding torque on the liquid in contact with it. Since action and reaction are equal and opposite, the inner cylinder, in its turn, experiences an equal and opposite torque  $\tau$  tending to rotate it through an angle  $\theta$ , say, until it is just balanced by the restoring torsional torque which is set in the suspension wire. If the couple per unit twist of the suspension wire be  $c$ , then we have

$$\tau = c\theta \quad (\text{iv})$$

Equating eqns. (iii) and (iv)

$$\tau = \frac{4\pi\eta/w_0 a^2 b^2}{b^2 - a^2} = c\theta \quad (\text{v})$$

In the above calculation (eqn. v), the viscous force acting on the base of the inner cylinder has not been taken into account. Let the magnitude of the torque due to the viscous force on the base of the cylinder A be  $\tau'$ . Then

the total torque on the liquid between A and B is given by

$$\tau_1 = \frac{4\pi\eta/w_0 a^2 b^2}{b^2 - a^2} + \tau' = c\theta \quad (\text{vi})$$

In actual practice  $\tau'$  is altogether eliminated by repeating the experiment with a different length or height  $l'$  of the liquid in between A and B. If the total torque now be  $\tau_2$  and the angle of rotation of the inner cylinder be  $\theta'$  then

$$\tau_2 = \frac{4\pi\eta/l'w_0 a^2 b^2}{b^2 - a^2} + \tau' = c\theta' \quad (\text{vii})$$

Subtracting (vii) from (vi), we get

$$\frac{4\pi\eta w_0 a^2 b^2}{b^2 - a^2} (l - l') = c (\theta - \theta')$$

Therefore,

$$\eta = \frac{(b^2 - a^2)c(\theta - \theta')}{4\pi w_0 a^2 b^2 (l - l')} \quad (\text{viii})$$

To find the value of  $c$ , the torsional couple per unit twist of the suspension wire, the inner cylinder A is first set into torsional vibration. The time period  $T_1$  of this vibration is noted. Next a circular disc of known moment of inertia  $I_2$  is placed over the cylinder A and the combination is then set into vibration. The time period  $T_2$  of this vibration is noted. Then, as we know

$$T_1 = 2\pi \sqrt{\frac{I_1}{c}} \quad (\text{ix})$$

$$\text{and } T_2 = 2\pi \sqrt{\frac{I_1 + I_2}{c}} \quad (\text{x})$$



where  $I_1$  is the moment of inertia of the inner cylinder alone about the suspension wire,

From eqn. (ix) and (x), we have

$$T_2^2 - T_1^2 = \frac{4\pi^2(I_1 + I_2 - I_1)}{c}$$

$$\text{or, } c = \frac{4\pi^2 I_2}{T_2^2 - T_1^2} \quad (\text{xi})$$

Substituting this value of  $c$ , as given by eqn. (xi) in eqn. (viii), we get

$$\eta = \frac{\pi I_2 (b^2 - a^2) (\theta - \theta')}{w_0 a^2 b^2 (l - l') (T_2^2 - T_1^2)} \quad (12.13)$$

The experiment is suitable for determining the coefficient of viscosity of both liquids and gases.

In case of a gas, the experiment is repeated with two different inner cylinders of *different lengths* but of the *same radius* instead of taking two observations with different heights of the liquid in between the same two cylinders. The distance of the bottom of each cylinder from the outer cylinder is kept the same. Then, in the expression for  $\eta$  as given by eqn. (xi),  $l$  and  $l'$  will now stand for the lengths of the two inner cylinders used, with  $a$  the radius of each inner cylinder and  $b$ , that of the outer cylinder.

**Example 12.14** A gas bubble of diameter 2 cm rises steadily through a solution of density 1.75 gm/c.c at the rate of 0.35 cm/sec. Calculate the coefficient of viscosity of the solution. Neglect density of the gas.

**Soln.**

$$\eta = \frac{2}{9} \frac{r^2 g (\rho - \sigma)}{v_i}$$

Here  $\rho = 0$

$V_i = -0.35$  cm/sec (since it moves upwards)

$g = 980$  cm/sec<sup>2</sup>

$$r = 1 \text{ cm}$$

$$s = 1.75 \text{ gm/c.c.}$$

$$\therefore \eta = \frac{2}{9} \frac{(1)^2 \times 980 \times (0 - 1.75)}{-1.35}$$

$$= \frac{2}{9} \frac{980 \times 1.75}{1.35}$$

$$= 1.089 \times 10^3 \text{ poise.}$$

**Example 12.15.** Determine the radius of the drop of water falling through air, if the terminal velocity of the drop is 1.2 cm/sec. Assume the coefficient of viscosity for air =  $1.8 \times 10^{-4}$  and the density of air =  $1.21 \times 10^{-3}$  gm/c.c.

**Soln.**

$$\eta = \frac{2}{9} \frac{r^2 g (\rho - \sigma)}{v_t}$$

$$\text{or, } r^2 = \frac{9 \eta v_t}{2g(\rho - \sigma)}$$

$$\text{Here } \eta = 1.8 \times 10^{-4}$$

$$r = 1 \text{ gm/c.c.}$$

$$\sigma = 1.21 \times 10^{-3} \text{ gm/c.c.}$$

$$v_t = 1.2 \text{ cm/sec.}$$

$$\therefore r^2 = \frac{9 \times 1.8 \times 10^{-4} \times 1.2}{2 \times 980 \times (1 - 1.21 \times 10^{-3})}$$

$$= \frac{19.44 \times 10^{-4}}{19.57 \times 10^2}$$

$$= 0.9933 \times 10^{-6}$$

$$\therefore r = 0.8866 \times 10^{-3}$$

$$\approx 0.001 \text{ cm.}$$

**Example 12.16.** Find the limiting velocity of a rain drop.

Assum, diameter =  $10^{-3}$  m

density of air relative to water =  $1.3 \times 10^{-3}$

coefficient of viscosity of air =  $1.81 \times 10^{-5}$  S.I. units

Density of water =  $10^3$  kg/m<sup>3</sup>.

**Soln.**

$$\eta = \frac{2}{9} \frac{r^2 g (\rho - \sigma)}{v_l}$$

$$= \frac{2r^2 \cdot \rho (1 - \frac{\sigma}{\rho})g}{9\eta}$$

Here  $r = 5 \times 10^{-4}$  m

$\rho = 10^3$  kg/m<sup>3</sup>

$\frac{\sigma}{\rho} = 1.3 \times 10^{-3}$

$\eta = 1.81 \times 10^{-5}$

$$\therefore v_l = \frac{2 \times (5 \times 10^{-4})^2 \times 2 \times 10^3 \times (1 - 1.3 \times 10^{-3}) \times 9.8}{9 \times 1.81 \times 10^{-5}}$$

$$= 30.04 \text{ m/sec.}$$

**Example 12.17.** Two equal drops of water, each of radius  $r$ , are falling through air (viscosity  $\eta$ ) with a steady velocity  $v$ . If the two drops coalesce, find the new velocity of fall.

**Soln.**

Let the radius of the bigger drop be  $R$

$$\therefore \frac{4}{3} \pi R^3 = 2 \left( \frac{4}{3} \pi r^3 \right)$$

$$\therefore R = (2)^{1/3} r = 1.26 r.$$

Let the velocity of the bigger drop be  $V_l$ .



Neglecting the density of air, we have

$$mg = 6\pi\eta rv_t \text{ for the smaller drop}$$

$$2mg = 6\pi\eta RV_t, \text{ for the bigger drop}$$

$$\text{whence } \frac{RV_t}{rv_t} = 2$$

$$\text{or, } V_t = \frac{2rv_t}{R}$$

$$= \frac{2rv_t}{1.26r} \quad (R = 1.26 r)$$

$$= 1.587 v_t.$$

**Example 12.18.** A rotating-cylinder viscometer is employed to measure the co-efficient of viscosity of castor oil at a temperature of  $20^\circ\text{C}$ . The radius of the inner cylinder is 4.00 cm and that of the outer cylinder is 4.28 cm. The inner cylinder is submerged in the oil to a depth of 10.2 cm. When the outer cylinder is rotating at 20.0 rev/min, a torsion balance reads a torque of  $3.24 \times 10^{-2} \text{ N.m}$ . Find the viscosity of the castor oil.

**Soln.**

For Art. 12.11, we have

$$\tau = \frac{4\pi\eta/w_0 a^2 b^2}{b^2 - a^2}$$

$$\text{or, } h = \frac{\tau(b^2 - a^2)}{4\pi/w_0 a^2 b^2}$$

Here  $a$  = radius of the inner cylinder

$$= 4.0 \text{ cm} = 0.04 \text{ m.}$$

$b$  = radius of the outer cylinder

$$= 4.28 \text{ cm} = 0.0428 \text{ m.}$$

$$l = 10.2 \text{ cm} = 0.102 \text{ m.}$$

$$w_0 = 20 \text{ rev/min}$$

$$= \frac{20.0 \text{ rev/min} \times 2\pi \text{ rad/rev}}{60 \text{ s/min}}$$

$$\tau = 3.24 \times 10^{-2} \text{ N.m.}$$

$$\therefore \eta = \frac{(3.24 \times 10^{-2} \text{ N.m}) (0.0428^2 \text{ m} - 0.04^2 \text{ m})}{4\pi \left( \frac{20.0 \text{ rev/min} \times 2\pi \text{ rad/rev}}{60 \text{ s/min}} \right) (0.102 \text{ m}) (0.0428 \text{ m}^2) (0.04 \text{ m})^2}$$

$$= \frac{0.75 \times 10^{-5}}{7.8609 \times 10^{-6}} \text{ Pl}$$

$$= \frac{0.75 \times 10^{-6}}{7.8609 \times 10^{-6}} \text{ Pl} = \mathbf{0.9542 \text{ Pl.}}$$

**Example 12.19.** A typical riverborne silt particle has a radius of  $20 \mu\text{m}$  and a density of  $2 \times 10^3 \text{ Kg/m}^3$ . If the viscosity of water is approximately  $1.0 \text{ m Pl}$ , find the terminal speed with which such a particle will settle to the bottom of a motionless volume of water.

**Soln.** From eqn. (12.10) we have,

$$v_t = \frac{2}{9} \frac{r^2 g (l - \sigma)}{\eta}$$

Here,  $r = 20 \times 10^{-6} \text{ m.}$ ,  $\rho = 2 \times 10^3 \text{ Kg/m}^3$ ,  $\sigma = 1 \times 10^3 \text{ Kg/m}^3$

$\eta = 1 \times 10^{-3} \text{ Pl}$ ,  $g = 9.8 \text{ m/sec}^2$ .

$$\therefore v_t = \frac{(2) (20 \times 10^{-6})^2 [(2.0 - 1.0) \times 10^3] (9.80)}{(9) (1 \times 10^{-3})}$$

$$= 8.7 \times 10^{-4} \text{ m/s.}$$

$$= \mathbf{0.87 \text{ mm/s.}}$$

**EXERCISES**

- [1] Explain the term viscosity of a fluid. Define coefficient of viscosity and give its dimensions in C.G.S. and S.I. units.
- [2] Describe the way in which the different parts of a viscous liquid move when flowing through a fine tube. What change takes place if the motion is increased?
- [3] Distinguish between streamline and turbulent motion. What is Reynold's number? Explain the significance of the Reynold's number.
- [4] Derive an expression for the rate of flow of a viscous liquid through a capillary tube. Discuss the sources of error and necessary corrections.
- [5] Describe the capillary flow method of determining the coefficient of viscosity of a liquid and deduce the formula you would use.  
What is the dimension of the coefficient of viscosity?
- [6] Derive Poiseuille's equation for the rate of flow of a viscous liquid through a capillary tube. Discuss its limitations.
- [7] Give Poiseuille's method of measuring the viscosity of liquids. Derive the theoretical expression on which the method is based.
- [8] Give the theory of rotation viscometer and derive the expression for the coefficient of viscosity of a liquid.
- [9] State Stokes' law and deduce it by the method of dimensions.  
Explain the terms critical velocity and terminal velocity.
- [10] Using Stokes' law, deduce an expression for the terminal velocity of a spherical ball falling under gravity through a viscous fluid.
- [11] Explain what is meant by the coefficient of viscosity of a liquid and define a unit in which it is expressed. Define briefly how this coefficient may be determined for a liquid like glycerine at room temperature by the falling sphere method, indicating the main sources of possible error.
- [12] Describe Poiseuille's method for determining the coefficient of viscosity of a liquid. Discuss the necessary corrections that must be applied to the theoretical expression on which the method is based.
- [13] Deduce the expression for the rate of steady flow of a liquid through a capillary tube of circular cross-section. Explain the limitations of the expression.



- [14] Derive Stokes' formula for the velocity of a small sphere falling through a viscous liquid. Explain how this is utilized to determine the viscosity of a liquid like castor oil. Mention one more application of Stokes' formula.
- [15] Calculate the horizontal force required to move a metal plate of area  $2\text{m}^2$  with a velocity of  $4.5\text{ m/s}$  when it rests on a layer of oil  $2 \times 10^{-3}\text{ m}$  thick. Coefficient of viscosity of oil =  $2\text{ N-s/m}^2$ . ( $9 \times 10^3\text{ N}$ ).
- [16] In a certain experiment on the flow of a liquid through a capillary tube the following data were obtained: Volume of liquid coming out per minute =  $15\text{ c.c.}$ ; head of liquid =  $30\text{ cms}$ ; length of the tube =  $25\text{ cms}$ ; radius of tube =  $1\text{ mm}$ . Calculate the coefficient of viscosity of the liquid; its density being  $2.3\text{ gm/c.c.}$  [ $0.424\text{ dyne - sec./cm}^2$ ].
- [17] A large bottle is fitted with a siphon made of capillary glass tubing. Compare the coefficients of viscosity of water and petrol, if the time taken to empty the bottle in the two cases is in the ratio  $2:5$ . Specific gravity of petrol =  $0.8$  ( $1:2$ ).
- [18] A cylindrical vessel of radius  $0.07\text{ m}$  is filled with water to a height of  $0.5\text{ m}$ . It has a capillary tube  $0.1\text{ m}$  long and  $2 \times 10^{-4}\text{ m}$  radius fixed horizontally at its bottom. Calculate the time in which the water level will fall to a height of  $0.25\text{ m}$ .  $\eta = 10^{-3}\text{ N-s/m}^2$ , and  $g = 9.8\text{ m/sec}^2$ . ( $48.14\text{ hours}$ ).
- [19] A cylindrical vessel of diameter  $10\text{ cm}$  has at its bottom a horizontal capillary tube of length  $20\text{ cm}$  and internal radius  $0.5\text{ mm}$ . If the vessel is filled with water, find the time in which the water level becomes half the initial height. Viscosity of water  $\text{N-s/m}^2$ . ( $1.257\text{ hrs.}$ )
- [20] Two capillary tubes LM and MN are joined end to end at M. Length of LM is  $0.16\text{ m}$  and its diameter is  $4 \times 10^{-3}\text{ m}$ . Length of MN is  $0.04\text{ m}$  and its diameter is  $2 \times 10^{-3}\text{ m}$ . The composite tube is held horizontally with L connected to a vessel of water having a constant head of  $0.3\text{ m}$ . N is open to the atmosphere. Calculate the pressure at M. ( $2.352 \times 10^3\text{ N/m}^2$ ).
- [21] A gas bubble of diameter  $1\text{ cm}$  rises steadily through a liquid of density  $1.5\text{ gms/c.c.}$  at the rate of  $0.45\text{ cm/sec}$ . Calculate the coefficient of viscosity of the liquid. Neglect the density of the gas. ( $181.5\text{ poise}$ ).
- [22] Calculate the terminal velocity of an air bubble of radius  $5 \times 10^{-4}\text{ m}$  rising in a liquid of viscosity  $0.15\text{ N-s/m}^2$ . Density of the liquid is  $900\text{ kg/m}^3$ . Density of air is negligibly small as compared to that of the liquid. ( $3.26 \times 10^{-3}\text{ m/s.}$ )
- [23] A spherical glass ball of mass  $1.34 \times 10^{-4}\text{ kg}$  and diameter  $4.4 \times 10^{-3}\text{ m}$  takes  $6.4\text{ seconds}$  to fall steadily through a height of  $0.381\text{ m}$  inside a large volume of oil of specific gravity  $0.943$ . Calculate the coefficient of viscosity of the oil. ( $0.3653\text{ N-s/m}^2$ ).

- [24] An air bubble is rising through water with a terminal velocity of  $8.72 \times 10^{-4}$  m/s. If the density and viscosity of water are  $10^3$  kg/m<sup>3</sup> and  $10^{-3}$  N-s/m<sup>2</sup> respectively, calculate the radius of the bubble. Neglect density of air in comparison to that of water. ( $2 \times 10^{-8}$  m).
- [25] Two equal drops of water are falling through air with a steady terminal velocity of 5 cm/sec. If the drops coalesce, what will be the new terminal velocity. (7.98 cm/sec.)
- [26] Three capillaries of the same length but internal radii  $3r$ ,  $4r$  and  $5r$  are connected in series and a liquid flows through them in streamline conditions. If the pressure across the third capillary is 8.1 mm. deduce the pressure across the first capillary.



## CHAPTER XIII

### SURFACE TENSION

*Molecular forces of cohesion and adhesion-Molecular range : sphere of influence-Surface tension-Molecular theory of surface tension : Surface film and surface energy-Dimensions of surface tension-Surface tension and free energy of a surface-Temperature variation of surface tension-Pressure difference across a liquid surface-Excess pressure inside a liquid drop-Excess pressure inside a soap bubble-Work done in blowing a bubble-Experimental determination of the surface tension of a liquid-Excess pressure across a curved film or membrane-Capillarity and the angle of contact-Capillary rise : determination of surface tension-Determination of surface tension of water-Determination of surface tension and angle of contact of mercury : Quincke's method-Solved problems-Exercises.*

#### 13.1 Molecular forces of cohesion and adhesion

Surface tension is essentially a molecular phenomenon. It would, therefore, be better to have first a clear idea as to what forces operate between molecules.

(i) *cohesion* : forces of cohesion or *cohesive force* is the force of attraction between the molecules of the same substance. Cohesive force is the greatest in the case of solids, less in the case of liquids and the least in the case of gases (almost negligible at ordinary temperature and pressure, when the molecules lie very much further apart for it to be appreciable). This explains why a solid has a definite shape and size, a liquid has a definite free surface and a gas has neither.

(ii) *adhesion* : forces of adhesion or the *adhesive force* is the force of attraction between molecules of different substances and is different for different pairs of substances. When one substance is placed on another substance the outcome is decided by the relative strengths of the cohesive and adhesive forces. When a drop of water is placed on a piece of clean glass it spreads and we say that water wets glass. This is because the force of adhesion between the molecules of water and glass is much too stronger than the cohesive force between the water molecules. For the same reason gum adheres to paper, glue to wood. But when a drop of mercury is placed on a clean sheet of glass, it will take up a nearly spherical



shape, the smaller the drop the more spherical the shape will be. In this case the cohesion between mercury molecules is stronger than the adhesion between the molecules of mercury and glass.

### 13.2 Molecular range : sphere of influence

Cohesive force is different from ordinary gravitational force and does not obey the ordinary inverse square law. However, the force does vary inversely as some power—probably the eighth power, of the distance between two molecules and thus decreases rapidly with distance. *It is appreciable when the distance between two molecules is inappreciable and vice versa.* The maximum distance between two molecules up to which the force of cohesion acts is called their *molecular range*. It is generally of the order of  $10^{-7}$  cm in the case of solids and liquids, being different for different substances. *A sphere drawn around a molecule with the molecule as the centre and a radius equal to the molecular range is called the sphere of influence of the molecule.* The molecule will attract or be attracted by other molecules only when these other molecules are within its sphere of influence, remaining unaffected by the molecules outside it.

### 13.3 Surface tension

The fact that a drop of liquid, e.g., water, can cling to the lower side of a horizontal glass plate illustrates two properties: *first* there is a force of adhesion between the molecules of liquid and glass and *second* the force of cohesion between the liquid molecules makes it behave as though it were enclosed in an *elastic bag or skin* which is always under tension tending to contract its surface, and forming a boundary between the liquid and the atmosphere which surrounds the drop. The following example will further illustrate this idea of elastic skin. Most people are aware that an ordinary sewing needle can be made to float on water even though the density of steel may be as much as ten times that of water. Usually, the needle is placed on a small piece of filter paper, which is then gently placed on the water surface. Within few seconds the paper sinks to the bottom and the needle is left floating. The experiment works best if the needle is slightly greasy.

On close examination it will be revealed that the needle rests in a slight depression. The surface of the water behaves as though covered with an elastic skin. This property of a liquid is called *surface tension*.

The concept of elastic skin and the pulls they are capable of exerting is useful in the understanding of surface tension. The portions of the free surface of any liquid in an open vessel lying on opposite sides of any straight line imagined on the surface, resist separation, showing the existence of tension in the surface. The liquid on either side of the imaginary line exerts a *pull* on the liquid on the other side. This pull lies on the plane on the surface and is perpendicular to the line. Thus *surface tension is quantitatively defined as the tangential force (pull) per unit length across a portion of the line on the surface film, acting perpendicularly to it at every point, and tending to pull the surface apart along the line.*

The phenomenon of surface tension and the *tension exerted* by a surface film can be beautifully illustrated by the following example. If a wire-ring is dipped in a soap solution and then taken out, a thin soap-film will be formed across the ring. If a moistened cotton loop is placed lightly on the film, it will remain in any form or position in which it is placed as shown in Fig. 13.1 (a) for the soap-film lies both inside and outside the loop. Thus there are equal and opposite forces at every point on the loop, one tending to pull it *outwards* (due to outside film) and the other tending to pull it *inwards* (due to the inside film) thus cancelling each other out. If the film inside the loop be now pricked, the film inside disappears and the loop at once gets stretched into the form of circular ring as shown in Fig. 13.1 (b). This happens because, the inward forces having all vanished, only the outward forces are left acting perpendicularly to it at every point. The area enclosed by the loop will be maximum since, for a given perimeter, a circle encloses a maximum area. In other words, the area of the soap-film, sandwiched between the wire ring and the cotton loop, will be minimum. This clearly demonstrates that the film has a tendency to contract or shrink, or that there is tension in it.



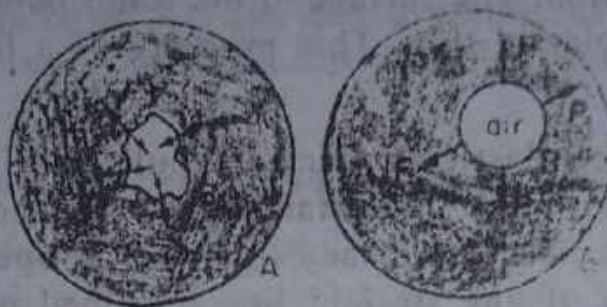


Fig. 13.1

A tent keeps out water owing to the tension which exists in the lower surface of the rain-water in contact with the canvas. If, after a shower of rain, the inside surface of a tent is examined a large number of water surface will be observed through the pores of the canvas. The water is held by the strong surface tension forces of water. It is important not to touch the inside of the canvas. If it is touched at any point, a large number of these tiny surfaces are brought into contact and they coalesce into a large drop of water. This cannot be supported by the surface tension. Consequently the tent starts leaking at the point where it is touched.

#### 13.4 Molecular theory of surface tension : surface film and surface energy

Laplace successfully applied the concepts of molecular ranges and sphere of influence to surface tension and was the first to explain the many observations in connection with surface tension.

Fig. 13.2 shows four molecules A, B, C and D with their spheres of influence drawn around them. Molecule A lies well inside the liquid mass. As its sphere of influence lies wholly within the liquid, it is attracted by equal forces in all directions by the other molecules. So there is no resultant force on the molecule to make it move in any particular direction. Impact with other molecules entirely determines its direction of motion.



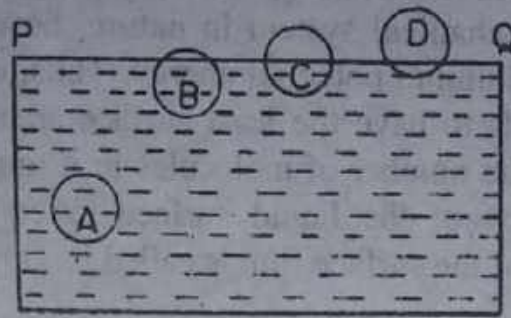


Fig. 13.2

Molecule B lies near the surface at a depth less than the range of molecular attraction. Part of it lies outside the liquid and the upper half contains fewer molecules than the lower half. Thus a resultant force acts on this molecule towards the interior of the liquid.

The molecule C lies on the surface of the liquid; full one-half of its sphere of influence lies outside the liquid molecules. Since there are liquid molecules in the entire lower half, the resultant downward force on the molecule C is maximum.

In case of the molecule D, only a part of its sphere of influence lies inside the liquid, so that the downward force on it decreases. When the sphere of influence completely passes out of the liquid surface, there is no downward force on the molecule at all and the molecule is then free to move as a molecule of gas or vapour, *i.e.*, the liquid evaporates.

From the discussion above, it becomes obvious that all the molecules lying in a surface film *will experience a resultant cohesive force in the downward direction, the magnitude of which increases as the free surface of the liquid is approached.* The surface-film may be defined as the layer of liquid lying in between a plane drawn parallel to the free surface layer and a plane drawn at a distance equal to molecular range from it. Due to this downward pull, the molecules in the surface film tend to move towards the interior of the liquid and so the surface of the liquid tends to contract or decrease and *behave like a stretched membrane.*

When a molecule is brought from the interior of the liquid to the surface-film, some extra work has to be done against the resultant downward pull towards the interior. The molecule thus, gains potential energy at the expense of its kinetic energy. The molecules in the surface-

film, therefore, possess more potential energy than those in the interior of the liquid. Every mechanical system in nature, however, tends to settle down to a state of minimum potential energy. Thus, to minimize potential energy, the film tends to have the least surface area (its thickness being fixed) in order that the number of molecules in it may be minimum. This explains the tendency of the liquid surface to contract. The potential energy per unit area of the surface film is called its *surface energy*.

All liquid surfaces, in fact, tend to contract and become as small as possible for any given volume of liquid. For example, liquid drops tend to be spherical as a sphere is that shape that has a minimum surface for a given volume. However, the decrease in area is arrested when repulsive forces arising from the molecules moving closer together balance the inward attraction.

The molecules lying side by side on the surface cling to one another and endow the surface with properties somewhat resembling those of a stretched membrane. This is the explanation of the surface tension of a liquid from the stand-point of molecular theory.

### 13.5 Dimensions of surface tension

Surface tension of a liquid is the force per unit length on an imaginary line on the surface of the liquid film, the force acting at right angle to this line.

Hence its dimensions are

$$\frac{\text{force}}{\text{length}} = \frac{ML}{T^2} \div L = \frac{M}{T^2} = M.T^{-2}.$$

### 13.6 Surface tension and free energy of a surface

If a surface of a liquid is to be increased, work has to be done. For instance take a rectangular frame-work of wire ABCD of which the side AD can slide over BA and CD (Fig. 13.3). Let a soap-film be formed across ABCD by dipping it in a soap solution. The film will have two surfaces one on either side. These surface will tend to contract and, therefore, the wire AD will tend to be drawn towards the closed end of the loop. Thus a force equal to that due to surface tension will have to be applied to keep it in position. Let this force be  $F$ . Remembering that the film has two surfaces, each having a surface tension  $T$ , the total force due



to surface tension is  $2T.l$  where  $l$  is the length of the wire AD. Since the film is in equilibrium

$$2T.l = F$$

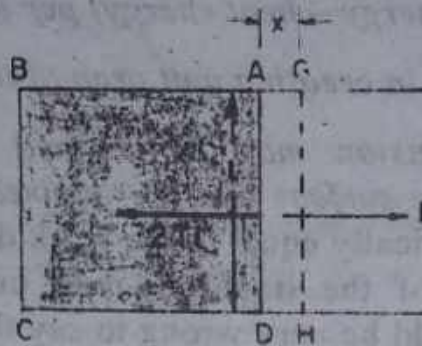


Fig. 13.3

Now, if the wire be pulled through a distance  $x$  so as to extend the film by an area  $l.x$  on each side, the work done will be

$$F.x = 2Tl.x$$

If the change is considered to have taken place isothermally, then the work done is stored in the film as *surface energy*. The increase in area of both sides of the film is  $2l.x$ . Hence the surface energy per unit area,  $E$ , will be

$$E = \frac{2Tl.x}{2lx} = T \quad (13.1)$$

*Surface tension is, therefore, numerically equal to the surface energy per unit area.*

If, however, the effect due to change in temperature is considered, the above expression ( $E = T$ ) will have to be slightly modified. The film gets cooled on being stretched and, therefore, takes up heat from the atmosphere to come to its original temperature. The mechanical work done in stretching the film together with the amount of heat absorbed now forms the energy of the new surface area  $2lx$  of the film formed.

If  $Q$  be the amount of heat energy absorbed per unit area of the new surface formed, then



$$E, 2lx = 2T.lx + Q.2lx$$

$$\text{or, } E = T + Q$$

$$\text{or, } T = (E - Q) \quad (13.2)$$

= (surface energy – heat energy) per unit area

= work done in creating unit area of the film

Thus surface tension may be defined in general as the mechanical part of the surface energy required to create unit area of the film. It is numerically equal to the work done in increasing the area of the surface of the liquid by one unit under isothermal conditions. But it would be very wrong to say that surface tension is surface energy per unit area.

The quantity of energy that is numerically equal to the surface tension is known as the *free surface energy* because the mechanical work done can be released when the surface contracts.

The stretching of the soap-film described in this experiment is different from the stretching of a sheet of elastic rubber in the sense that the molecules at the surface do not get stretched. What happens is the molecules inside the body of the film get to the surface forming fresh film. During the process of this increase in the area of the surface film, the force due to surface tension  $2l.T$  remains constant. In the case of rubber sheet, the stretch increases the tension also.

From thermodynamic considerations, eqn.

$$E = T + \theta \text{ may be written as}$$

$$\text{where } Q = -\theta . \frac{dT}{d\theta}$$

With all liquids the surface tension decreases with the rise of temperature, and thus  $\frac{dT}{d\theta}$  is negative.  $E$  is, therefore, greater than  $T$ .

Fig. 13.4 shows graphs of surface energy per unit area and surface tension, as function of temperature, for water. The temperature  $374^\circ\text{C}$  where both become zero is the *critical temperature*.

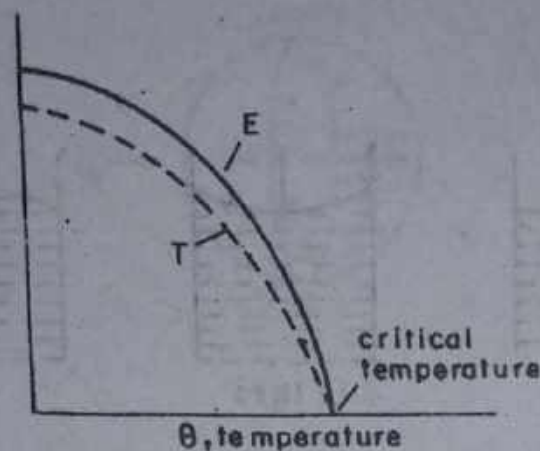


Fig. 13.4

### 13.7 Temperature variation of surface tension

Surface tension depends on temperature. Experiment shows that the surface tension of liquids, and water in particular, *decreases* with increase of temperature. Various formulae relating the surface tension to temperature have been proposed, but none has been found to be completely satisfactory. For small ranges of temperature the relation is fairly linear and is given by,  $T = T_0 (1 - \alpha t)$ , where  $T_0$  and  $T$  are surface tensions at  $0^\circ\text{C}$  and  $t^\circ\text{C}$  respectively, and  $\alpha$  is the temperature coefficient of surface tension for the liquid.

The decrease of surface tension with temperature may be attributed to the greater average separation of the molecules at higher temperature. The force of attraction between the molecules is then reduced, thereby, reducing the surface tension. The surface tension practically vanishes at a temperature, called the *critical temperature*, at which the interface between a liquid and its vapour disappears.

### 13.8 Pressure difference across a liquid surface

A liquid surface may be plane, concave or convex as shown in (Fig. 13.5).

(i) Suppose the free surface of a liquid is *plane* as shown in Fig. 13.5 (i). Then the resultant force due to surface tension on a



molecule on the surface is zero. The cohesion-pressure is, therefore, just *nominal*.

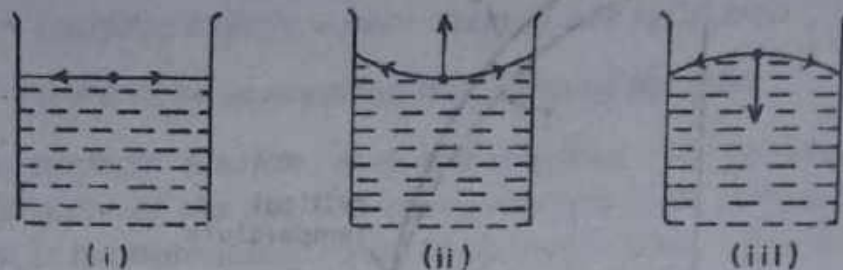


Fig. 13.5

(ii) When the free surface of the liquid is concave outside [Fig. 13.5(ii)], the resultant force due to surface tension on a molecule on the surface is upwards. The cohesion-pressure is, therefore, *decreased*.

(iii) When the free surface of the liquid is convex outside [Fig. 13.5 (iii)], the resultant force due to surface tension on a molecule on the surface is downwards. The cohesion-pressure in this case is, therefore, *increased*.

Thus for a curved liquid surface, the pressure on the *concave side is greater than that on the convex surface*. The difference of pressure depends on surface tension of the liquid and the radius of curvature of its surface.

### 13.9 Excess pressure inside a liquid drop

Let us consider a spherical liquid drop of radius  $r$ . The molecules near the surface of the drop, which is *convex* in shape, experience a resultant pull *inwards* due to surface tension. The drop, therefore, tends to contract. But as the drop contracts, it compresses the liquid within the drop, thereby increasing the interior pressure to a point which prevents further contraction. At this point, the pressure inside the drop must be greater than the pressure outside the drop. This is so, because, *for equilibrium the inward thrust due to pressure outside and force of surface tension must be equal to the outward thrust due to the pressure inside*.



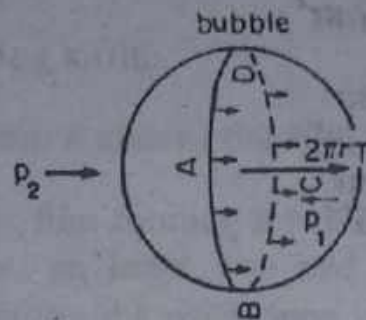


Fig. 13.6

Let the drop be divided into two equal hemispheres by a horizontal plane ABCD (Fig. 13.6). Considering the equilibrium of the upper hemisphere, let the pressure inside and outside be  $p_1$  and  $p_2$ .

Now the (outward) thrust due to  $p_1 = p_1 \times \pi r^2$  and the (inward) thrust due to  $p_2 = p_2 \times \pi r^2$

where  $\pi r^2$  is the area obtained by projecting one-half of the drop on a plane perpendicular to the direction of the force (Fig. 13.7). Now the other half of the drop exerts a force to the left equal to the surface tension times the circumference (perimeter) of the face ABCD and is equal to  $T.2\pi r$ .

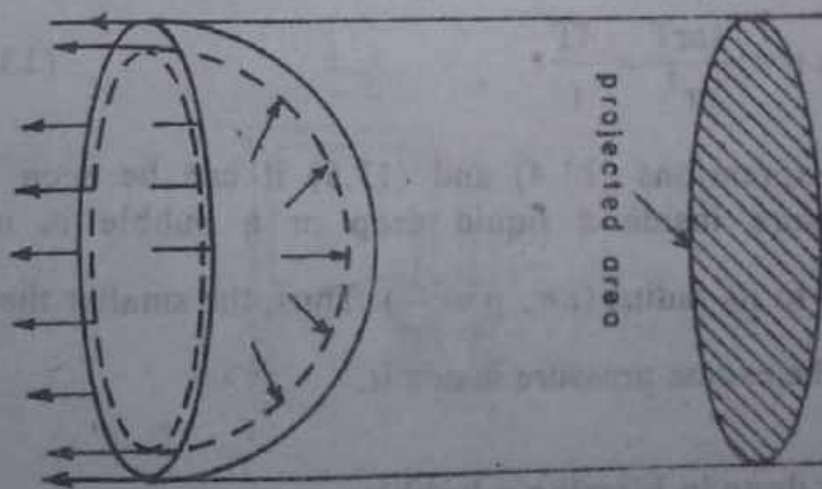


Fig. 13.7

Therefore, for equilibrium,

$$T \cdot 2\pi r + p_2 \cdot \pi r^2 = p_1 \cdot \pi r^2$$

$$\text{or, } (p_1 - p_2) \pi r^2 = T \cdot 2\pi r$$

$$\text{or, } p = (p_1 - p_2) = \frac{T \cdot 2\pi r}{\pi r^2}$$

$$= \frac{2T}{r} \quad (13.4)$$

where  $p = p_1 - p_2$  is the excess pressure inside over the pressure outside.

### 13.10 Excess pressure inside a soap-bubble

If, instead of a drop of liquid, we consider a bubble, say a soap-bubble, there are two surfaces to be considered, and *not one*. The bubble is formed of a certain volume of air bounded by a thin spherical shell or a hollow cylinder, having two surfaces in contact with air, one inside the bubble and the other outside the bubble with liquid (say, soap solution) in between. Therefore, the force due to surface tension in this case is  $2 \times 2\pi r \cdot T$  ( $2\pi r \cdot T$  being the force due to each surface). For equilibrium of the hemisphere, we, therefore, have in this case,

$$p \cdot \pi r^2 = 2 \times 2\pi r \cdot T$$

$$\text{whence } p = \frac{4\pi r T}{\pi r^2} = \frac{4T}{r} \quad (13.5)$$

From expressions (13.4) and (13.5) it can be seen that the excess pressure inside a liquid drop or a bubble is inversely proportional to its radius (*i.e.*,  $p \propto \frac{1}{r}$ ). Thus, the smaller the bubble, the greater the excess pressure inside it.

### 13.11 Work done in blowing a bubble

The work done in blowing a bubble can be easily calculated as follows :

We know that if the cooling produced while stretching a film is neglected (art. 13.6), then

work done in creating a film

= surface tension  $\times$  area of the film created.

Now, the area of the film forming a bubble of radius  $r$  is  $2 \times 4\pi r^2$ , for it has two surfaces, an inner one and an outer one, each of surface area  $4\pi r^2$ . Therefore, the work done in blowing the bubble

$$= T \times 2 \times 4\pi r^2.$$

### 13.12 Experimental determination of the surface tension of a liquid

The expression for the excess pressure inside a bubble as deduced in Art. 13.10, affords a simple method for the determination of the surface tension of the liquid. The vertical arm AB of a glass tube having the shape shown in (Fig. 13.8) is dipped into the experimental liquid. The tube AB has a fine *orifice* or aperture at its lower end. On dipping into the liquid, a liquid film is formed at this orifice. This film is then blown into a bubble at B, by opening the stop-cock S in the top arm for a while and allowing some air to come into AB. The remaining limb of the tube is connected to a special type of manometer for observing small changes of pressure.

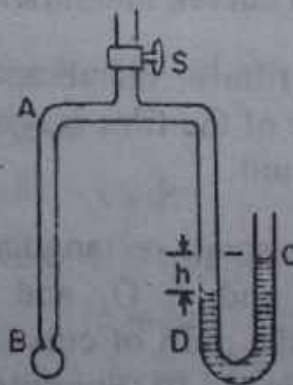


Fig. 13.8



Let  $h$  be the difference of levels of the liquid in the manometer. Then, if  $\rho$  be the density of the manometer liquid, the excess pressure inside the bubble is clearly equal to  $h\rho g$ . But as we know, the excess pressure inside the bubble is also equal to  $4T/r$  the surface tension of the liquid. Hence, combining the two, we have

$$h\rho g = \frac{4T}{r}$$

$$\text{or, } T = \frac{h\rho gr}{4}$$

The radius  $r$  can be measured very accurately with the help of a travelling microscope. Then knowing  $h$ ,  $\rho$ ,  $g$  and  $r$ , the surface tension  $T$  can be found out.

The accuracy of the measurement demands that  $h$  should be large *i.e.*, the excess pressure inside the bubble should be large. This would be so, if the bubble be small or the size of the aperture at B be small.

### 13.13 Excess pressure across a curved film or membrane

When the pressures on both sides of the surface of a liquid film are equal, the surface remains plane; if the pressures are unequal, the surface takes the shape of a curved membrane.

In the case of curved film or membrane, the total force due to excess pressure on one side of the film is equal to the total force due to surface tension of the liquid.

Let ABCD represent a small rectangular portion of the curved liquid surface of sides  $x$  and  $y$ .  $O_1$  and  $O_2$  are the centres of curvature and  $r_1$  and  $r_2$  be the radii of curvature of the two surfaces AB and BC respectively (Fig. 13.9). Suppose the excess pressure on the concave surface be  $p$ .

Initial surface area of the elementary portion ABCD =  $xy$ .

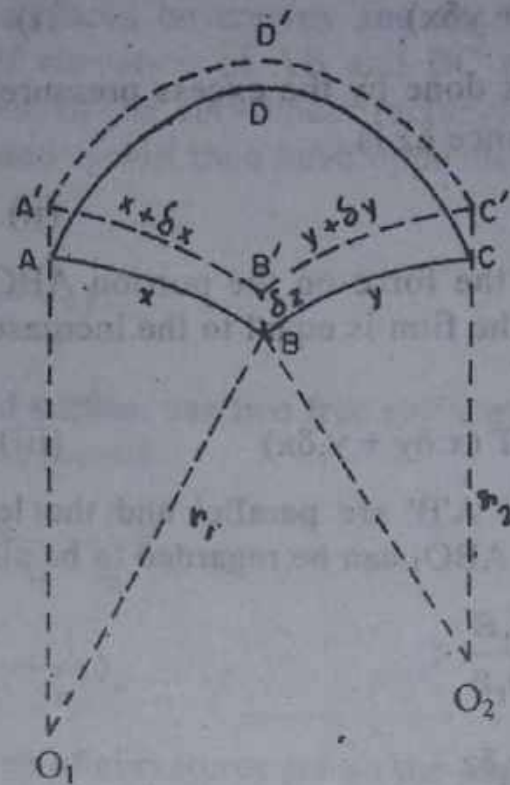


Fig. 13.9

Let this area ABCD be given a small displacement  $\delta z$  by the excess pressure  $p$  along the outward drawn normal to the new position  $A'B'C'D'$  parallel to itself. As a result of this displacement, let the sides of the rectangle be increased from  $x$  to  $x + \delta x$  and from  $y$  to  $y + \delta y$  respectively.

After stretching, the new area of the elementary film  $A'B'C'D'$

$$= (x + \delta x)(y + \delta y)$$

$$= xy + x\delta y + y\delta x + \delta x \delta y$$

$$= xy + x\delta y + y\delta x \text{ (neglecting } \delta x \delta y \text{)}$$

Therefore, the increase in area of the liquid film

$$= (xy + x\delta y + y\delta x) - xy$$

$$= x\delta y + y\delta x$$

Let the surface tension be  $T$ . Since the surface tension may be defined as the work (or energy) required to stretch a surface film by unit area under isothermal condition (Art. 13.16), the increase in surface energy

$$= T (x\delta y + y\delta x) \quad (i)$$

Also, the work done by the excess pressure  $p$  in stretching the film through a distance  $\delta z$  is

$$p \cdot xy \cdot \delta z \quad (ii)$$

where  $p \cdot xy$  is the force on the portion ABCD. Since the work done in stretching the film is equal to the increase in surface energy, we have

$$p \cdot xy \cdot \delta z = T (x \cdot \delta y + y \cdot \delta x) \quad (iii)$$

Again AB and A'B' are parallel and the length AB is small; hence the triangles ABO<sub>1</sub> can be regarded to be similar.

$$\therefore \frac{A'B'}{AB} = \frac{O_1B'}{O_1B};$$

$$\text{or, } \frac{x + \delta x}{x} = \frac{r_1 \delta z}{r_1}$$

$$\text{or, } 1 + \frac{\delta x}{x} = 1 + \frac{\delta z}{r_1}$$

$$\text{or, } \delta x = \frac{x}{r_1} \cdot \delta z \quad (iv)$$

Similarly it can be proved that

$$\delta y = \frac{y}{r_2} \cdot \delta z \quad (v)$$

Substituting the values of  $\delta x$  and  $\delta y$  in eqn. (iii), we get

$$p \cdot xy \cdot \delta z = T \left( \frac{xy \delta z}{r_2} + \frac{xy \delta z}{r_1} \right)$$

$$\text{or, } p = T \left( \frac{1}{r_1} + \frac{1}{r_2} \right) \quad (13.6)$$

Eqn. (13.6) is also true for a *liquid drop* or an *air bubble inside a liquid*.



If one of the surfaces be convex and the other concave, *i.e.*, when the centres of curvature of AB and BC are on the opposite sides of the film, one of the curvilinear rectangle increases and the other decreases.  $r_1$  and  $r_2$  will then have opposite signs and we shall have

$$p = T \left( \frac{1}{r_1} - \frac{1}{r_2} \right)$$

when the liquid surface has two free surfaces as in the case of a soap-film, the excess pressure

$$\begin{aligned} p &= 2 \times T \left( \frac{1}{r_1} + \frac{1}{r_2} \right) \\ &= 2T \left( \frac{1}{r_1} + \frac{1}{r_2} \right) \end{aligned} \quad (13.7)$$

when the centres of curvatures are on the same side.

If the centres of curvatures are on the opposite sides,

$$p = 2T \left( \frac{1}{r_1} - \frac{1}{r_2} \right)$$

A few special cases will now be considered :

(i) *spherical surface* : If the surface is spherical as in the case of a liquid drop, or an air bubble inside a liquid, we have  $r_1 = r_2 = r$ , say ; so that the excess pressure inside it is given by

$$\begin{aligned} p &= T \left( \frac{1}{r} + \frac{1}{r} \right) \\ &= \frac{2T}{r} \end{aligned} \quad (13.8)$$

the same expression as that given by eqn. (13.4)

In the case of a *soap-bubble*, because there are two surfaces, an *inner* and an *outer* one, we have

$$p = 2 \times \frac{2T}{r} = \frac{4T}{r} \quad (13.9)$$

the same as that given by eqn. (13.5)

(ii) *cylindrical surface* : In the case of a cylindrical surface, one of the radii is infinite while the other is the same as the radius  $r$  of the cylinder.

Thus the excess pressure is,

for

$$\text{one single surface} \quad p = T \left( \frac{1}{r} + \frac{1}{\infty} \right) = \frac{T}{r}$$

$$\text{and two surfaces} \quad p = 2 \times \frac{T}{r} = \frac{2T}{r}$$

**Example 13.1.** What is the work done in blowing a soap bubble of radius 10 cm? What additional work will be performed in further blowing it, so that its radius becomes 15 cm?  $T = 30$  dynes/cm.

**Soln.**

Work done in blowing a bubble

= surface area of the bubble  $\times$  free surface energy

= surface area  $\times$  surface tension.

( $\because$  free surface energy = surface tension)

A soap-bubble surrounded by air has two – not one, surface (Art. 13.10). So in case of a soap-bubble,

Work done = 2  $\times$  surface area  $\times$  surface tension

$$= 2 \times 4\pi r^2 \times T$$

$$= 2 \times 4 \times 3.14 \times (10)^2 \times 30$$

$$= 7.536 \times 10^4 \text{ ergs.}$$

Work done when the radius of the soap-bubble is 15 cm

$$= 2 \times 4\pi r^2 \times T$$

$$= 2 \times 4 \times 3.14 \times (15)^2 \times 30$$

$$= 16.956 \times 10^4 \text{ ergs.}$$

Hence additional work done

$$= (16.956 \times 10^4 - 7.536 \times 10^4) \text{ ergs}$$

$$= 9.42 \times 10^4 \text{ ergs.}$$

*Alternately*

when the soap-bubble is blown to a radius of 15 cm, the total increase in surface area

$$= 2 \times 4\pi(15)^2 - 2 \times 4\pi(10)^2$$

$$= 1800\pi - 800\pi.$$

$$= 1000\pi.$$

Hence increase in the work done (additional work)

$$= 1000\pi \times T$$

$$= 1000 \times 3.14 \times 30$$

$$= 9.42 \times 10^4 \text{ ergs.}$$

**Example 13.2.** Calculate the excess pressure inside a soap-bubble of radius  $3 \times 10^{-3}$  m. Surface tension of soap solution is  $20 \times 10^{-3}$  N/m. Calculate also the surface energy of the soap-bubble.

**Soln.**

$$(i) \quad p = \frac{4T}{r}$$

$$= \frac{4 \times 20 \times 10^{-3}}{3 \times 10^{-3}}$$

$$= 26.67 \text{ N/m}^2.$$

(ii) surface energy = (total surface area)  $\times$  (surface tension)

$$= 2 \times 4\pi r^2 \times T$$

$$= 2 \times 4 \times 3.14 \times (3 \times 10^{-3})^2 \times 20 \times 10^{-3}$$

$$= 4.52 \times 10^{-6} \text{ Joules.}$$

**Example 13.3.** What would be the pressure inside a small air bubble of 0.1 mm radius situated just below the surface of water?



Surface tension of water = 72 dynes/cm, atmospheric pressure =  $1.013 \times 10^6$  dynes/cm<sup>2</sup>.

**Soln.**

The excess pressure inside the air bubble,

$$\begin{aligned} p &= \frac{2T}{r} = \frac{2 \times 72}{0.01} \quad (r = 0.1 \text{ mm} = 0.01 \text{ cm}) \\ &= 144 \times 10^2 \text{ dynes/cm}^2 \\ &= 0.0144 \times 10^6 \text{ dynes/cm}^2 \end{aligned}$$

Now excess pressure,  $p = p_1 - p_2$

where  $p_1$  = pressure inside the bubble and  $p_2$  = pressure outside the bubble (Art. 13.9)

Since, the bubble is just below the surface of water, the pressure outside the bubble,

$$\begin{aligned} p_2 &= \text{atmospheric pressure} \\ &= 1.013 \times 10^6 \text{ dynes/cm}^2. \end{aligned}$$

$$\begin{aligned} \therefore p_1 &= p + p_2 \\ &= 1.013 \times 10^6 + 0.0144 \times 10^6 \\ &= 1.0274 \times 10^6 \text{ dynes/cm}^2. \end{aligned}$$

**Example 13.4.** Calculate the depth of water at which an air bubble of radius  $4 \times 10^{-4}$  m may remain in equilibrium. Surface tension of water =  $70 \times 10^{-3}$  N/m.

**Soln.**

Obviously, for the air bubble to remain in equilibrium, the excess pressure inside the bubble must be equal to the pressure due to the given depth of water, say  $h$  metre.

$$\text{Now } p = \frac{2T}{r}$$

Again pressure due to  $h$  metre of water

$$= h\rho g.$$

$$\therefore p = h\rho g$$

$$\text{or, } \frac{2T}{r} = h\rho g$$

$$\text{or, } h = \frac{2T}{r\rho g}$$

$$\text{here } r = 4 \times 10^{-4} \text{ m.}$$

$$\rho = 10^3 \text{ kg/m}^3$$

$$g = 9.8 \text{ m/sec}^2$$

$$\begin{aligned} \therefore h &= \frac{2 \times 70 \times 10^{-3}}{4 \times 10^{-4} \times 10^3 \times 9.8} \\ &= 3.571 \times 10^{-3} \times 10^{-3} \times 10^4 \\ &= 3.571 \times 10^{-2} \text{ m.} \end{aligned}$$

**Example 13.5.** The pressure of air inside a soap bubble of 0.7 cm diameter is 8 mm of water above the atmospheric pressure. Calculate the surface tension of soap solution.

**Soln.**

$$p = \frac{4T}{r} ; \quad \text{or, } T = \frac{pr}{4}$$

$$\text{Here } p = h\rho g = 0.8 \times 1 \times 980$$

$$r = 0.35 \text{ cm.}$$

$$\therefore T = \frac{0.8 \times 1 \times 980 \times 0.35}{4}$$

$$= \frac{274.4}{4}$$

$$= 68.8 \text{ dynes/cm.}$$

**Example 13.6.** The pressure inside a soap-bubble of radius 1 cm balances a 1.4 mm column of oil of specific gravity 0.80. Calculate the surface tension of the soap-solution.

**Soln.**

Excess pressure inside a soap bubble,

$$p = \frac{4T}{r}$$

This is balanced by an oil column of height 1.4 mm.

$$\begin{aligned}\therefore p &= 0.14\rho \times g \\ &= 0.14 \times 0.80 \times 980 \\ &= \frac{4T}{r}\end{aligned}$$

$$\begin{aligned}\therefore T &= \frac{r \times 0.14 \times 0.8 \times 980}{4} \\ &= \frac{1 \times 0.14 \times 0.8 \times 980}{4} \\ &= \frac{109.76}{4} \\ &= 27.44 \text{ dyne/cm.}\end{aligned}$$

**Example 13.7.** A soap bubble 2 mm in diameter is blown at the end of a tube which is connected at the other end to a U-tube manometer containing oil of specific gravity 0.8. Find the difference in the level in the two limbs of the manometer. Surface tension of the soap solution = 25 dynes/cm.

**Soln.**

Let the difference in level of the oil column be  $h$  cm. Then pressure corresponding to this difference in the level,  $p = h\rho g$ .

Obviously this is equal to the excess pressure inside the soap bubble.

$$\text{Now the excess pressure, } p = \frac{4T}{r}$$

$$\therefore h\rho g = \frac{4T}{r}$$

$$\text{or, } h = \frac{4T}{r\rho g}$$

Here  $T = 25$  dynes/cm.

$r = 1 \text{ mm} = 0.1 \text{ cm}$

$\rho = 0.8$

$g = 980 \text{ cm/sec}^2$



$$\therefore h = \frac{4 \times 25}{0.1 \times 0.8 \times 980}$$

$$= 1.2755 \text{ cm.}$$

**Example 13.8.** A minute spherical air bubble is rising slowly through a column of mercury contained in a deep jar. If the radius of the bubble at a depth of 100 cm is 0.1 mm, calculate its depth when its radius is 0.126 mm. Atmospheric pressure = 760 mm of mercury; surface tension of mercury = 567 dynes/cm.

**Soln.**

$$\begin{aligned} \text{Total pressure on the air bubble at a depth of 100 cms of mercury} \\ &= \text{atmospheric pressure} + \text{pressure due to 100 cms. of mercury} \\ &= 76 \times 13.6 \times 980 + 100 \times 13.6 \times 980 \\ &= (76 + 100) \times 13.6 \times 980 \\ &= 2345728 \text{ dynes/cm}^2 \end{aligned}$$

Excess pressure inside the air bubble,

$$p = \frac{4T}{r} = \frac{2 \times 567}{0.01} = 113400 \text{ dynes/cm}^2.$$

$$\begin{aligned} \therefore \text{the total pressure inside the air bubble} \\ &= \text{excess pressure} + \text{pressure outside} \\ &= 113400 + 2345728 \\ &= 2459128 \text{ dynes/cm}^2 \end{aligned}$$

$$\begin{aligned} \text{volume of the bubble} &= \frac{4}{3} \pi r^3 \\ &= \frac{4}{3} \times \pi \times (0.01)^3. \end{aligned}$$

Let the depth of the mercury column when the radius of the air bubble is 0.126 mm be  $h$  cm.

$$\begin{aligned} \text{Then the total pressure on the bubble at a depth of } h \\ &= (76 + h) \times 13.6 \times 980 \text{ dynes/cm.} \end{aligned}$$

excess pressure inside the bubble,

$$= \frac{2T}{r} = \frac{2 \times 567}{0.0126}$$

$$= 90000 \text{ dynes/cm}^2.$$

Hence, the total pressure inside the bubble

$$= (76 + h) \times 13.6 \times 980 + 90000$$

$$= 1012928 + 13328h + 90000$$

$$= 1102928 + 13328h.$$

volume of the bubble when

$$r = 0.0126 \text{ cm}$$

$$= \frac{4}{3} \pi (0.0126)^3.$$

According to Boyle's law  $PV = \text{constant}$ .

$$\therefore 2459128 \times \frac{4}{3} \pi (0.01)^3$$

$$= (1102928 + 13328h) \times \frac{4}{3} \pi (0.0126)^3$$

$$\text{or, } 2.459128 = 2.2063 + 0.0267h.$$

$$\text{or, } 0.0267h = 2.459128 - 2.2063$$

$$= 0.2528$$

$$\text{or, } h = \frac{0.2528}{0.0267} \simeq 9.47 \text{ cm}$$

Thus, the required depth = 9.47 cm.

**Example 13.9.** A liquid drop of radius  $R$  breaks up into 64 small drops. Calculate the change in energy.

**Soln.**

Breaking the liquid drop means an increase in the surface area. Since work has to be done for this purpose, this means that additional amount of energy will be needed. The work done which becomes the surface energy of this increased surface area is given by surface tension  $\times$  increase in the surface area.

Let  $T$  be the surface tension of the liquid.

The surface area of the bigger drop  $= 4\pi R^2$  and the volume of the bigger drop  $= \frac{4}{3} \pi R^3$ .

Let the radius of each smaller drop  $= r$ .

Since the volume of the liquid remains unchanged,

$$\frac{4}{3} \pi R^3 = 64 \times \frac{4}{3} \pi r^3$$

$$\text{or, } R = 4r$$

$$\text{or, } r = \frac{R}{4}$$

Hence the surface area of a smaller drop  $= 4\pi r^2$ . Thus, the total surface area of the 64 drops

$$= 64 \times 4\pi r^2 = 64 \times 4\pi \left(\frac{R}{4}\right)^2$$

$$= 16 \pi R^2.$$

Hence increase in the surface area

$$= 16 \pi R^2 - 4\pi R^2$$

$$= 12 \pi R^2$$

$\therefore$  Change in energy (energy needed)

$$= \text{surface tension} \times \text{increase in surface area}$$

$$= T \times 12 \pi R^2$$

$$= 12 \pi T R^2.$$

**Example 13.10.** A mercury drop of radius 1 cm is sprayed into one million drops of the same size. Calculate the energy expended. Surface tension of mercury  $= 35 \times 10^{-3} \text{ N/m}$ .

**Soln.**

Let the radius of each of the smaller drops  $= r$

$$\therefore \frac{4}{3} \pi (R)^3 = 10^6 \times \frac{4}{3} \pi r^3$$



where  $R$  is the radius of the bigger drop.

$$\therefore R = 10^2 r$$

$$\text{or, } r = \frac{R}{100} = \frac{1}{100} = 0.01 \text{ cm.} = 10^{-4} \text{ m.}$$

$\therefore$  increase in area

$$= 10^6 4\pi r^2 - 4\pi R^2$$

$$= 10^6 4\pi (10^{-4})^2 - 4\pi (10^{-2})^2$$

$$(R = 1 \text{ cm} = 10^{-2} \text{ m})$$

$$= 4\pi 10^{-2} - 4\pi 10^{-4}$$

$$= 4\pi (10^{-2} - 10^{-4})$$

$$= 4\pi (100 \times 10^{-4} - 10^{-4})$$

$$= 4\pi \times 99 \times 10^{-4}$$

$\therefore$  Energy expended

$$= (\text{increase in area}) \times (\text{surface tension})$$

$$= 4 \times 3.14 \times 99 \times 10^{-4} \times 35 \times 10^{-3}$$

$$= 4.352 \times 10^{-2} \text{ Joules.}$$

**Example 13.11.** Calculate the amount of energy needed to break a drop of petrol of volume  $10^{-6} \text{ m}^3$  into a thousand million drops of equal size. Surface tension of petrol =  $26 \times 10^{-3} \text{ N/m}$ .

**Soln.**

Let the radius of the drop of petrol =  $R$ .

$$\therefore \frac{4}{3} \pi R^3 = 10^{-6}$$

$$\text{or, } R^3 = \frac{3}{4\pi} \times 10^{-6}$$

$$\text{or, } R = \left( \frac{3}{4\pi} \right)^{1/3} \times 10^{-2} \text{ m.}$$

If  $r$  is the radius of each of the smaller drops, then

$$\frac{4}{3} \pi R^3 = 1000 \times 10^6 \times \frac{3}{4\pi} \pi r^3$$

$$\text{or, } r = \left(\frac{3}{4\pi}\right)^{1/3} \times 10^{-5} \text{ m.}$$

Increase in surface area

$$= 1000 \times 10^6 \times 4\pi r^2 - 4\pi R^2$$

$$= 10^9 \times 4\pi \times \left(\frac{3}{4\pi}\right)^{2/3} \times 10^{-10} - 4\pi \times \left(\frac{3}{4\pi}\right)^{2/3} \times 10^{-4}$$

$$= 4\pi \times \left(\frac{3}{4\pi}\right)^{2/3} [10^{-1} - 10^{-4}]$$

$$= 4\pi \times \left(\frac{3}{4\pi}\right)^{2/3} [1000 \times 10^{-4} - 10^{-4}]$$

$$= 4\pi \times \left(\frac{3}{4\pi}\right)^{2/3} \times 999 \times 10^{-4}$$

Hence, energy required,

$$= (\text{increase in area}) \times (\text{surface tension})$$

$$= 4\pi \times \left(\frac{3}{4\pi}\right)^{2/3} \times 999 \times 10^{-4} \times 26 \times 10^{-3}$$

$$= 0.01163 \text{ Joules.}$$

**Example 13.12.** 100 drops of water, each of diameter 0.2 mm, combine to form a single drop. Calculate the loss of energy (or energy liberated) in forming the larger drop. Surface tension of water = 72 dynes/cm.

**Soln.**

Let  $r$  be the radius of each of the smaller drops and  $R$  the radius of the bigger drop.

Since volume remains unchanged,

$$1000 \times \frac{4}{3} \pi r^3 = \frac{4}{3} \pi R^3$$

$$\text{or, } 1000 \times \frac{4}{3} \pi \left( \frac{0.02}{2} \right)^3 = \frac{4}{3} \pi R^3$$

$$(r = 0.2/2 \text{ mm} = 0.02/2 \text{ cm}).$$

$$\text{whence } R = 0.1 \text{ cm} = 10^{-1} \text{ cm}$$

Decrease in surface area

$$= 1000 \times 4\pi r^2 - 4\pi R^2$$

$$= 1000 \times 4\pi \times (10^{-2})^2 - 4\pi \times (10^{-1})^2$$

$$= 4\pi [10 \times 10^{-2} - 10^{-1}]$$

$$= 4\pi \times 9 \times 10^{-2}$$

Hence loss of energy or energy liberated

$$= (\text{decrease in area}) \times (\text{surface tension})$$

$$= 4\pi \times 9 \times 10^{-2} \times 72$$

$$= 8138.9 \times 10^{-2} \text{ ergs}$$

$$= 81.389 \text{ ergs.}$$

**Example 13.13.** A soap-bubble is slowly enlarged from a radius of 0.01 m to 0.1 m. Calculate the work done in the process. Surface tension of soap-solution =  $26 \times 10^{-3} \text{ N/m}$ .

**Soln.**

$$\text{radius of the smaller drop, } r = 0.01 \text{ m} = 10^{-2} \text{ m}$$

$$\text{radius of the larger drop, } R = 0.1 \text{ m} = 10^{-1} \text{ m.}$$

Increase in surface area,

$$= 2 [4\pi R^2 - 4\pi r^2]$$

since a soap-bubble has two surfaces.

$$= 2 \times 4\pi \times (10^{-1})^2 - 4\pi \times (10^{-2})^2$$

$$= 2 \times 4\pi [100 \times 10^{-4} - 10^{-4}]$$

$$= 2 \times 4\pi \times 99 \times 10^{-4}$$

Surface energy per unit area



= surface tension

Hence increase in energy

= (increase in area)  $\times$  (surface tension)

$$= 2 \times 4\pi \times 99 \times 10^{-4} \times 26 \times 10^{-3}$$

$$= 64658.88 \times 10^{-7}$$

$$= 6.47 \times 10^{-3} \text{ Joules.}$$

**Example 13.14.** A ring is cut from a platinum tube of 8.5 cms internal and 8.7 cms external diameters. The ring is supported horizontally from the pan of a balance so that it comes in contact with water in a glass vessel. It is found that an extra weight of 3.97 gms is required to pull the ring away from water. Calculate the surface tension of water.

**Soln.**

Let  $r_1$  and  $r_2$  be the internal and external radii of the ring and  $T$  the surface tension.

The mean radius of the ring

$$r = \frac{4.25 + 4.35}{2} = \frac{8.6}{2} = 4.3 \text{ cms.}$$

Total force exerted on the ring due to surface tension

$$= 4\pi r T.$$

$$\text{But } 4\pi r T = mg = 3.97 \times 980$$

$$\text{or, } T = \frac{3.97 \times 980}{4\pi r}$$

$$= \frac{3.97 \times 980}{4 \times 3.14 \times 4.3}$$

$$= 72.04 \text{ dynes/cm.}$$

**Example 13.15.** Two soap bubbles of radii  $a$  and  $b$  coalesce to form a single bubble of radius  $r$ . If the external pressure is  $P$ , prove that the surface tension of the solution from which the bubble are formed is given by

$$T = \frac{P(r^3 - a^3 - b^3)}{(a^2 + b^2 - r^2)}$$

Assume that the temperature remains constant.

Let  $P_r$ ,  $P_a$  and  $P_b$  be the internal pressures of the bubbles of radii  $r$ ,  $a$  and  $b$  respectively. At constant temperature

$$P_r V_r = P_a V_a + P_b V_b \text{ (Boyle's law)}$$

$$\text{or, } P_r \frac{4}{3} \pi r^3 = P_a \frac{4}{3} \pi a^3 + P_b \frac{4}{3} \pi b^3$$

$$\text{or, } P_r r^3 = P_a a^3 + P_b b^3 \quad (i)$$

Also,

$$P = P_r - P = \frac{4T}{r} ; \text{ or, } P_r = P + \frac{4T}{r}$$

$$P_a - P = \frac{4T}{a} ; \text{ or } P_a = P + \frac{4T}{a}$$

$$P_b - P = \frac{4T}{b} ; \text{ or } P_b = P + \frac{4T}{b}$$

Substituting these values of  $P_r$ ,  $P_a$  and  $P_b$  in eqn. (i), we get

$$\left(P + \frac{4T}{r}\right) r^3 = \left(P + \frac{4T}{a}\right) a^3 + \left(P + \frac{4T}{b}\right) b^3$$

$$\text{or, } P(r^3 - a^3 - b^3) = 4T(a^2 + b^2 - r^2)$$

$$\text{or, } T = \frac{P(r^3 - a^3 - b^3)}{(a^2 + b^2 - r^2)} \quad (ii)$$

**Example 13.16.** There is a minute circular hole (or orifice) at the bottom of a small hollow vessel. The vessel has to be immersed in water to a depth of 0.4 m so that no water penetrates inside. Calculate the radius of the hole. Surface tension of water =  $72 \times 10^{-3} \text{ N/m}$ .

**Soln.**

$$p = \frac{2T}{r}$$

But  $p = h\rho g$ .

$$\therefore h\rho g = \frac{2T}{r}$$

$$\text{or, } r = \frac{2T}{h\rho g}$$

Here  $T = 72 \times 10^{-3} \text{ N/m}$ .

$h = 0.4 \text{ m}$

$\rho = 10^3 \text{ kg/m}^3$

$g = 9.8 \text{ m/s}^2$

$$\begin{aligned}\therefore r &= \frac{2 \times 72 \times 10^{-3}}{0.4 \times 10^3 \times 9.8} \\ &= 1.3674 \times 10^{-5} \text{ m.}\end{aligned}$$

**Example 13.17.** Assuming the surface tension of rain water to be 72 dynes/cm, find the difference of pressure inside and outside a rain drop of diameter 0.02 cm. What would this difference of pressure amount to, if the drop were to be decreased by evaporation to a diameter of 0.00002 cm?

**Soln.**

Difference of pressure between the inside and outside of a rain drop is given by

$$p = \frac{2T}{r}$$

Here,  $T = 72 \text{ dynes/cm}$

$r = 0.02/2 \text{ cm} = 0.01 \text{ cm}$

$$\begin{aligned}\therefore p &= \frac{2 \times 72}{0.01} \\ &= 144 \times 10^2 \text{ dynes/cm}^2.\end{aligned}$$

When the diameter is reduced to 0.00002 cm, we have

$$p = \frac{2T}{r}$$



$$= \frac{2 \times 72}{0.00001}$$

$$= 144 \times 10^5 \text{ dynes/cm}^2.$$

**Example 13.18.** A U-tube has two limbs of internal radii 0.5 cm and 1 cm respectively. Calculate the difference in the water levels in the two limbs. Surface tension of water =  $7.2 \times 10^{-2}$  N/m and pressure in the two limbs is given by

$$p = 2T \left( \frac{1}{r_1} - \frac{1}{r_2} \right)$$

**Soln.**

Again  $p = h\rho g$

where  $h$  is the difference in water levels in the two limbs.

$$\therefore h\rho g = 2T \left( \frac{1}{r_1} - \frac{1}{r_2} \right)$$

Here  $r_1 = 0.5 \text{ cm} = 5 \times 10^{-3} \text{ m}$

$r_2 = 1 \text{ cm} = 10^{-2} \text{ m}$

$T = 7.2 \times 10^{-2} \text{ N/m}$

$\rho = 10^3 \text{ kg/m}^3$

$g = 9.8 \text{ m/sec}^2$

$$\therefore h = \frac{2T}{\rho g} \left( \frac{1}{r_1} - \frac{1}{r_2} \right)$$

$$= \frac{2 \times 7.2 \times 10^{-2}}{10^3 \times 9.8} \left( \frac{1}{5 \times 10^{-3}} - \frac{1}{10^{-2}} \right)$$

$$= \frac{14.4 \times 10^{-2}}{9.8 \times 10^3} (2 \times 10^2 - 10^2)$$

$$= 1.47 \times 10^{-2} \times 10^{-3} \times 10^2$$

$$= 1.47 \times 10^{-3} \text{ m.}$$

**Example 13.19.** A glass vessel of internal radius  $5 \times 10^{-4}$  m is dipped vertically into a vessel containing mercury such that the lower end of the tube is  $10^{-2}$  m below the surface of mercury. Calculate the gauge pressure (excess of pressure) of air inside the tube to blow a hemispherical bubble at the lower end of the tube. Surface tension of mercury =  $3.5 \times 10^{-2}$  N/m. Density of mercury =  $1.36 \times 10^4$  kg/m<sup>3</sup>.

**Soln.**

Here, gauge pressure or excess of pressure

$$= \frac{2T}{r} + h\rho g$$

$$T = 3.5 \times 10^{-2} \text{ N/m}$$

$$r = 5 \times 10^{-4} \text{ m}$$

$$h = 10^{-2} \text{ m}$$

$\therefore$  Gauge pressure

$$\rho = 1.36 \times 10^4 \text{ kg/m}^3$$

$$g = 9.8 \text{ m/sec}^2$$

$$= \frac{2 \times 3.5 \times 10^{-2}}{5 \times 10^{-4}} + 10^{-2} \times 1.36 \times 10^4 \times 9.8$$

$$= 140 + 1332$$

$$= 1472 \text{ N/m}^2.$$

**Example 13.20.** A spherical soap bubble of radius 0.01 m is formed inside another soap bubble of radius 0.02 m. Calculate the radius of a single soap-bubble which will have an excess of pressure equal to the difference in pressure between the inside of the inner bubble and the outside of the outer bubble.

**Soln.**

Let the pressure inside the smaller bubble be  $P_1$ , pressure in between the two bubbles be  $P_2$  and atmospheric pressure be  $P$ .

(i) For the smaller bubble

$$P_1 - P_2 = \frac{4T}{r_1} \quad \text{(i)}$$

(ii) For the larger bubble

$$P_2 - P = \frac{4T}{r_2} \quad \text{(ii)}$$

Adding (i) and (ii)

$$\begin{aligned}
 P_1 - P &= \frac{4T}{r_1} + \frac{4T}{r_2} \\
 &= 4T \left( \frac{1}{r_1} + \frac{1}{r_2} \right) \quad \text{(iii)}
 \end{aligned}$$

(iii) For a bubble whose excess pressure is  $P_1 - P$  and radius  $R$ ,

$$P_1 - P = \frac{4T}{R}$$

Equating (iii) and (iv) we have

$$\frac{4T}{R} = 4T \left( \frac{1}{r_1} + \frac{1}{r_2} \right)$$

$$\begin{aligned}
 \text{or, } \frac{1}{R} &= \frac{1}{r_1} + \frac{1}{r_2} \\
 &= \frac{r_1 + r_2}{r_1 r_2}
 \end{aligned}$$

$$\text{or, } R = \frac{r_1 r_2}{r_1 + r_2}$$

$$= \frac{0.1 \times 0.02}{0.01 + 0.02}$$

$$= 6.67 \times 10^{-3} \text{ m.}$$

### 13.14 Capillarity and the angle of contact

If a capillary tube (from the latin word, *capillus* – a hair), which is a glass tube of fine bore open at both ends, be dipped vertically into water or any other liquid that wet it, it will be observed that some of the water rises in the tube to a level higher than the free surface outside the tube. It will be further noticed that the surface of water in the tube which is exposed to the pressure of the atmosphere is concave in shape *i.e.*, shaped like a cup. This cup is termed the *meniscus*. This phenomenon of rise of liquid in a tube of fine bore is what is known as *capillarity*. If the tube is dipped in a liquid that does not wet it, *e.g.*, mercury, then the level of liquid



inside the tube falls to a level below the free surface of the liquid outside the tube. The liquid inside the tube then terminates in a *convex surface*. This capillary rise or fall depends on the nature of the liquid and the bore of the tube. Further, the liquid rises or descends not only in the inside of the tube, it will do so also against the outer wall of the tube. The angle which the tangent to the liquid surface at the point where it meets the tube makes with the wall of the tube *inside the liquid* is called the *angle of contact* (Fig. 13.10). In case of a liquid that does not wet glass the tangent to the liquid at the point where the liquid touches the glass makes an obtuse angle with the glass when measured through the liquid. A liquid will rise in a capillary tube if the angle of contact is acute whereas the liquid will be depressed in the tube if the angle of contact is obtuse.

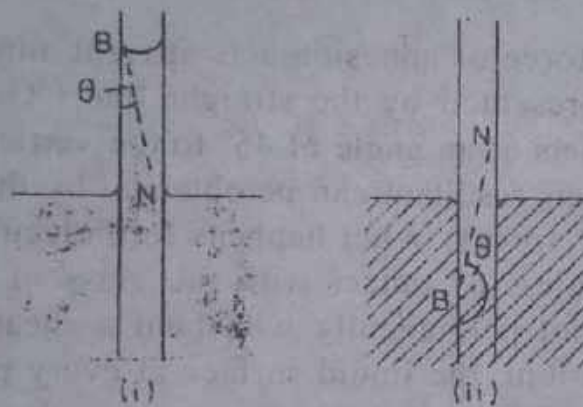


Fig. 13.10

The phenomena of capillary rise and fall and the shape of the meniscus can be explained by molecular theory as follows:

Let the horizontal surface of the liquid meet the surface of plate P at the point O. Then, due to the *force of adhesion*, a liquid molecule at O, in contact with the tube there, will be attracted outwards along the direction OG by the solid molecules of the tube near to it. The same molecule will be attracted inwards along the direction OL by the near molecules of the liquid, due to the *force of cohesion*. The resultant force on it will, therefore, be the resultant of these two forces of adhesion and cohesion.

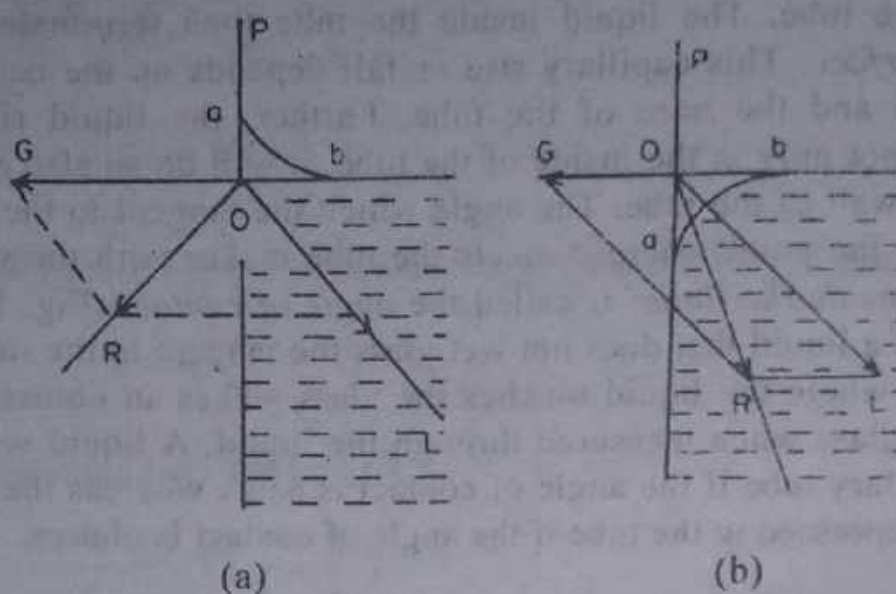


Fig. 13.11

The resultant force of adhesion acts at right angles to the tube at the point P, represented by the straight line OG. The resultant force of cohesion acts at an angle of  $45^\circ$  to the vertical, represented by the line OL. Their resultant can be obtained by the ordinary law of parallelogram of forces. What happens to molecules O, happens to all other molecules in contact with the glass of the tube. And since a liquid cannot permanently withstand a shearing stress, for the sake of equilibrium, the liquid surface at every point will be at right angles to the resultant force there.

In case of a liquid that wets glass [Fig. 13.11 (a)], the adhesive force is greater than the cohesive force, the resultant force, as represented by OR lies outside the liquid. For the sake of equilibrium, the molecules of the liquid near the walls of the tube must rise up against the side of the tube so as to be at right angles to the resultant force at every point, the molecules in the middle, however, remain practically unaffected, thus making the liquid surface *concave upwards*.

In the case of a liquid that does not wet glass [Fig. 13.11 (b)], the adhesive force is smaller than the cohesive force. The resultant force, as represented by OR, lies *inside* the liquid. Thus, for the sake of equilibrium, the liquid molecules near the walls of the tube are depressed. The molecules in the middle, however, remain practically unaffected thus making the liquid surface *convex upwards*.

When the magnitudes of the adhesive and cohesive forces are such that the resultant force act along the vertical, the molecules of the liquid are neither raised nor lowered and the liquid surface remains *flat or plane*.

### 13.15 Capillary rise : determination of surface tension

Let a capillary tube be *dipped* partially in a liquid, say water, which wets it. Due to the force of surface tension, the liquid is found to rise to a certain height inside the capillary terminating in a concave surface upwards. If the tube be a *fine* one, the shape of the liquid meniscus is spherical and concave upwards (Fig. 13.12). We shall obtain an expression for this capillary rise.

Let  $r$  be the radius of the tube at B, the point up to which the liquid rises into it, and as such it will also be the radius of the concave meniscus. The force due to surface tension will act all along the line of contact between the liquid and the glass, *i.e.*, the direction of the force will make an angle  $\theta$  with the walls of the tube where  $\theta$  is the angle of contact between the liquid and the glass. There is thus exerted an *inwards* pull on the glass in *this* direction at all these points. Since in according with Newton's third law of motion,

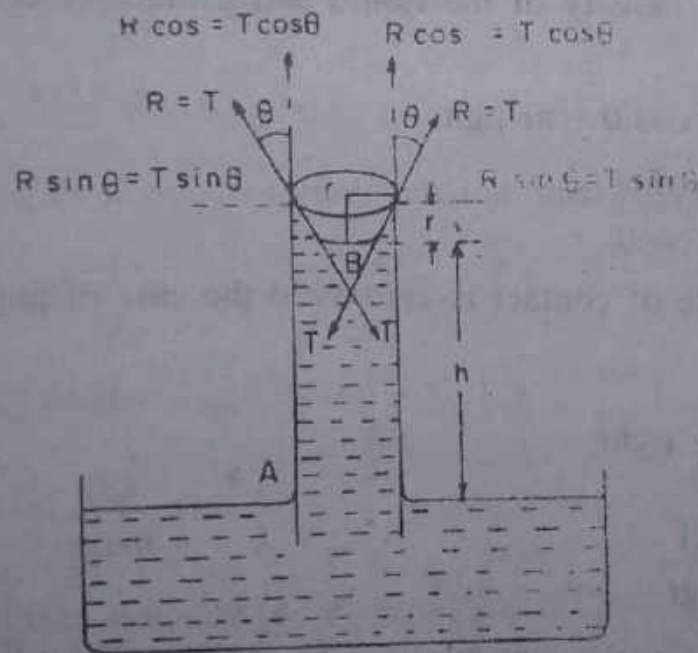


Fig. 13.12



motion, action and reaction are equal and opposite, there is an *equal* and *opposite* reaction  $R$  exerted by the glass on the liquid. This reaction  $R$  (equal to  $T$ ) may be resolved into two rectangular components :

(i)  $R \cos \theta = T \cos \theta$  per cm along the vertical in the upward direction, and

and (ii)  $R \sin \theta = T \sin \theta$  per cm at right angles to it, in the outward direction as shown in the figure. Taking the whole meniscus into consideration, the horizontal or outward components all cancel each other out, and only the vertical components remain to be effective. As the vertical components are all in the same direction, they add up. Since the radius of the tube is  $r$ , the meniscus of the capillary rise touches the tube along a length  $2\pi r$ , the circumference of the circle of radius  $r$ , so the total upward force on the liquid in the tube is  $2\pi r T \cos \theta$ . It is this force which supports the weight of the column of the liquid inside tube.

If  $h$  is the length of the column of the liquid from the horizontal surface  $A$  of the liquid outside the tube to the bottom of the meniscus at  $B$ , then the weight of the liquid is

$$\pi r^2 \cdot h \cdot \rho \cdot g$$

where  $\rho$  is the density of the liquid and  $g$  the acceleration due to gravity. Hence,

$$2\pi r T \cos \theta = \pi r^2 \rho g h$$

$$\therefore T = \frac{\rho g h r}{2 \cos \theta} \quad (13.10)$$

If the angle of contact is zero as in the case of pure water and clean glass,

$$T = \frac{1}{2} \rho g h r$$

$$\text{or, } h = \frac{2T}{\rho g r} \quad (13.11)$$

If a curve is drawn by plotting the height of the liquid column inside the capillary tube against the radius of the tube, the curve is found to be a rectangular hyperbola. This shows that the product of

the radius of the tube and the height of the liquid column is a constant. This is known as *Jurin's law* which states that *the height of the ascent or descent of a liquid in a capillary tube varies inversely as the radius of the capillary tube.*

In deducing the expression for the capillary rise we have not taken into account the weight of the liquid inside the meniscus itself. Very often this weight is not negligible and, therefore, a correcting term is necessary for equation (13.11). If it is assumed that the angle of contact is  $\theta$  and that the meniscus is exactly hemispherical in shape of radius nearly equal to that of the tube at that place, then the volume of the liquid in the meniscus is clearly the difference between the volume of a cylinder of radius  $r$  and length  $r$  and a hemisphere of radius  $r$ . Thus the volume of the liquid inside the meniscus

$$= \pi r^2 \times r - \frac{1}{2} \left( \frac{4}{3} \pi r^3 \right)$$

$$= \pi r^3 - \frac{2}{3} \pi r^3$$

$$= \frac{\pi r^3}{3}$$

The weight of this volume of liquid

$$= \frac{1}{3} \pi r^3 \rho \cdot g$$

Hence the force due to surface tension must balance a weight

$$\pi r^2 \rho g h + \frac{1}{3} \pi r^3 \rho \cdot g = \pi \rho g r^2 \left( h + \frac{r}{3} \right)$$

$$\text{or, } 2\pi r T \cos\theta = \pi \rho g r^2 \left( h + \frac{r}{3} \right)$$

$$\text{or, } T = \frac{1}{2} \frac{\rho g r}{\cos\theta} \left( h + \frac{r}{3} \right) \quad (13.12)$$

Again, if  $\theta$  is assumed to be zero, then

$$T = \frac{1}{2} \rho g r \left( h + \frac{r}{3} \right) \text{ dynes/cm.} \quad (13.13)$$

Eqn. (13.13) can be rearranged as

$$\begin{aligned} T &= \rho g \left( \frac{rh}{2} + \frac{r^2}{6} \right) \\ &= \frac{\rho g r}{2} \left( h + \frac{r}{3} \right) \end{aligned} \quad (13.14)$$

If  $r$  is very small

$$T = \frac{r h \rho g}{2} \text{ dynes/cm.} \quad (13.15)$$

the same as eqn. (13.11)

If  $\theta$  is greater than  $90^\circ$ ,  $\cos\theta$  is negative. According to eqn. (13.10)  $h$  is negative which means that the liquid column in the tube will be depressed below the liquid level outside it. The angle of contact  $\theta$  between mercury and a capillary tube is  $140^\circ$ . Hence it is so difficult to introduce mercury into a fine capillary tube.

Again, the angle of contact  $\theta$  between a capillary tube and glass can be made greater than  $90^\circ$  by coating the inside of the capillary tube with paraffin wax. If such a tube is dipped into water, the surface of the water column inside the tube lies below the surface of water outside the tube.

Further, two liquids may have the same surface tension but different angles of contact. If the same capillary tube is introduced into them, the water column will naturally rise to different heights.

### 13.16 Determination of the surface tension of water

Surface tension of water can be determined experimentally using capillary rise method. A capillary tube of uniform bore is washed thoroughly with nitric acid and potash solution so as to free it from dirt and grease respectively. A straight portion of the tube is pasted on a glass plate and the glass plate is then clamped vertically above a glass beaker containing water in such a way that the lower end of the capillary tube dips into water (Fig. 12.13). A long pointer is also pasted in the glass plate and held vertically along the side of the capillary tube in such a manner that its lower end just touches the surface of water in the beaker. The reading corresponding to the



bottom of the meniscus of the water column inside the capillary tube is taken with a travelling microscope. The beaker is then removed and the reading for the lower end of the pointer is taken. The difference in these readings gives the height of water column in the tube above the surface of water in the beaker. The capillary tube is then cut off with a file at the level of the meniscus and held horizontally. By moving the microscope horizontally, the vertical cross-wire is made tangential first to the left side of the bore and then to the right side of the bore. The difference in readings gives the diameter of the bore from where its radius can be determined. The surface tension of water at room temperature can be determined with the help of eqn. (13.14).

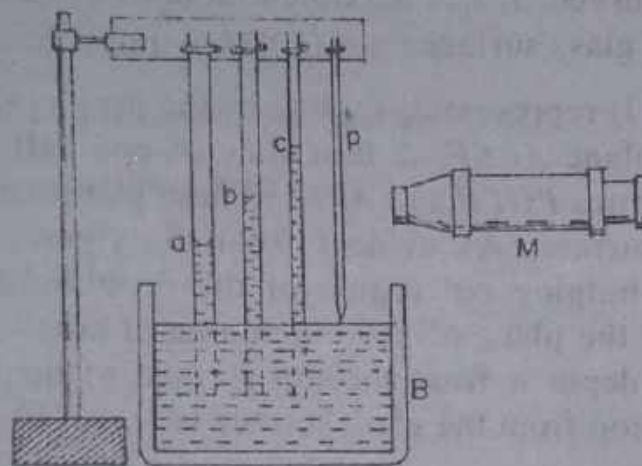


Fig. 13.13

In the actual experiment, three capillary tubes of different diameters are taken and the mean value of surface tension is then calculated.

#### Sources of error and precautions to be taken

- i. The tube should be thoroughly cleansed of dirt and grease.
- ii. The bore of the tube should be as uniform as possible.
- iii. The tube must be exactly vertical.
- iv. Temperature correction for density of water should be taken into consideration.
- v. The angle of contact has been taken to be zero, because it cannot be measured inside the tube.

vi. Distilled water should not be used as it may contain grease which will vitiate the results to a great extent.

vii. Microscope readings should be accurately taken.

### 13.17 Determination of surface tension and angle of contact of mercury : Quincke's method

If a small drop of mercury, which does not wet glass, is placed on a glass plate, the drop assume spherical shape due to surface tension. If the drop be fairly large, force of gravity will dominate and the drop flattens out until its top surface becomes practically horizontal. It means that the top and bottom surface are plane and the sides are curved. It has maximum area of cross-section at some plane from the glass surface.

(Fig. 13.14) represents a section of the drop cut into two halves by a vertical plane  $ABEF$ . A thin slice of one-half is again cut by two vertical planes  $PGEB$  and  $AFKL$ . These planes are perpendicular to the glass surface. As evident from the figure,  $IJ$  is the most protruding or bulging out region of the drop and the plane  $IJCD$  corresponds to the plane of maximum area of cross-section. Let this plane be at a depth  $h$  from the top surface of the drop. The total height of the drop from the glass surface is  $H$ .

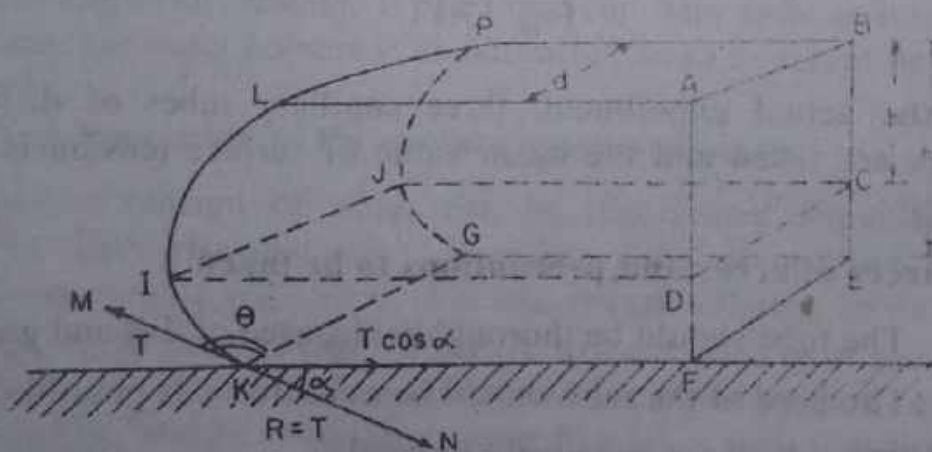


Fig. 13.14

Consider the equilibrium of the portion of the drop above the plane  $IJCD$ . The different forces acting on  $ABCDILPG$  are

(i) The force due to surface tension acting perpendicularly to the line AB from left to right (towards right).

$$\text{Force} = T \times AB$$

$$= T \times d$$

where  $AB = d$ .

(ii) The hydrostatic pressure due to rest of the drop which acts from right to left on the cross-section ABCD.

As the hydrostatic pressure increases from zero to  $h\rho g$  from the top plane AB to the bottom plane CD, the mean hydrostatic pressure  $= \frac{1}{2} h\rho g$ .

$$\text{Area of cross-section ABCD} = h \times d$$

$\therefore$  total force (or thrust) on the area ABCD

$$= \text{mean pressure} \times \text{area}$$

$$= \frac{1}{2} h\rho g \times hd$$

$$= \frac{1}{2} h^2 \rho g d$$

As the surface tension acting vertically along IL has no horizontal component, for equilibrium

$$T \times d = \frac{1}{2} h^2 \rho g d$$

$$\text{or, } h^2 = \frac{2T}{\rho g} \quad (\text{ii.})$$

$$\text{and } T = \frac{h^2 \rho g}{2} \quad (13.16)$$

As can be seen from eqn. (ii) the height  $h$  is independent of the size of the drop.

Let us now consider the equilibrium of the whole section ABEF. At the point K, the surface tension acts along KM. The



reaction  $R$  (equal to  $T$ ) acts along  $KN$ . Its component parallel to the surface of glass along  $KF$  (towards right) is  $T \cos \alpha$  where  $\alpha$  is the supplement of the angle of contact for the liquid and solid surfaces.

Then the force towards the right due to horizontal component of surface tension acting at the point  $K$  is

$$T \cos \alpha \cdot d$$

And the force towards the right due to surface tension acting perpendicularly over the line  $AB$  is  $T \cdot d$ .

Hence the total force due to surface tension from left to right

$$= Td + Td \cos \alpha$$

$$= Td (1 + \cos \alpha)$$

This force is opposed by the horizontal hydrostatic thrust over the face  $ABEF$ , acting towards left. The total force (or thrust) due to this hydrostatic pressure

$$= \text{mean pressure} \times \text{area}$$

$$= \frac{1}{2} H \rho g \times Hd$$

$$= \frac{1}{2} H^2 \rho g d$$

There being no other horizontal forces to be considered, we have, for the equilibrium of the slice,

$$Td (1 + \cos \alpha) = \frac{1}{2} H^2 \rho g d$$

$$\text{or, } T = \frac{H^2 \rho g}{2 (1 + \cos \alpha)}$$

From eqns. (13.16) and (13.17), we have

$$\frac{h^2 \rho g}{2} = \frac{H^2 \rho g}{2 (1 + \cos \alpha)}$$

$$\text{or, } \cos \alpha = \left( \frac{H^2}{h^2} - 1 \right) \quad (13.18)$$

Therefore, by determining experimentally the values of  $h$  and  $H$ , the value of  $\cos \alpha$  – and hence that of  $\alpha$  can be easily calculated.

Now the angle of contact  $\theta$  for the given liquid and solid is given by

$$\theta = (180^\circ - \alpha) \quad (13.19)$$

Hence by measuring  $H$  and  $h$ , the value of the angle of contact  $\theta$  can be measured.

#### *experimental determination*

Clean a glass plate with caustic soda and nitric acid and finally with distilled water. Then dry it. Pour some mercury, free from any impurity, on it so as to form a large circular drop.

Arrange the cross-wires of a travelling microscope so that one of them is exactly vertical in the eye-piece. Focus the microscope on the edge of the drop. Adjust the microscope such that the vertical cross-wire is tangential at the point of maximum curvature of the mercury drop and put the inter-section of the cross-wires at this point by vertical movement of the microscope. Note this reading (i).

Sprinkle some lycopodium powder on the top of the drop. Raise the microscope and move it towards the flat surface of the drop and adjust the microscope such that the horizontal cross-wire coincides with the upper surface of the drop. Note the reading (ii). The difference of the readings (i) and (ii) gives  $h$ .

Lower the microscope and focus it on the line of separation of the glass plate and mercury. Note the reading (iii). The difference of the readings (i) and (iii) gives  $H$ .

From the known values of  $\rho$  and  $g$ , the values of surface tension  $T$  and the angle of contact  $\theta$  can be calculated with the help of eqns. (13.18) and (13.19) respectively.

---

**Example 13.21.** A capillary tube of 0.5 mm bore stands vertically in a wide vessel containing a liquid of surface tension 30 dynes/cm. The liquid wets the tube and has a specific gravity of 0.8. Calculate the rise of the liquid in the tube.

**Soln.**

$$T = \frac{r h \rho g}{2 \cos \theta}$$

The liquid wets the tube; so the angle of contact,  $\theta = 0$ . Hence  $\cos \theta = 1$ .

$$\therefore T = \frac{r h \rho g}{2} ; \text{ or, } h = \frac{2T}{r \rho g}$$

$$\begin{aligned} \text{Here } r &= (0.5 \text{ mm})/2 = (0.05 \text{ cm})/2 \\ &= 0.025 \text{ cm.} \end{aligned}$$

$$T = 30 \text{ dynes/cm.}$$

$$\rho = 0.8 ; g = 980 \text{ cm/sec}^2.$$

$$\therefore h = \frac{r h \rho g}{2 \cos \theta}$$

$$= 3.061 \text{ cm.}$$

**Example 13.22.** In a capillary tube water rises to a height of 0.1 m. In the same capillary tube mercury is depressed by  $3.42 \times 10^{-2}$  m.

Angle of contact for water =  $0^\circ$

Angle of contact for mercury =  $135^\circ$

Calculate the surface tension of mercury given that the surface tension of water is  $72 \times 10^{-3}$  N/m and density of mercury is  $13.6 \times 10^3$  kg/m<sup>3</sup>.

**Soln.**

$$\text{We have, } T = \frac{r h \rho g}{2 \cos \theta}$$

$$(i) \text{ for water, } T_1 = \frac{h_1 \rho_1 g r}{2 \cos \theta_1}$$

$$(ii) \text{ for mercury, } T_2 = \frac{r_2 h_2 \rho_2 g}{2 \cos \theta_2}$$



$$\therefore \frac{T_2}{T_1} = \left( \frac{h_2}{h_1} \right) \left( \frac{\rho_2}{\rho_1} \right) \left( \frac{\cos \theta_1}{\cos \theta_2} \right)$$

Here,  $h_1 = 0.1 \text{ m}$ ,  $h_2 = -3.42 \times 10^{-2} \text{ m}$

$\rho = 13.6 \times 10^3 \text{ kg/m}^3$ ,  $\rho_1 = 10^3 \text{ kg/m}^3$

$\cos \theta_1 = 1$  ( $\because \theta_1 = 0$ )

$\cos \theta_2 = \cos 135^\circ = -0.707$

$T_1 = 72 \times 10^{-3} \text{ N/m}$

$$\begin{aligned} \therefore T_2 &= 72 \times 10^{-3} \times \frac{-3.42 \times 10^{-2}}{0.1} \times \frac{13.6 \times 10^3}{10^3} \times \frac{1}{-0.707} \\ &= 72 \times 10^{-3} \times 0.342 \times 13.6 \times 1.412 \\ &= 472.86 \times 10^{-3} \\ &= 0.4728 \text{ N/m.} \end{aligned}$$

**Example 13.23.** A U-tube is made up of two capillaries of bore 1 mm and 2 mm respectively. The tube is held vertically and partially filled with a liquid of surface tension 49 dynes/cm and zero contact angle. Calculate the density of the liquid, if the difference in the levels of the meniscus is 1.25 cm.

**Soln.**

We have

$$T = \frac{r h \rho g}{2 \cos \theta} = \frac{r h \rho g}{2} \quad (\theta \text{ being } = 0, \cos \theta = 1)$$

Let  $h_1$  and  $h_2$  be the heights of the liquid columns in the two limbs, and  $r_1$  and  $r_2$  be their respective radii.

Then,

$$T = \frac{r_1 h_1 \rho g}{2} ; \text{ or, } h_1 = \frac{2T}{r_1 \rho g}$$

$$\text{again, } T = \frac{r_2 h_2 \rho g}{2} ; \text{ or, } h_2 = \frac{2T}{r_2 \rho g}$$

$$\therefore h_1 - h_2 = \frac{2T}{\rho g} \left( \frac{1}{r_1} - \frac{1}{r_2} \right)$$

$$\text{Here } r_1 = \frac{1}{2} = 0.5 \text{ mm} = 0.05 \text{ cm}$$

$$r_2 = \frac{2}{2} = 1 \text{ mm} = 0.1 \text{ cm}$$

$$T = 49 \text{ dynes/cm}$$

$$g = 980 \text{ cm/sec}^2$$

$$\text{and } h_1 - h_2 = 1.25 \text{ cm}$$

$$\begin{aligned} \therefore 1.25 &= \frac{2 \times 49}{\rho \times 980} \left[ \frac{1}{0.05} - \frac{1}{0.1} \right] \\ &= \frac{89}{\rho \times 980} (20 - 10) = \frac{1}{\rho} \end{aligned}$$

$$\therefore \rho = \frac{1}{1.25} = 0.8 \text{ gm/c.c.}$$

**Example 13.24.** Find the difference in the levels of mercury in the two limbs of a U-tube, if the diameter of the bore of one limb is 1 mm and of the other 8 mm. The surface tension of mercury is 440 C.G.S. units, its density 13.6 gm/c.c. and the angle of contact with the walls of the tube  $140^\circ$ .

**Soln.**

We have

$$T = \frac{r h \rho g}{2 \cos \theta} ; \text{ or, } h = \frac{2T \cos \theta}{r \rho g}$$

Let the depression of mercury in one limb be  $h_1$  and that in the other  $h_2$ . Then since  $r_1 = 0.5 \text{ mm} = 0.05 \text{ cm}$ ,  $r_2 = 4 \text{ mm} = 0.4 \text{ cm}$ ,  $T = 440 \text{ dynes/cm}$ , and  $\cos \theta = \cos 140^\circ = -0.7660$ , we have

$$h_1 = \frac{2T \cos \theta}{r_1 \rho g} = \frac{2 \times 440 \times (-0.7660)}{0.05 \times 13.6 \times 980}$$

$$\text{and } h_2 = \frac{2T \cos \theta}{r_2 \rho g} = \frac{2 \times 440 \times (-0.7660)}{0.04 \times 13.6 \times 980}$$

$$\begin{aligned}
 \therefore h_1 - h_2 &= \frac{2 \times 440 \times 0.7660}{13.6 \times 980} \times \left( \frac{1}{0.05} - \frac{1}{0.4} \right) \\
 &= \frac{2 \times 440 \times 0.7660}{13.6 \times 980} \times \frac{35}{2} \\
 &= 0.8850 \text{ cm.}
 \end{aligned}$$

**Example 13.25.** A liquid of density 1.05 gm/c.c. and angle of contact  $20^\circ$  has a vertical capillary tube of 2 mm diameter dipping into it. If the surface tension of the liquid be 235 dynes/cm, find the rise of the liquid in the capillary tube.

**Soln.**

We have

$$T = \frac{r h \rho g}{2 \cos \theta} ; \text{ or, } h = \frac{2T \cos \theta}{r \rho g}$$

Here,  $r = 1 \text{ mm} = 0.1 \text{ cm}$ ,

$\rho = 1.05 \text{ gm/c.c.}$ ,  $T = 235 \text{ dynes/cm}$

$g = 980 \text{ cm/sec}^2$ ,  $\theta = 20^\circ$

or,  $\cos 20^\circ = 0.9397$

$$\begin{aligned}
 \therefore h &= \frac{2 \times 235 \times 0.9387}{0.1 \times 1.05 \times 980} \\
 &= \frac{441.659}{102.9} = 4.29 \text{ cm.}
 \end{aligned}$$

### EXERCISES

- [1] Explain the term surface tension. Show that the surface tension of a liquid is equal to the mechanical part of its energy.
- [2] Explain surface tension from the viewpoint of molecular structure of matter.



- [3] Define surface energy. Show that the surface energy per unit area is numerically equal to the surface tension per unit length.
- [4] Explain the terms (a) surface tension (b) surface energy and (c) angle of dimensions of surface tension.
- [5] Define surface tension, and show how it can be regarded as involving potential energy.
- [6] Derive an expression for the excess of pressure inside a spherical liquid drop.
- [7] Show that the difference in pressure between the inside and outside of a spherical liquid drop is  $\frac{2T}{r}$  where  $T$  is the surface tension of the liquid and  $r$  is the radius of the drop.
- [8] Find the relation between the radius of (a) a spherical drop, (b) a spherical bubble of a liquid, the surface tension and pressure.
- [9] Find an expression for the excess pressure inside a spherical air bubble of radius  $r$  inside a liquid of surface tension  $T$ .
- [10] Explain the term surface energy. How is it related to surface tension? Deduce an expression for the excess of pressure inside a spherical soap bubble.
- [11] Define surface tension. Show that the excess pressure acting on the curved surface of a curved membrane is given by  $p = T \left( \frac{1}{r_1} + \frac{1}{r_2} \right)$  where  $r_1$  and  $r_2$  are the radii of curvature and  $T$ , the surface tension of the membrane.
- What will be the excess pressure in case of a liquid drop and a soap bubble?
- [12] Show that the excess pressure inside a soap bubble of radius  $r$  over the atmospheric pressure outside it is equal to  $4T/r$ , where  $T$  is the surface tension of soap solution.
- [13] Find an expression for the difference of pressure between the interior and the exterior of a spherical soap-bubble. Give a rough estimate of the energy of the bubble.
- [14] Starting from the expression for the excess pressure across the curved surface of a curved membrane, obtain expressions of the excess pressure inside (i) a spherical liquid drop and (ii) a spherical soap-bubble.

- [15] Distinguish between surface tension and capillarity. Find an expression for the height to which a liquid may rise in a capillary tube. State Jurin's law and briefly describe an experimental method by which you can verify it.
- [16] Describe capillary rise method of determining the surface tension of a liquid. Deduce the necessary formula you would use.
- [17] Define surface tension. Describe Quincke's method of measuring the surface tension and the angle of contact of mercury in the form of large flat drop. Derive the necessary formula.
- [18] Describe with necessary theory the laboratory method of measuring the surface tension of a liquid by the rise of the liquid in the capillary tube.
- [19] Explain why a liquid meniscus is (i) plane (ii) concave or (iii) convex.
- [20] Give the theory and method of determining the surface tension of water by capillary rise.
- [21] What would be the pressure inside a small air bubble of 1mm radius, situated just below the surface of water? Surface tension of water may be taken to be 70 dynes/cm and the atmospheric pressure to be  $1.012 \times 10^6$  dynes/cm<sup>2</sup>. [ $1.0134 \times 10^6$  dynes/cm<sup>2</sup>].
- [22] An air bubble of radius 0.2 mm is situated just below the surface of water. Calculate the gauge pressure (excess of pressure) inside the air bubble. Surface tension of water =  $7.2 \times 10^{-2}$  N/m. [ $720 \text{ N/m}^2$ ].
- [23] Calculate the work done on the film in blowing a soap bubble from a diameter of 4 cm to one of 30 cm, if its surface tension be 45 C.G.S. units [ $2.5 \times 10^5$  ergs].
- [24] The pressure for air in a soap bubble of diameter  $7 \times 10^{-3}$  m is  $8 \times 10^{-3}$  m of water column above the atmospheric pressure. Calculate the surface tension of soap solution. [ $68.61 \times 10^{-3} \text{ N/m}$ ].
- [25] If the excess pressure inside a soap-bubble of radius 1.0 cm is balanced by that due to a column of oil (sp. gr. 0.8) 2 mm high, find the surface tension of the soap bubble. [ $39.24$  dynes/cm].
- [26] A spherical soap bubble of radius 2 cm is formed inside another soap bubble of radius 3 cm. Calculate the radius of a single soap bubble which will have an excess pressure between the inside of the inner bubble and the outside of larger bubble. [ $1.2 \text{ cm}$ ].
- [27] Calculate the amount of energy needed to break a drop of water 2 mm in diameter into  $10^9$  droplets of equal size, taking the surface tension of water as 73 dynes/cm. [ $9160$  ergs].



- [28] Calculate the work done in spraying a spherical drop of mercury of radius  $10^{-3}$  m into a million drops of equal size. Surface tension of mercury =  $550 \times 10^{-3}$  N/m. [ **$1.6844 \times 10^{-4}$  J**].
- [29] Calculate the energy liberated if 1000 drops of water, each of diameter 2 mm coalesce to form one large drop. The surface tension of water is equal to 72 dynes/cm. [**8143 ergs**].
- [30] What amount of energy will be liberated if 1000 droplets of water each  $10^{-8}$  m in diameter coalesce to form one large spherical drop. Surface tension of water =  $72 \times 10^{-3}$  N/m. [ **$2.035 \times 10^{-14}$  J**].
- [31] A capillary tube is dipped in a liquid. The liquid rises to a height of 4 cm above the surrounding liquid. If the angle of contact is zero and the radius of the tube is .0.1 mm, what is the surface tension of the liquid? [**19.63 dynes/cm**].
- [32] What will be the pressure in a spherical cavity within a mass of paraffin oil? The cavity is at a depth of 20 cm below the surface and has a diameter of 0.0026 cm. The specific gravity of the oil is 0.85 and its surface tension is 26 C.G.S. units. The pressure of the air over the oil surface is equal to 76 cm of mercury. [ **$10.7067 \times 10^5$  dynes/cm<sup>2</sup>**].
- [33] What should be the radius of a capillary tube so that a liquid (surface tension 70 dynes/cm) will rise to a height of 8 cm in it? [**0.018 cm**].
- [34] A capillary tube of internal diameter 1 mm and external diameter 5 mm hangs vertically from the arm of a balance, the lower end of the tube being in a liquid of surface tension 40 dynes/cm. Assuming that the liquid wets the tube, what is the change in the apparent weight of the tube due to surface tension?
- [35] A U-tube has two limbs of internal diameters 0.5 cm and 1 cm respectively. Calculate the difference in the water levels in the two limbs. Surface tension of water =  $7.2 \times 10^{-2}$  N/m, and density of water =  $10^3$  kg/m<sup>3</sup>. [ **$2.944 \times 10^{-3}$  m**].
- [36] Water rises to a height of 5.0 cm in a certain capillary tube. In the same tube the level of mercury surface is depressed by 1.54 cm. Compare the surface tensions of water and mercury. sp. gr. mercury = 13.6, angle of contact for water =  $0^\circ$  and angle of contact for mercury =  $130^\circ$ . [**1:6.516**].
- [37] A U-tube, whose ends are open and whose limbs are vertical, contains oil of sp. gravity 0.85 and surface tension 28 dynes/cm. If one limb has a diameter of 2.2 mm and the other a diameter of 0.8 mm, what is the difference in level of the oil in the two limbs? Assume the angle of contact between the oil and the glass to be zero. [**1.07 cm**].



## CHAPTER XIV

### THERMOMETRY

*Introduction – Thermometry – Constant volume hydrogen thermometer – Platinum resistance thermometer – Thermo-couple – Thermo-electric thermometer – Pyrometers – Fery's total radiation pyrometer – Disappearing filament optical pyrometer – Solved problems – Exercises.*

#### 14.1. Introduction

The simplest way to distinguish hot bodies from cold bodies is to use the sense of touch. Bodies can be arranged in order of their hotness by touch, deciding that A is hotter than B, B than C, etc.; or in other words, A is at a higher temperature than B, B at a higher temperature than C, etc. This is referred to as our temperature sense. However, this can be very unreliable. For example, let a person immerse his hands, one in hot water, the other in cold. Let him then put both hands in water of intermediate hotness. The water will appear cooler to the first hand and warmer to the second hand. Thus the judgement of temperature by the sense of touch can be misleading. An objective numerical measure of temperature is needed.

To begin with one should try to understand the meaning of temperature. Let an object which feels cold to the hand be placed in contact with an identical object which feels hot to the hand. If a sufficient length of time be allowed to pass the two objects will give rise to the same sensation of hotness or coldness (temperature). The two objects are said to be in thermal equilibrium with one other which means that the two bodies are in such states that if the two were connected, there would be no net transfer of heat from one body to the other. The logical or operational test for thermal equilibrium is to use a third or a test body, such as a thermometer. If two bodies, say A and B, are in thermal equilibrium with a third body (say the thermometer), then A and B are in thermal equilibrium with each other. This is referred to as the *zeroth law of thermodynamics*.

Thus temperature of a system can be regarded as a property that determines whether or not the body is in thermal equilibrium with the neighbouring systems. If a number of systems are in thermal equilibrium,

this common property of the system can be represented by a single numerical value called the temperature. It means that if two systems are not in thermal equilibrium, then they are at different temperatures.

## 14.2 Thermometry

The branch of heat relating to the measurement of temperature of a body is known as thermometry. An instrument used to measure the temperature of a body is called a thermometer. *Measurement of temperature depends upon some physical property of matter which varies with temperature preferably in a uniform manner.* Amongst these properties are :

(i) *Changes in length, volume or pressure.* Liquid (mercury or alcohol) – in-glass thermometer, constant volume hydrogen thermometer, etc., are examples of this category.

(ii) *Electrical changes such as change in resistance* (platinum resistance thermometer) or *thermo-electric effect* (thermo-couples or thermo-electric thermometer).

(iii) *Changes of state, colour, etc.,* (radiation and optical pyrometers).

The principle, construction and working of the thermometers, listed below, will be discussed in this chapter :

(i) *Constant-volume hydrogen thermometer*

(ii) *Platinum – resistance thermometer*

(iii) *Thermo-electric thermometer or thermo-couple*

(iv) *Radiation pyrometer*

and (v) *Optical pyrometer.*

It may be noted that for calibration of any type of thermometer, it is necessary to record the indication of the thermometer at two fixed points. The two fixed points normally chosen are :

(i) *lower fixed point* – the temperature at which ice melts (melting point of ice).

(ii) *upper fixed point* – the temperature at which pure water boils at normal atmospheric pressure i.e., at a pressure of 760 mm of mercury (boiling point of water).



The difference in temperature between the two fixed points is known as *fundamental interval*.

### 14.3 Constant Volume Hydrogen Thermometer (International Standard Gas Thermometer).

The thermometer was first designed by Harker and Chappius and is known as the international standard thermometer. The thermometer makes use of the principle that when the volume is kept constant, the pressure of a gas increases with the increase in temperature.

The thermometer consists of a cylindrical bulb C of platinum-irridium alloy, 110 cm long by 3.6 cm in diameter and capacity 1 litre. The bulb is mounted horizontally in a heating chamber having a glass window as shown in Fig. 14.1 and is filled with pure dry hydrogen which is at a pressure of 100 cms of mercury at  $0^{\circ}\text{C}$ . It is connected to one limb A of a manometer through a capillary tube 1 metre long and 0.7 mm in diameter.

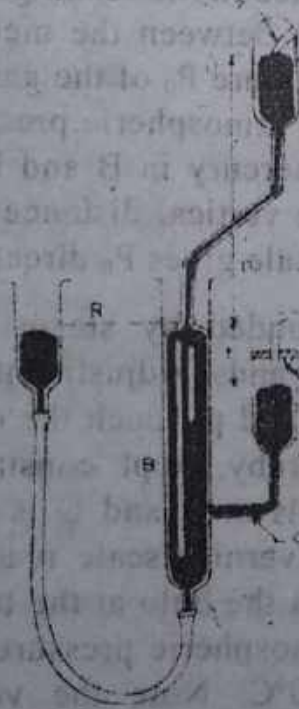


Fig. 14.1



The manometer has two vertical glass limbs A and B connected to each other by a cross-piece and contains pure and dry mercury. The manometer is also connected by a rubber tubing to a reservoir of mercury R whose position can be adjusted. A barometer in the form of a bent glass tube D about a metre long and having a bulb at one end is incorporated in the instrument to measure the pressure of hydrogen in C directly. The open end of D dips under mercury of the uncorked limb B of the manometer and is so bent to be vertically above A. There are two platinum pointers one ( $p_1$ ) inside the corked limb A and the other ( $p_2$ ) inside the bulb of the tube D. A scale S is placed vertically by the side of the tube D such that the zero of the scale coincides with the tip of the pointer  $p_1$  inside the corked limb of the manometer. A sliding vernier V is fixed to the barometer tube D such that its zero coincides with the platinum tip  $p_2$ .

#### working

To use the instrument, the bulb C is surrounded by pounded ice and distilled water. The positions of the reservoir R and the bent tube D are so adjusted that the mercury in the manometer limb touches the tip of the pin  $p_1$  and the mercury in the barometer tube touches the tip  $p_2$ . Let the difference in the readings of the scale (the main scale and the vernier scale) between the mercury surfaces in tubes A and B be  $h_0$ . Then the pressure  $P_0$  of the gas in the bulb C at  $0^\circ\text{C}$  is  $P_0 = P + h_0$  where  $P$  is the atmospheric pressure. This means that the difference in levels of mercury in B and D is equal to the atmospheric pressure  $P$ . Then the vertical distance between the tips of  $p_1$  and  $p_2$  as measured on the scale gives  $P_0$  directly.

The bulb C is next surrounded by steam thus heating up hydrogen which consequently expands. Adjustments of R and D are again made so that the tips of  $p_1$  and  $p_2$  touch the mercury surfaces. The volume of the gas is, thereby, kept constant. The vertical difference between mercury levels in A and D is again noted with the help of the main scale and vernier scale readings. This gives directly the pressure of the gas in the bulb at the temperature of the steam as calculated from the atmospheric pressure. For the sake of simplicity let its value be  $100^\circ\text{C}$ . Note the vertical difference between mercury levels in A and B. If this difference is  $h_{100}$ , then the pressure of hydrogen  $P_{100}$  at  $100^\circ\text{C}$  is given by

$$P_{100} = P + h_{100}$$

Next, the bulb C<sub>1</sub> is surrounded by the bath whose unknown temperature  $t$  is to be measured. The pressure  $P_1$  of hydrogen is again noted after making due adjustments of the positions of R and D so that the tips  $p_1$  and  $p_2$  touch the mercury surfaces. If  $h_1$  is the vertical difference between mercury levels in A and B, then  $P_1 = P + h_1$ .

### calculations

If  $\gamma$  represents the coefficient of increase of pressure at constant volume, then

$$P_{100} = P_0 (1 + \gamma \cdot 100)$$

$$\text{or, } P_{100} - P_0 = P_0 \times 100\gamma \quad (14.1)$$

Similarly

$$P_1 = P_0 (1 + \gamma \cdot t)$$

$$\text{or, } P_1 - P_0 = P_0 \gamma \cdot t \quad (14.2)$$

Dividing (14.2) by (14.1), we get

$$\frac{t}{100} = \frac{P_1 - P_0}{P_{100} - P_0}$$

$$\text{or, } t = \frac{P_1 - P_0}{P_{100} - P_0} \times 100$$

$$= \frac{(P - h_1) - (P + h_0)}{(P - h_{100}) - (P + h_0)} \times 100$$

$$= \frac{h_1 - h_0}{h_{100} - h_0} \times 100 \quad (14.3)$$

The unknown temperature  $t$  can be calculated from eqn. (14.3).

### corrections

To get correct temperature with a constant volume hydrogen thermometer, corrections must be made for the errors mentioned below :



(i) *dead-space correction* : The temperature of the gas in the *dead-space* is not the same as the gas in the bulb. The space inside the capillary tube which connects the bulb with the manometer tube is referred to as the *dead-space*.

(ii) *volume correction* : The bulb C also expands on heating and its volume is different at different temperatures.

(iii) *density correction* : The density of mercury varies with temperature.

(iv) *correction for deviation from perfect gas condition* : The real gases at ordinary temperature and pressure shows a slight deviation from the ideal gas. So the temperature obtained by a gas thermometer requires a correction which, however, is small for ordinary temperatures.

### ***advantages***

(i) A gas thermometer can be used almost down to its liquefaction temperature. On the high temperature side a constant volume hydrogen thermometer can be used upto  $500^{\circ}\text{C}$ . Above this temperature hydrogen diffuses through platinum. Thus a hydrogen gas thermometer has a wide range from  $-200^{\circ}\text{C}$  to  $500^{\circ}\text{C}$ . Nitrogen is used above  $500^{\circ}\text{C}$  upto  $1500^{\circ}\text{C}$ .

(ii) Since expansion of hydrogen gas is considerable, the thermometer is very sensitive. It has an accuracy of  $0.005^{\circ}\text{C}$  between  $0^{\circ}\text{C}$  and  $100^{\circ}\text{C}$  and  $0.1^{\circ}\text{C}$  upto  $500^{\circ}\text{C}$ .

(iii) Pure hydrogen is easy to obtain and the irregularities in the expansion of the bulb have no appreciable effect on the measurements. Hence any two properly constructed thermometers will always agree in their readings. That is why the thermometer is quite suitable as a *standard of reference*.

(iv) The readings of a gas thermometer agree closely with the thermodynamic scale of Lord Kelvin which is the ultimate standard of reference.

### ***disadvantages***

A gas thermometer is very bulky and so cannot be used for ordinary



temperature measurements. Its only purpose is to serve as a standard of reference for more convenient and compact thermometers like mercury thermometers, resistance thermometers, thermo-couples, etc.

**Example 14.1.** The pressures of a gas indicated by a constant volume hydrogen thermometer are 76 cms and 105 cms at  $0^{\circ}\text{C}$  and  $100^{\circ}\text{C}$  respectively. When the bulb is placed in liquid air, the pressure is 25 cms. Calculate the temperature of liquid air.

**Soln.**

$$P = 25 \text{ cm of Hg}$$

$$P_{100} = 105 \text{ cm of Hg}$$

$$P_0 = 76 \text{ cm of Hg.}$$

$$t = \left[ \frac{P_t - P_0}{P_{100} - P_0} \right] \times 100$$

$$= \left[ \frac{25 - 76}{105 - 76} \right]$$

$$= \frac{-51}{29} \times 100 = -175.86^{\circ}\text{C.}$$

**Example 14.2.** The pressure of a gas at  $0^{\circ}\text{C}$  in a constant volume hydrogen thermometer is 65.00 cms of mercury; at  $100^{\circ}\text{C}$  it is 91.60 cms. When it is used to measure the temperature of a liquid, the pressure becomes 108.70 cms. What is the temperature of the liquid?

**Soln.**

$$P_t = 108.70 \text{ cm of Hg}$$

$$P_{100} = 91.60 \text{ cm of Hg}$$

$$P_0 = 65.00 \text{ cm of Hg}$$

$$t = \frac{P_t - P_0}{P_{100} - P_0} \times 100$$

$$= \frac{108.70 - 65.00}{91.60 - 65.00} \times 100$$

$$= \frac{43.7}{26.6} \times 100 = 164.28^\circ\text{C}.$$

**Example 14.3** When a constant volume hydrogen thermometer is used to measure the temperature of a furnace, the excess pressure in the bulb is found to be 152 cm of Hg above the atmospheric pressure. At  $0^\circ\text{C}$  the pressure in the bulb is equal to that of the atmospheric pressure. Calculate the temperature of the furnace, assuming that the atmospheric pressure throughout the experiment remains constant.

**Soln.**

$$P = 76 \text{ cm of Hg}$$

$$P_1 = (76 + 152) \text{ cm of Hg}$$

$$T_0 = 273^\circ\text{K}; T = ?$$

$$\begin{aligned} \frac{P_1}{T} &= \frac{P_0}{T_0}; \text{ or, } T = \frac{P_1}{P_0} \times T_0 \\ &= \frac{228}{76} \times 273 \\ &= 819^\circ\text{K} = 546^\circ\text{C}. \end{aligned}$$

#### 14.4 Platinum resistance thermometer

##### *principle*

The platinum resistance thermometer is based on the principle that the electrical resistance of a metallic wire is found to increase gradually and fairly uniformly with temperature over a wide range. Clausius was the first who gave a relation of the type given by

$$R_t = R_0 (1 + \alpha t) \quad (14.4)$$

where  $R_0$  and  $R_t$  are the resistances of the wire at  $0^\circ\text{C}$  and  $t^\circ\text{C}$  respectively and  $\alpha$  the coefficient of increase of resistance with temperature whose value depends upon the nature and properties of the wire. According to eqn. (14.4), the resistance of a wire should theoretically become zero at absolute zero. The relation is, therefore, not true for all ranges of temperature; it does not represent experimental observations accurately even at ordinary temperature.



Callendar was, however, able to prove that pure platinum wire, free from alloy with silicon, carbon, tin or other impurities and not subjected to strain or rough usage, always possesses the same resistance at the same temperature. The variation of its resistance with temperature could be fairly represented by the relation

$$R_t = R_0 (1 + \alpha t + \beta t^2) \quad (14.5)$$

where  $\alpha$  and  $\beta$  are constants for the material of the wire used. For pure platinum  $\alpha = 3.94 \times 10^{-3}$  and  $\beta = -5.8 \times 10^{-7}$ .

If  $R_t$  and  $R_0$  are known, the value of temperature  $t$  can be obtained by solving eqn. (14.5) which is quadratic in nature. For small ranges of temperature,  $\beta$  is negligibly small, so that the resistance is practically a linear function of the temperature. Eqn. (14.5), therefore, reduces to the more simpler form of Clausius *viz.*,  $R_t = R_0 (1 + \alpha t)$  which represents equal change in resistance for equal change in temperature. If  $R_0$ ,  $R_{100}$  and  $R_t$  are resistances of the platinum wire at  $0^\circ\text{C}$ ,  $100^\circ\text{C}$  and a temperature  $t_p$  on the *platinum resistance scale of temperature*, we may write

$$R_{100} = R_0 (1 + \alpha \cdot 100)$$

$$\text{or, } R_{100} - R_0 = R_0 \cdot \alpha \cdot 100$$

$$\therefore \alpha = \frac{R_{100} - R_0}{100 R_0} \quad (14.6)$$

and

$$R_t = R_0 (1 + \alpha \cdot t_p)$$

$$\text{or, } R_t - R_0 = R_0 \cdot \alpha \cdot t_p$$

$$\text{or, } t_p = \frac{R_t - R_0}{R_0 \cdot \alpha}$$

$$= \frac{R_t - R_0}{R_{100} - R_0} \times 100 \quad (14.7)$$

$R_{100} - R_0$  is known as the *fundamental interval* of the thermometer and is usually adjusted to be *one ohm*.



The difference between the gas scale temperature  $t$  and resistance scale temperature  $t_p$  is given by

$$\begin{aligned}
 t - t_p &= t - \frac{R_t - R_0}{R_{100} - R_0} \times 100 \\
 &= t - \frac{R_0(1 + \alpha t + \beta t^2) - R_0}{R_0[1 + 100t + (100)^2\beta] - R_0} \times 100 \\
 &= t - \frac{\alpha t + \beta t^2}{100\alpha + (100)^2\beta} \times 100 \\
 &= t - \frac{t(\alpha + \beta t)}{\alpha + 100\beta} \\
 &= \frac{1}{\alpha + 100\beta} \times [t(\alpha + 100\beta) - t(\alpha + \beta t)] \\
 &= \frac{1}{\alpha + 100\beta} (100t - t^2) \\
 &= \frac{\beta(100)^2}{\alpha + 100\beta} \cdot \left[ \left( \frac{t}{100} \right)^2 - \frac{t}{100} \right]
 \end{aligned}$$

$$\text{or, } t - t_p = \delta \left[ \left( \frac{t}{100} \right)^2 - \frac{t}{100} \right] \quad (14.8)$$

where  $\delta = -\frac{\beta(100)^2}{\alpha + 100\beta}$  is a constant which depends on the nature of the

wire employed. As  $\beta$  is a negative quantity and  $\alpha$  is a positive quantity,  $\delta$  is, therefore, a positive quantity. For pure platinum the value of  $\delta$  is approximately equal to 1.5. The temperature on the resistance scale as given by eqn. (14.7) can be converted to temperature on the gas scale with the help of eqn. (14.8).

### construction

The first resistance thermometer was constructed by Sir William Siemens in 1871. But the constructional details of his instrument was very unsatisfactory. Later improvements by Callendar and Griffiths have

been able to produce a modern form of platinum resistance thermometer, which is described below.

It consists of a platinum wire ( $R$ ) free from silicon, carbon, tin and other impurities and from strain or chemical alteration. The wire is doubled on itself to avoid the effects of *self induction* and wound on a frame  $M$  consisting of two rectangular pieces of mica intersecting at right angles at their centres along the length (Fig. 14.2). In order that the mica sheets may hold the wire, their edges are notched. The free ends of the platinum leads  $L_1$  and  $L_2$  which are connected to two binding screws  $P, P$  fixed on the ebonite cap  $E$  at the top. As the leads  $L_1$  and  $L_2$  are long and have appreciable resistance which changes with temperature, it is necessary to compensate the effect due to variation of this resistance. For this purpose, an exactly similar pair of leads  $C_1$  and  $C_2$  having the same length and cross-section as  $L_1$  and  $L_2$ , and with their lower ends joined together is placed close to the leads  $L_1$  and  $L__2$ . The free

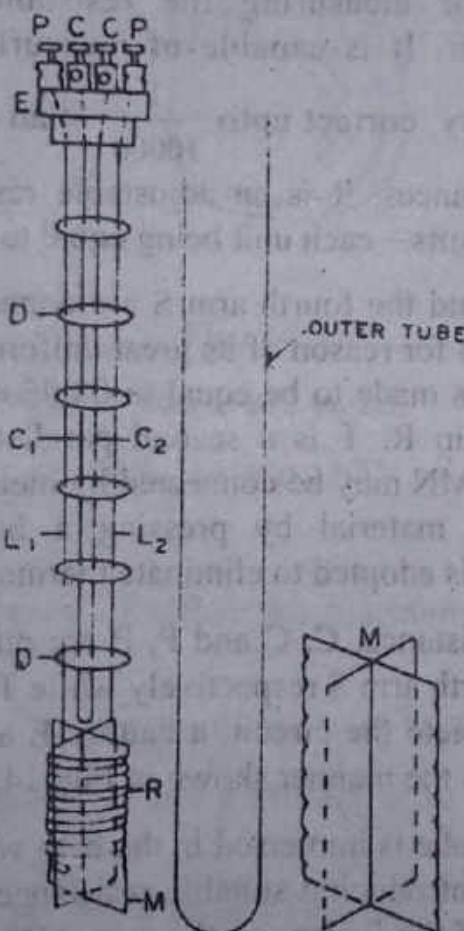


Fig. 14.2



ends of  $C_1$  and  $C_2$  are joined to two other binding screws  $C, C$  on the cap  $E$ . These are called the *compensating leads*. The four wires are passed through mica discs  $DD$  closely fitted along the length of the outer tube. The whole arrangement is enclosed up to  $E$  in a tube made of porcelain or pyrex glass with thin wall. The tube is exhausted of air and hermetically sealed to prevent the oxidation of the wires at high temperatures. For temperatures upto  $700^\circ\text{C}$ , copper leads may be used and the whole may be enclosed in a tube of hard glass. But at higher temperatures platinum leads and a tube of glazed porcelain must be used.

### working

The resistance of the wound platinum coil of the platinum resistance thermometer can be measured with the help of a Callender and Griffith's bridge. It is a modified form of Wheatstone's bridge specially designed for measuring the resistance of a platinum resistance thermometer. It is capable of measuring resistance to a

high degree of accuracy, correct upto  $\frac{1}{10000}$  of an ohm. In Fig. 14.3,  $P$  and  $Q$  are equal resistances.  $R$  is an adjustable resistance, capable of giving 10, 20, 40, etc. units – each unit being equal to 0.01 ohm.

The third arm  $R$  and the fourth arm  $S$  are connected to a stretched bridge wire  $MN$ , chosen for reason of its great uniformity. The resistance per centimetre of  $MN$  is made to be equal to 0.005 ohm *i.e.*, half of the above mentioned unit in  $R$ .  $T$  is a second parallel wire of the same material as  $MN$ .  $T$  and  $MN$  may be connected by means of a short length of wire of the same material by pressing a button jockey. This arrangement of contact is adopted to eliminate thermo-electric forces.

For measuring resistance,  $C, C$  and  $P, P$  are put in series with the third arm  $R$  and the fourth arm  $S$  respectively while  $T$  is connected to the galvanometer. To complete the circuit, a battery  $E$  and a tap key  $K$  are included in the circuit in the manner shown in Fig. 14.3.

The thermometer tube is immersed in the bath whose temperature is to be measured. After introducing suitable resistances in the arm  $R$  and adjusting the position of the jockey on the wire  $MN$ , let a balance point  $D$  be obtained for no deflection of the galvanometer.



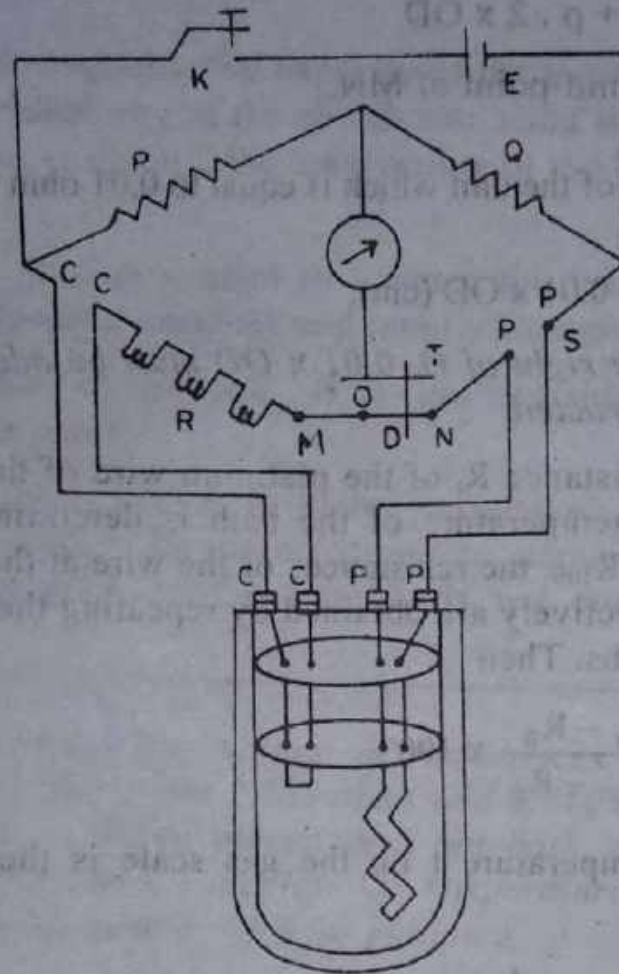


Fig. 14.3

Since the ratio arms P and Q are equal,

$$R + r + \rho \cdot MD = R_1 + r + \rho \cdot ND$$

where

$r$  is the resistance of either the platinum or the compensating leads,

$\rho$  is the resistance per cm of the wire MN

$R_1$  is the resistance of the platinum wire at the temperature of the bath.

$$\text{or, } R_1 = R + \rho (MD - ND)$$

$$= R + \rho (OM + OD - ON + OD)$$

$$= R + \rho \cdot 2 \times OD$$

O being the mid-point of MN.

Since  $\rho = \frac{1}{2}$  of the unit which is equal to 0.01 ohm

$$R_t = R + 0.01 \times OD \text{ (cm).}$$

If D is to the right of O,  $0.01 \times OD$  must be added to R ; if to the left of O, subtracted.

Thus the resistance  $R_t$  of the platinum wire of the thermometer at the unknown temperature of the bath is determined accurately. Similarly  $R_0$  and  $R_{100}$ , the resistances of the wire at the ice-point and steam-point respectively are obtained by repeating the experiment in ice and steam baths. Then

$$t_p = \frac{R_t - R_0}{R_{100} - R_0} \times 100$$

The true temperature  $t$  on the gas scale is then obtained by using the relation

$$t - t_p = \delta \left[ \left( \frac{t}{100} \right)^2 - \frac{t}{100} \right]$$

### advantages

(i) The platinum resistance thermometer can be conveniently used to measure the temperature over a wide range of temperature - from  $-200^\circ\text{C}$  to  $1200^\circ\text{C}$ , as in the case of a gas thermometer.

(ii) It is extremely accurate over this whole range. Its readings are reliable to  $0.01^\circ\text{C}$  upto  $500^\circ\text{C}$  and  $0.1^\circ\text{C}$  upto  $1200^\circ\text{C}$ .

(iii) It has a large size; because of which it can be conveniently used for measuring the average temperature of a large volume.

(iv) Once the platinum resistance thermometer has been standardized by comparison with a gas thermometer, it serves as a reliable secondary standard of reference.

**disadvantages**

(i) Due to large thermal capacity of the thermometer wire and low thermal conductivity of the sheath enclosing it, the thermometer takes a long time to acquire the temperature of the bath in which it is placed.

(ii) Some time is needed to balance the bridge. So it is not suitable for measuring unsteady and rapidly changing temperature.

(iii) Because of its size, it is not suitable for measuring temperature at a point.

(iv) The thermometer is rather expensive as highly pure platinum is necessary for its construction. Impurities in platinum introduce much error as the formula is not correct for impure platinum.

**Example 14.4.** The values of resistances of a platinum resistance thermometer are 2.585 ohms and 3.510 ohms at 0°C and 100°C respectively. When placed in a hot bath, the resistance is found to be 9.098 ohms. Calculate the temperature of the hot bath on the gas scale. Assume  $\delta = 1.5$  for platinum.

**Soln.**

$$R_0 = 2.585, R_{100} = 3.510 \quad R_t = 9.098$$

$$\begin{aligned} \therefore t_p &= \frac{R_t - R_0}{R_{100} - R_0} \times 100 \\ &= \frac{9.098 - 2.585}{3.510 - 2.585} \times 100 = 704^\circ \end{aligned}$$

$$t - t_p = \delta \left[ \left( \frac{t}{100} \right)^2 - \frac{t}{100} \right]$$

$$\text{or, } t - 704 = 1.5 \left[ \left( \frac{t}{100} \right)^2 - \frac{t}{100} \right]$$

Solving, we have  $t = 785^\circ\text{C}$ .



**Alternate solution**

The method of successive approximation may be used for solving the above equation which may be written as

$$t - t_p = \delta \left\{ \frac{t}{100} \left[ \frac{t}{100} - 1 \right] \right\}$$

As a first approximation, let  $t = t_p$  on the right hand side of the above equation.

Then

$$\begin{aligned} t - t_p &= 1.5 \left[ \frac{704}{100} \left( \frac{704}{100} - 1 \right) \right] \\ &= 1.5 \times 7.04 \times 6.04 = 63.8 \end{aligned}$$

$$\therefore t = t_p + 63.8 = 704 + 63.8 = 767.8^\circ\text{C} \approx 768^\circ\text{C}.$$

For the second approximation, let  $t = 768$  on the right hand side.

$$\begin{aligned} \therefore t - t_p &= 1.5 \left\{ \frac{768}{100} \left( \frac{768}{100} - 1 \right) \right\} \\ &= 1.5 \times 7.68 \times 6.68 = 77^\circ \end{aligned}$$

$$\begin{aligned} \therefore t &= t_p + 77 = 704 + 77 \\ &= 781^\circ\text{C}. \end{aligned}$$

For the third approximation, let  $t = 781$  on the right hand side.

$$\begin{aligned} \therefore t - t_p &= 1.5 \left\{ \frac{781}{100} \left( \frac{781}{100} - 1 \right) \right\} \\ &= 1.5 \times 7.81 \times 6.81 = 80 \end{aligned}$$

$$\therefore t = t_p + 80 = 704 + 80 = 784^\circ\text{C}.$$

For a fourth approximation,  $t = 784$

$$\begin{aligned} \therefore t - t_p &= 1.5 \left\{ \frac{784}{100} \left( \frac{784}{100} - 1 \right) \right\} \\ &= 1.5 \times 7.84 \times 6.84 = 80.44 \end{aligned}$$

$$\therefore t = 704 + 80.44 = 784.44^\circ\text{C}.$$

As will be observed, further approximations after the third approximation will not alter the values much.

Hence  $t = 784^\circ\text{C}$  (approx.)

**Example 14.5.** The resistances of a platinum resistance thermometer are 2.56, 3.56 and 6.78 ohms at  $0^\circ\text{C}$  and the boiling point of sulphur on the gas scale ( $444.6^\circ\text{C}$ ). The thermometer when placed in an unknown temperature bath is found to have a resistance of 5.23 ohms. Calculate the temperature of the bath on the gas scale.

**Soln.**

$$R_0 = 2.56, R_{100} = 3.56, R_t = 6.78, \text{ when } t = 444.6$$

$$\begin{aligned} t_p &= \frac{R_t - R_0}{R_{100} - R_0} \times 100 \\ &= \frac{6.78 - 2.56}{3.56 - 2.56} \times 100 = 422 \end{aligned}$$

$$\text{From, } t - t_p = \delta \left\{ \frac{t}{100} \left( \frac{t}{100} - 1 \right) \right\}$$

we have

$$444.6 - 422 = \delta \left\{ \frac{444.6}{100} \left( \frac{444.6}{100} - 1 \right) \right\}$$

$$\text{or, } 22.6 = \delta \times 4.446 \times 3.446$$

$$\text{or, } \delta \simeq 1.48$$

For the unknown temperature bath,

$$R_t = 5.23$$

$$t_p = \frac{5.23 - 2.56}{3.56 - 2.56} \times 100 = 267$$

Putting  $t = t_p$  on the right hand side as a first approximation, we have

$$t - t_p = \delta \left\{ \frac{t}{100} \left( \frac{t}{100} - 1 \right) \right\}$$

$$= 1.48 \left\{ \frac{267}{100} \left( \frac{267}{100} - 1 \right) \right\}$$

$$= 1.48 \times 2.67 \times 1.67 = 6.6$$

$$\therefore t = 267 + 6.6 = 273.6$$

For the second approximation, let  $t \approx 274$

$$t - t_p = 1.48 \left\{ \frac{274}{100} \left( \frac{274}{100} - 1 \right) \right\}$$

$$= 1.48 \times 2.74 \times 1.74 = 7.056$$

$$\therefore t = t_p + 7.056 = 267 + 7.056$$

$$= 274.056$$

Hence the temperature of the unknown temperature bath.

$$t = 274^\circ\text{C (approx.)}$$

**Example 14.6.** The resistance  $R_t$  of a platinum wire at temperature  $t^\circ\text{C}$ , measured on the gas scale, is given by  $R_t = R_0 (1 + \alpha t + \beta t^2)$  where  $\alpha = 3.8 \times 10^{-3}$  and  $\beta = -5.6 \times 10^{-7}$ . What will be the temperature on the platinum scale when the temperature on the gas scale is  $200^\circ\text{C}$ ?

**Soln.**

$$R_{100} = R_0 (1 + \alpha t + \beta t^2)$$

$$= R_0 [1 + 100 \cdot \alpha + (100)^2 \cdot \beta]$$

$$= R_0 [1 + 3.8 \times 10^{-3} \times 100 + (100)^2 \cdot (-5.6 \times 10^{-7})]$$

$$= 1.3744 R_0$$

$$R_{200} = R_0 [1 + 200 \times 3.8 \times 10^{-3} + (200)^2 \cdot (-5.6 \times 10^{-7})]$$

$$= 1.7376 R_0$$



$$\begin{aligned}
 \therefore t_p &= \frac{R_{200} - R_0}{R_{100} - R_0} \times 100 \\
 &= \frac{1.7376 R_0 - R_0}{1.3744 R_0 - R_0} \times 100 \\
 &= \frac{(1.7376 - 1.0) R_0}{(1.3744 - 1.0) R_0} \times 100 \\
 &= \frac{0.7376}{0.3744} \times 100 = 197^\circ.
 \end{aligned}$$

**Example 14.7.** If the platinum temperature is  $50.25^\circ$  when the temperature on the gas scale is  $50^\circ\text{C}$ , what will be the temperature on the platinum scale corresponding to  $150^\circ\text{C}$  on the gas scale?

**Soln.**

Let  $t_p$  be the temperature on the platinum scale and  $t$  on the gas scale.

Then

$$t - t_p = \delta \left[ \frac{t}{100} \left( \frac{t}{100} - 1 \right) \right]$$

$$\text{or, } 50 - 50.25 = \delta \left[ \frac{50}{100} \left( \frac{50}{100} - 1 \right) \right] \quad (\text{i})$$

Again let  $t_p = x$  when  $t = 150^\circ$

Then

$$\begin{aligned}
 150 - x &= \delta \left[ \frac{t}{100} \left( \frac{t}{100} - 1 \right) \right] \\
 &= \delta \left[ \frac{150}{100} \left( \frac{150}{100} - 1 \right) \right] \quad (\text{ii})
 \end{aligned}$$

Dividing (ii) by (i) and simplifying

$$x = t_p = 149.25^\circ\text{C}.$$

**Example 14.8.** The resistances of a platinum resistance thermometer are 5.5, 7.5 and 14.5 ohms at  $0^\circ\text{C}$ ,  $100^\circ\text{C}$  and  $444.6^\circ\text{C}$  (boiling point of sulphur) respectively. The resistance of a platinum wire at a temperature  $t^\circ\text{C}$  is given by the equation  $R_t = R_0 (1 + \alpha t + \beta t^2)$ . Find the value of  $\alpha$  and  $\beta$ .

**Soln.**

$$R_0 = 5.5, R_{100} = 7.5, R_{444.6} = 14.5$$

$$R_{100} = R_0 [1 + \alpha (100) + \beta (100)^2] \quad (i)$$

$$R_{444.6} = R_0 [1 + \alpha (444.6) + \beta (444.6)^2] \quad (ii)$$

Substituting the values in equations (i) and (ii)

$$7.5 = 5.5 [1 + 100\alpha + (100)^2\beta] \quad (iii)$$

$$14.5 = 5.5 [1 + 444.6\alpha + \beta (444.6)^2] \quad (iv)$$

Simplifying equations (iii) and (iv)

$$\alpha + 100\beta = \frac{2}{5.5 \times 100} \quad (v)$$

$$\alpha + 444.6\beta = \frac{9}{5.5 \times 444.6} \quad (vi)$$

Subtracting equation (v) from (vi)

$$344.6\beta = \frac{900 - 2 \times 444.6}{5.5 \times 444.6 \times 100}$$

$$\text{or, } \beta = 1.281 \times 10^{-7}/^\circ\text{C}.$$

Substituting the value of  $\beta$  in eqn. (v)

$$\alpha + 100 (1.281 \times 10^{-7}) = \frac{2}{5.5 \times 100}$$

$$\alpha = 363.6 \times 10^{-5} - 1.281 \times 10^{-5}$$

$$= 362.319 \times 10^{-5}$$

$$= 3.62 \times 10^{-3}/^\circ\text{C}.$$

**Example 14.9.** The resistances of a platinum resistance thermometer are 5 and 5.93 ohms at 0°C and 100°C respectively. The pressures indicated by a gas in a constant volume gas thermometer are 100 cm and 136.6 cm of Hg at 0°C and 100°C respectively. When both the thermometers are inserted in an unknown temperature bath, the resistance of the platinum resistance thermometer is 5.795 ohms and the pressure indicated by the gas is 131.11 cm of Hg. Calculate the temperature of the bath (i) on the platinum scale and (ii) on the gas scale.

**Soln.**

(i) On the platinum scale

$$R_0 = 5, R_{100} = 5.93, R_t = 5.795$$

$$\begin{aligned} t_p &= \frac{R_t - R_0}{R_{100} - R_0} \times 100 \\ &= \frac{5.795 - 5.0}{5.93 - 5.0} \times 100 = 85.48^\circ \end{aligned}$$

(ii) On the gas scale

$$P_0 = 100 \text{ cm of Hg}$$

$$P_{100} = 136.6 \text{ cm of Hg}$$

$$P_t = 131.11 \text{ cm of Hg}$$

$$\begin{aligned} t &= \frac{P_t - P_0}{P_{100} - P_0} \times 100 \\ &= \frac{131.11 - 100}{136.6 - 100} \times 100 = 85^\circ \text{C.} \end{aligned}$$

## 14.5 Thermo-couple

If two dissimilar metals are joined at two ends and if one of the junction is kept at a higher temperature than the other, i.e., there is a temperature difference between the two junctions, an electromotive force (e.m.f.) is set up. If the circuit is closed, this e.m.f. will cause a current to flow in the circuit as indicated by a galvanometer. This



phenomenon was discovered by Seebeck, a Berlin physicist, and is known as *Seebeck effect*. The two dissimilar metals joined at the two ends constitute what is known as a *thermo-couple* and the electromotive force developed is referred to as *thermoelectromotive force*. The direction and magnitude of the current depend on the nature of the material constituting the thermo-couple and the temperature difference between the two junctions. Seebeck arranged different metals in a particular order such that if a thermo-couple be made of any two of this series, known as *Seebeck series*, then with the junctions at different temperatures, a current flows through the *cold* junction from the metal coming earlier in the series to one coming later. Seebeck's original series has now been extended and is known as *thermo-electric series*. This is as follows :

Antimony (*Sb*), *Fe*, *Cd*, *Zn*, *Ag*, *Au*, *Rb*, *Mo*, *Cr*, *Sn*, *Pb*, *Hg*, *Cu*, *Pt*, *Co*, *Ni*, Bismuth (*Bi*).

It is useful to remember that current flows from A (*Antimony*) to B (*Bismuth*) through C (the *cold junction*). For a copper-iron thermo-couple it is useful to remember the word *chi* indicating that current flows from *copper* to *iron* across the *hot* junction.

Metals to the right of *Pb* (lead) are called *thermo-electrically positive*, while those to the left are called *thermo-electrically negative*. The *more* any two metals are separated in the series the *greater* will be the thermo-electromotive force developed in the couple made of these metals. A rough idea of the amount of *e.m.f.* produced for different thermo-couples is given below. Lead is taken as one of the metals to form a thermo-junction with other metal, and the temperature of the cold junction is kept at  $0^{\circ}\text{C}$ , and that of the hot junction at  $100^{\circ}\text{C}$ .

Sb-Pb	$31\ \mu\text{v}/^{\circ}\text{C}$	Pb-Cu	$-3\ \mu\text{v}/^{\circ}\text{C}$
Fe-Pb	$14\ \mu\text{v}/^{\circ}\text{C}$	Pb-Ni	$-18\ \mu\text{v}/^{\circ}\text{C}$
Pb-Rhodium	$3\ \mu\text{v}/^{\circ}\text{C}$	Pb-Constantan	$-39\ \mu\text{v}/^{\circ}\text{C}$
Pb-Pt	$4\ \mu\text{v}/^{\circ}\text{C}$	Pb-Bi	$-69\ \mu\text{v}/^{\circ}\text{C}$

Thus the *e.m.f.* developed for a Sb-Bi couple is  $31 - (-69) = 100$  microvolts per degree celcius difference of temperature between the junctions at  $0^{\circ}\text{C}$  and  $100^{\circ}\text{C}$ .

### neutral temperature and temperature of inversion of a thermocouple

Keeping the cold junction at  $0^{\circ}\text{C}$ , as the temperature of the hot junction is gradually increased, the *e.m.f.* developed at first goes on increasing, reaches a maximum and then begins to decrease until at a certain temperature the *e.m.f.* is reversed and goes on increasing in the reversed direction without showing any sign of again coming to a maximum.

If the *e.m.f.* is plotted against the temperature of the hot junction, a curve OABD which is approximately parabolic in shape is obtained as shown in Fig. 14.4. The temperature C at which *e.m.f.* is maximum is called the *neutral temperature*, and is a constant for the pair of metals. The temperature B beyond which the *e.m.f.* is reversed is known as the *temperature (of point) of inversion* for the particular couple. If the cold junction be at any temperature higher

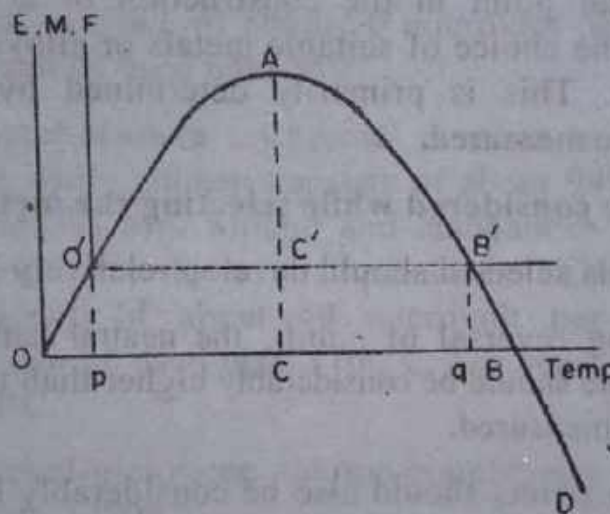


Fig. 14.4

then  $0^{\circ}\text{C}$ , the origin O is shifted to O' corresponding to the temperature Op of the cold junction. The curve is now O'AB'D, so that the neutral temperature remains unchanged but the temperature of reversal B' advances towards the neutral temperature by an amount Bq equal to Op, the temperature of the cold junction. Thus while the temperature at which maximum *e.m.f.* is produced, is constant for the pair, the temperature at which reversal takes place is



variable, being as much above the neutral temperature as the cold junction is below it.

#### 14.6 Thermo-electric thermometer

A thermo-electric thermometer essentially consists of the following elements :

- (i) the two dissimilar metals constituting the thermo-couple
- (ii) the electrical insulation of the metallic wires constituting the thermo-couple
- (iii) a outer protecting tube for the thermo-couple
- (iv) a potentiometer or an indicator for measuring the thermo-*e.m.f.* developed in the thermo-couple.

##### *choice of suitable metals for the thermo-couple*

The essential point in the construction of a thermo-electric thermometer is the choice of suitable metals or alloys, known as the *thermo-elements*. This is primarily determined by the range of temperature to be measured.

**The points to be considered while selecting the metals are :**

- (i) The materials selected should develop relatively large *e.m.f.*
- (ii) For avoiding reversal of *e.m.f.*, the neutral temperature of the thermo-couple should be considerably higher than the temperature range to be measured.
- (iii) The melting points should also be considerably higher than the temperature range for which they are to be used.
- (iv) They should also be *resistant to such actions as oxidation, corrosion, etc.*, which may destroy the metals.

For ordinary laboratory or industrial uses, base metals and base metal alloys are used for the construction of thermo-couples. At higher temperatures noble metals must be used. Listed below are some of these thermo-couples along with the ranges of temperatures over which they could be used and the *e.m.f.* developed per degree temperature difference between the junctions.



1. **Copper-constantan** : Constantan is an alloy of 60% Cu and 40% Ni. This thermo-couple, which can be used within the range  $-200^{\circ}\text{C}$  to  $400^{\circ}\text{C}$ , develops a large *e.m.f.* of about 45 microvolt per  $^{\circ}\text{C}$  difference in temperature between the junctions. Its *temperature vs e.m.f.* curve is practically linear and can be used both in the laboratory and for ordinary industrial work.

2. **Iron-constantan** : This thermo-couple develops an *e.m.f.* of about 55 microvolt per  $^{\circ}\text{C}$  temperature difference and can be used in the range of  $-200^{\circ}\text{C}$  to  $800^{\circ}\text{C}$ . The *temperature vs e.m.f.* curve is also practically linear for this thermo-couple.

3. **Iron-nickel** : This thermo-couple may be used between  $400^{\circ}\text{C}$  to  $800^{\circ}\text{C}$ , over which its *temperature vs e.m.f.* curve is practically linear. It develops an *e.m.f.* of about 30 microvolt per  $^{\circ}\text{C}$  temperature difference.

4. **Chromel-copel** : Chromel consists of 11% Cr, 25% Fe and 64% Ni while copel consists of 45% Ni and 55% Cu. This thermo-couple develops an *e.m.f.* of about 70 microvolt per  $^{\circ}\text{C}$  temperature difference and can be used upto  $550^{\circ}\text{C}$ .

5. **Chromel-alumen** : Chromel consists of 90% nickel and 10% chromium while alumen consists of about 94% Ni and 2% Al with 1% Si and 3% Mn. Silicon and manganese are necessary to reduce brittleness at lower temperatures. The thermo-couple develops an *e.m.f.* of about 40 microvolt per  $^{\circ}\text{C}$  temperature difference and can be used upto  $1100^{\circ}\text{C}$ . For short periods it can be used upto  $1300^{\circ}\text{C}$ .

6. A **nickel-nickrome** thermo-couple can be used in the range of  $600^{\circ}\text{C}$  to  $1200^{\circ}\text{C}$ .

At higher temperatures, base metals quickly get oxidized and can not, therefore, be used. In that case thermo-couples made with noble metals and their alloys are used.

**Platinum and an alloy of platinum with rhodium** (90% Pt and 10% Rh) is used upto  $1600^{\circ}\text{C}$ , the lower limit being  $600^{\circ}\text{C}$ . Thermo-couples of **iridium and an alloy of iridium with rubeidum** (90% Ir and 10% Ru) can be used in the range of  $1600^{\circ}\text{C}$  to  $2100^{\circ}\text{C}$ .

**Tungsten-molybdenum** thermo-couples can be used between  $2000^{\circ}\text{C}$  to  $3000^{\circ}\text{C}$ . *E.m.f.s.* developed in these thermo-couples are

rather small, of the order of 5mV per degree temperature difference. But the temperature difference involved in measurement with these thermo-couples are so large that the overall *e.m.fs.* developed are of sufficient magnitude to make it possible to employ these thermo-couples for scientific work.

### *description of a thermo-electric thermometer*

The two wires chosen as the elements of the thermo-couple are welded together electrically to form the hot junction A (Fig. 14.5). Near the hot junction, the two wires are insulated from each other by passing them through the capillary tube C of hard glass. To keep the

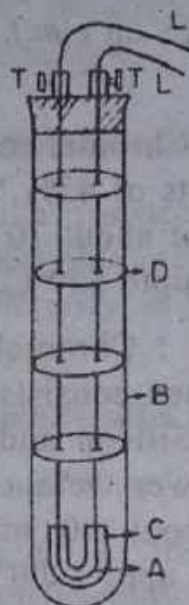


Fig. 14.5

wires in position, they are again passed through several mica discs D closely fitted in the outer protecting tube B. The two ends of the wire are attached to the terminals T, T fixed on the cap closing the mouth of the tube. To these terminals are connected long, flexible leads, usually of the same materials as the elements, which form the cold junction immersed in ice at  $0^{\circ}\text{C}$  at a fairly distant place.

Upto  $600^{\circ}\text{C}$  pyrex glass can be used both as capillary tube C and protecting tube B. Above this temperature, glazed porcelain must be used for these purposes.



**measurement of *e.m.f.***

Development of *e.m.f.* means the flow of the current in the circuit when it is closed. However, it is preferable to measure the *e.m.f.* instead of the circuit current because *the e.m.f. developed depends upon the temperature difference only* whereas the current depends on the resistance of the circuit which changes considerably with temperature. For accurate scientific work, where temperature should be measured with the greatest possible precision, a sensitive potentiometer capable of reading in microvolts is used.

Fig. 14.6 shows the arrangement for the measurement of *e.m.f.* with the help of a potentiometer. AB is a wire of uniform cross-section of resistance  $\rho$  ohm/cm and a total length of usually 10 metres. The wire is connected in series with a resistance box  $R$  and a battery of *e.m.f.*  $E$ .

The temperature  $T_1$  of the cold junction is maintained at  $0^\circ\text{C}$  by placing it in melting ice. The temperature  $T_2$  of the hot junction is gradually increased by placing the junction in a suitable bath. For any temperature of the hot junction, let the *e.m.f.* developed in the thermo-couple be balanced against the potential drop over a length AC of the potentiometer wire. Since the wire is of uniform cross-section, the fall of potential across any length of it is proportional to the length.

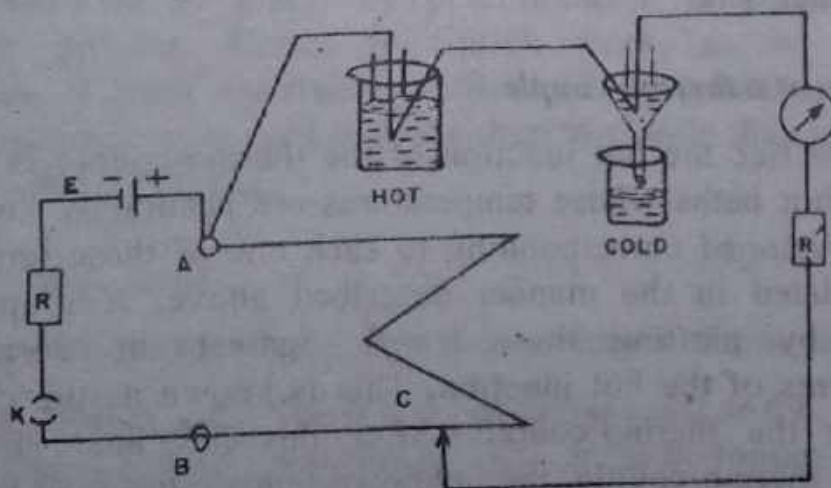


Fig. 14.6



If  $r$  be the resistance of the total length of the wire, then the total resistance in series with the battery is  $R+r$ ,  $R$  being the value of the resistance as used in the resistance box. Therefore, the steady current flowing through the circuit is

$$i = \frac{E}{R+r}$$

Hence the potential drop over the entire length  $AB$  of the wire is  $i.r = \frac{E}{(R+r)} \cdot r$ . If the total length of the potentiometer wire be 1000 cms, then the potential drop per cm of the potentiometer wire is

$$\frac{Er}{(R+r) \times 1000}$$

If the length of  $AC$  is equal to  $l$  cms, then the *e.m.f.* developed in the couple is

$$\frac{Er}{(R+r) \times 1000} \times l$$

Normally  $E$  and  $r$  are both fixed. Therefore,  $R$  is chosen in such a way that the potential drop per cm should be nearly 5 to 10 microvolts. Then the *e.m.f.* developed for any temperature of the hot junction can be calculated by balancing the *e.m.f.* against any given length of the potentiometer wire.

### *calibration of a thermo-couple*

In practice the hot junction of the thermo-couple is placed in different hot baths whose temperatures are accurately known. The *e.m.fs.* developed corresponding to each one of these temperatures are calculated in the manner described above. A graph is then obtained by plotting these *e.m.fs.* against the corresponding temperatures of the hot junction. This is known as the *calibration curve* for the thermo-couple. After this has been done for a particular thermo-couple, the unknown temperature can be read off corresponding to the *e.m.f.* at that temperature.

If the cold junction is kept at  $0^\circ\text{C}$ , then the relationship between the *e.m.f.*  $E$ , developed and the corresponding temperature  $\theta$  of the

hot junction is approximately given by the relation

$$E = a\theta + b\theta^2$$

where  $a$  and  $b$  are constants having different values for different thermocouples. In order to evaluate the constants  $a$  and  $b$ , the junction is successively placed at two known fixed temperatures and the corresponding *e.m.f.s.* are measured. Substituting these values in the above equation, two equations are obtained from which the values of the constants  $a$  and  $b$  may be determined. Then by measuring the *e.m.f.* at any unknown temperature, the temperature itself can be calculated from the known values of  $a$ ,  $b$  and  $E$ .

However, the nature of variation of *e.m.f.* with temperature is different at different ranges of temperature. This must be taken into consideration while calibrating a thermo-couple. Thus for a  $Pt - Pt + Rh$  thermo-couple, there are three different relations at three different ranges of temperature as shown below :

From  $0^\circ\text{C} - 400^\circ\text{C}$  ;  $E = a\theta + b(1 - e^{c\theta})$

From  $300^\circ\text{C} - 1200^\circ\text{C}$  ;  $E = a'\theta + b'\theta + C'\theta^2$

From  $1100^\circ\text{C} - 1750^\circ\text{C}$  ;  $E = -a'' + b''\theta + C''\theta^2$

where even the constants have different values at different temperatures.

Measurement of *e.m.f.* by potentiometer is rather a time-consuming process. Hence for quick work, as in industrial applications, a high resistance millivoltmeter, graduated to read temperature directly, is used with the thermo-couple thus making the instrument direct reading.

### **chief sources of error**

The chief sources of error in a thermoelectric thermometer are

(i) *Cold junction error if it is not maintained at  $0^\circ\text{C}$*  : Suitable means for keeping the cold junction at  $0^\circ\text{C}$  must be found; otherwise a correction has to be applied.

(ii) *Thermo-elements must be protected against contamination by the use of suitable sheaths of metal or refractory material like porcelain.*



(iii) *Parasitic e.m.fs. developed in the circuit due to Peltier and Thomson effects.* These are eliminated by the use of materials free from such effects.

(iv) *The moving-coil millivoltmeter used should be of high resistance so that the increase of resistance of the leads or thermo-couple does not affect the total circuit resistance appreciably.* Millivolt-meters of 500 ohm resistance giving a full-scale deflection for 40 mV give an accuracy of  $\pm 5^\circ\text{C}$ . For higher accuracy, it is necessary to use a suitable potentiometer.

### **advantages**

(i) Thermo-couples are rugged, simple and sensitive. They are also cheap and can be easily constructed, inspected and maintained.

(ii) Since the two junctions can be physically at a considerable distance from each other, they are suitable for measuring temperature from a distance.

(iii) Due to the low thermal capacity of the hot junction and hence less time lag, thermo-couples are quite suitable for measuring rapidly varying temperatures.

(iv) Thermo-couples have a wide range. With suitable choice of thermo-elements, thermo-couples can be used to measure temperature from  $-200^\circ\text{C}$  to  $3000^\circ\text{C}$ .

### **disadvantages**

(i) The chief disadvantage lies in the fact that there is no satisfactory theoretical expression relating  $E$  and  $\theta$  for the whole range of temperature. Thus every thermo-couple requires separate calibration curves for different ranges of temperature thus rendering it only a subsidiary instrument.

(ii) Observations obtained with these thermo-couples are vitiated to a great extent by *spurious e.m.fs. due to Thomson and other effects.*

(iii) The existence of a neutral temperature also limits the ranges of utility.

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**Example 14.10.** A copper constantan-couple having a mean thermo-electric power of  $40\mu\text{V}/^\circ\text{C}$  is used to measure an unknown



temperature. The potentiometer wire is 1000 cms long having a resistance 0.01 ohm per cm and is used in series with a cell of e.m.f. 2 volts and a resistance of 2000 ohms. The balance point is obtained with a length of 402 cms of the wire between the thermo-couple terminals. Calculate the value of the unknown temperature. The cold junction is at  $0^{\circ}\text{C}$ .

**Soln.**

The potential drop per cm of the wire is

$$= \frac{E_r}{(R + r) \times 1000}$$

$E = \text{e.m.f. of the cell} = 2 \text{ volt}$

$R = \text{resistance in series} = 2000 \text{ ohms}$

$r = \text{resistance of the potentiometer wire}$

$$= 1000 \times 0.01 = 10 \text{ ohm}$$

Hence the potential drop in 402 cm. of the wire

$$= \frac{2 \times 10 \times 402}{(2000 + 10) \times 1000} = 4 \times 10^{-3} \text{ volts}$$

$$= 4 \times 10^{-3} \times 10^6 \mu\text{V} = 4 \times 10^3 \mu\text{V}$$

The temperature difference between the two junctions

$$= \frac{4 \times 10^3 \mu\text{V}}{40 \mu\text{V}/^{\circ}\text{C}} = 100^{\circ}\text{C}.$$

Hence the unknown temperature is

$$(0 + 100)^{\circ}\text{C} = 100^{\circ}\text{C}.$$

## 14.7 Pyrometers

Thermometers which are used for measuring temperatures above  $500^{\circ}\text{C}$  i.e., above the range of mercury thermometers, are referred to as *pyrometers*. According to this definition, platinum resistance thermometer and thermo-electric thermometer are also pyrometers and should be known as resistance pyrometers and thermo-electric pyrometers.

Two types of pyrometers which measure high temperature by the method of radiation will now be discussed. In one type, the total amount of heat radiations emitted by the hot source are measured and the temperature deduced therefrom by the application of Stefan's law. This type of pyrometer is known as *total radiation pyrometer*. Fery's total radiation pyrometer is an example. In the other type, the intensity of heat radiation from the hot body lying within a certain small range of wavelength  $d\lambda$  is compared with the intensity of emission within the same range of wavelength from a standard lamp. This type of pyrometer is known as *optical pyrometers*. *One of the most important and distinguishing features of the radiation and optical pyrometers is that unlike other thermometers, they have not to be placed in touch with the hot body.*

#### 14.8 Fery's Total Radiation Pyrometer

The instrument, first devised by Fery, is based on Stefan's law. It is merely a thermo-couple so arranged that it is robust, quick in response and is capable of giving direct reading. The instrument is used to measure the total energy of radiation of all wavelength and has arrangement which renders the reading independent of the distance between the pyrometer and the hot body within certain limits.

A sketch of Fery's total radiation pyrometer is shown in Fig. 14.7. It consists of a concave mirror C made of copper and nickel plated on the front side. The mirror has a small opening in the centre. The radiations from the hot body, whose temperature is to be measured, falls on the gold-plated mirror C after passing through the aperture AA. The mirror is pierced at the centre to allow light to reach the eyepiece, E. The position of the mirror can be adjusted with the help of a rack and pinion arrangement R so that the radiation, after reflection, is focussed on the hot junction of a thermo-couple which is attached at the back surface of a metal strip S. The front surface of S, facing the mirror, is painted black (so as to collect all the radiations that fall on it). There are two small semi-circular mirrors MM in front of S inclined to one another at an angle of  $5^\circ$  to  $10^\circ$ . There is an opening of about 1.5 mm at the centre of the mirrors and immediately in front of S. This acts as a limiting diaphragm and serves to focus the radiation on S accurately. The junction S and the inclined mirrors are enclosed inside a box Q so



that radiation is not incident directly on S. The cold junction is protected from radiations and is connected to a millivoltmeter which

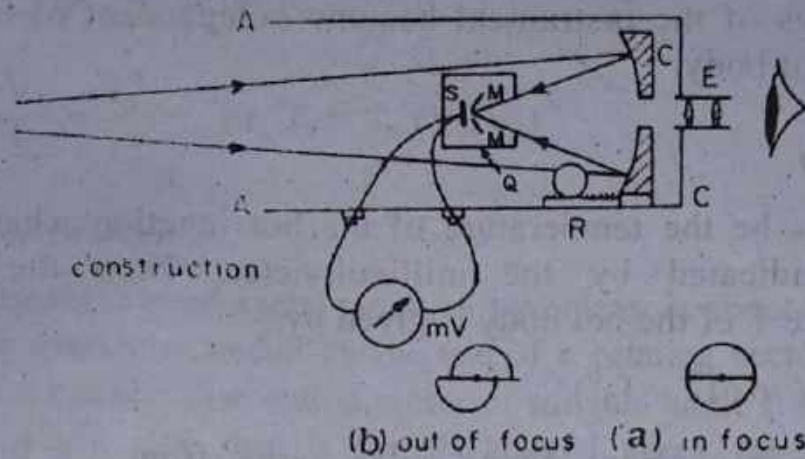


Fig. 14.7

measures the *e.m.f.* developed in the couple. When the image of the hot body, say a hole in a furnace, is focussed with the help of C on the junction S, the millivoltmeter shows a deflection. The image can be observed through an eye-piece E fitted in the opening of the concave mirror. How accurately the image has been focussed can be judged with the help of the two inclined mirrors MM. When focussing is correct, the images in the two mirrors would overlap exactly to form a complete circle [Fig. 14.7 (a)]. If the focussing is not correct, the two halves appear displaced as shown in Fig. 14.7(b).

The heat image formed by the mirror C must completely cover the aperture in the diaphragm. This is important because so long as the image is large than the aperture in the diaphragm, it is the *intensity* of the heat image and not the *total heat* reflected that is measured by the pyrometer. It can be shown that this intensity is independent of the distance of the pyrometer from the hot body. For example, if the distance between the hot body and the pyrometer is doubled, the amount of radiation received by the mirror is reduced to one-fourth. But at the same time the area which the image covers is also reduced to one-fourth. Thus the actual heat intensity of the image remains the same. For this to happen, *i.e.*, for the image to be larger than the aperture in the diaphragm, a relation can be found between the size of the object (a hole in the furnace) and the



distance of the pyrometer. The size of the object as calculated from the geometry of the pyrometer should be at least 40 mm in diameter for every 1 metre of distance between the pyrometer and the object. The pyrometer is then adjusted accordingly and after this adjustment the readings of the instrument become independent of its distance from the hot body.

### calibration

Let  $T_0$  be the temperature of the hot junction when  $V$  is the reading indicated by the millivoltmeter. Then the unknown temperature  $T$  of the hot body is given by

$$V \propto (T^b - T_0^b)$$

where  $b$  is a constant whose value varies from 3.8 to 4.2. This deviation from Stefan's fourth power law (the rate of emission  $R$  of radiant energy by unit area of a perfectly black body is directly proportional to the fourth power of its absolute temperature  $T$  i.e.,  $R = \sigma T^4$  where  $\sigma$  is called Stefan's constant), is due to the conduction along the wires of the thermo-couple and stray *e.m.f.* generated. Moreover, because  $T$  is very much large than  $T_0$ , the millivoltmeter readings are not strictly proportional to  $(T^4 - T_0^4)$ . The instrument is, therefore, calibrated sighting it on hot bodies kept at various temperatures. A graph is then obtained between the logarithm of the deflection  $\theta$  of the millivoltmeter and the logarithm of the absolute temperature  $T$ ,

$$\log \theta = K \log T$$

The graph will be a straight line. From the graph, for a particular value of the voltage in the millivoltmeter, the unknown temperature can be read off from the calibration curve. The method gives accurate result. However, in case absolute accuracy is not required, then the temperature of the source can be calculated as follows :

Let  $V_1$  and  $V_2$  be the reading of the millivoltmeter when the pyrometer is sighted on a body of known temperature  $T_1$  and a body of unknown temperature  $T_2$  respectively. Then

$$V_1 \propto (T_1^b - T_0^b) \text{ and } V_2 \propto (T_2^b - T_0^b)$$

$$\text{or, } \frac{V_2}{V_1} = \frac{T_2^b - T_0^b}{T_1^b - T_0^b}$$

Since  $T_0$  is very much smaller and generally negligible as compared to  $T_1$  and  $T_2$ ,

$$\frac{V_2}{V_1} = \frac{T_2^b}{T_1^b} \quad \text{or, } T_2 = T_1 (V_2/V_1)^{1/b}$$

### rotating sector device

The usual range of a total radiation pyrometer is about  $1500^\circ\text{C}$ . But this can be greatly extended by the use of a rotating sector device. It consists of a circular disc with a sector of suitable angle  $\theta$  cut from the disc (Fig. 12.8). The disc is rapidly rotated about the axis of the pyrometer and it limits the amount of radiation incident on the mirror.

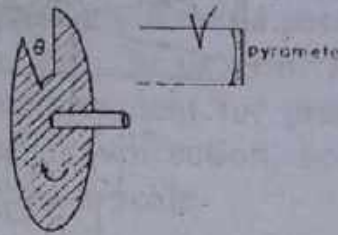


Fig. 14.8

If  $T_1$  is the temperature measured with the disc, according to Stefan's fourth power law,

$$T_1^4 \propto \theta$$

If  $T$  is the actual temperature of the hot body, then

$$T^4 \propto 360$$

$$\therefore \frac{T^4}{T_1^4} = \frac{360}{\theta}$$

$$\text{or, } T = T_1 \left[ \frac{360}{\theta} \right]^{1/4}$$

For example if  $\theta = 22.5^\circ$  and

$T_1 = 1500^\circ\text{K}$ , then

$$T = 1500 [360/22.5]^{1/4} = 3000^\circ\text{K}.$$



### 14.9 Disappearing Filament Optical Pyrometer

In an optical pyrometer, the intensity of radiations emitted by a hot body in a certain range of wavelength (say, red) is compared and matched with that given out by a small electric lamp. The pyrometer consists of a telescope with its cross-wires replaced by the filament  $F$  of an electric lamp. The objective  $O$  of the telescope is at one end of the telescope while the eye-piece  $E$  is at the other end (Fig. 14.9).  $DD$  are two limiting diaphragms on either side of the lamp. The filament is heated with the help of a battery and the temperature can be controlled by adjusting the current through it with the help of a rheostat  $Rh$ . The light from both the filament and the hot body passes through a filter of red glass  $G$  before reaching the eye to restrict radiations from both the hot body and the lamp within a narrow band of red wavelengths.

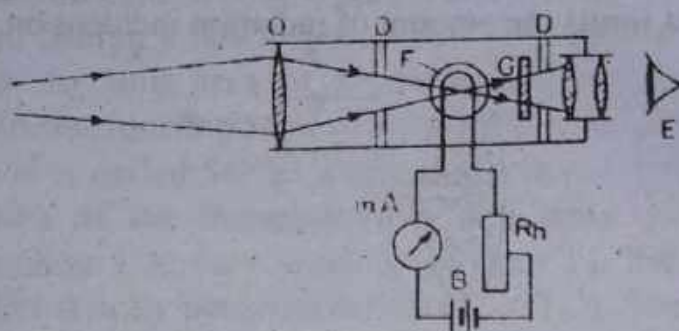


Fig. 14.9

By adjusting the objective lens  $O$ , the radiations from the hot body are focussed on the filament  $F$  of the lamp. Next, current is passed through the filament which is viewed through the eye-piece. Thus the filament is viewed against the background of the hot body. If the current through the filament is too strong, the filament would appear brighter than the image of the hot body. If on the other hand, the current through the filament is weak, then it would appear darker. The current through the filament is so adjusted with the help of the rheostat  $Rh$  that the filament just becomes invisible against the image of the hot body. The current at this point is noted. Since the brightness or *heat energy emitted per square centimeter* is the same in the two cases, the temperature of the hot body is the same as that of the filament. The temperature of the filament can be calculated from the relation

$$i = a + bT + cT^2$$



where  $i$  is the filament current and  $a$ ,  $b$  and  $c$  are constants which can be determined by calibrating the pyrometer against three known temperatures.

Initially the pyrometer is calibrated by noting the filament current against hot bodies at known temperature. A graph is plotted between the current and the temperature. From the graph, for a particular value of filament current the corresponding unknown temperature can be determined. For industrial work the milliammeter may be directly calibrated in degrees.

Temperature upto  $3200^{\circ}\text{C}$  can be directly measured with this pyrometer. However, by rotating a sectored disc in front of the pyrometer, as outlined in Art. 14.8, the range can be considerably extended.

#### 14.10 International practical temperature scale

It was shown by Lord Kelvin that a perfect gas scale is identical with absolute thermodynamic scale. But not only no real gas is perfect, they are also inconvenient for practical uses. Each of them requires its own characteristic correction, however small, to convert its readings to thermodynamic scale.

In the light of above difficulties with accurate gas thermometry, the International Committee on Weights and Measures, defined a temperature scale called the *International Practical Temperature Scale* at their Ninth General Conference in 1948. Slight refinement was subsequently incorporated to this scale which could be used readily with ease to calibrate scientific and industrial instruments. The international scale is based upon a number of reproducible fixed points to which numerical values are assigned. The temperature can be computed from the readings of specific instruments with the help of recommended formulas. The *recommended* primary and fundamental *fixed points* that are in use are given below:

(i) Oxygen point : Normal boiling point (NBP) of liquid oxygen :  $-182.97^{\circ}\text{C}$ ;

(ii) Ice point (fundamental fixed point) : Temperature of equilibrium between ice and air saturated water at normal pressure :  $0.00^{\circ}\text{C}$ ;

(iii) Steam point (fundamental fixed point) : Temperature of equilibrium between liquid water and its vapour at normal pressure (NBP of water) :  $100.00^{\circ}\text{C}$ ;

- (iv) Zinc point : Normal melting point (NMP) of zinc :  $419.51^{\circ}\text{C}$ ;
- (v) Antimony point : NMP of antimony :  $630.50^{\circ}\text{C}$ ;
- (vi) Silver point : NMP of silver :  $961.90^{\circ}\text{C}$ ;
- (vii) Gold point : NMP of gold :  $1064.5^{\circ}\text{C}$ .

The scale is divided into four distinct ranges for the purpose of determining the temperature between two fixed points. The instruments recommended for different ranges and the formulas to be used for computation are given below:

(1) *From oxygen point to the triple point of water* ( $-190^{\circ}\text{C}$  to  $0^{\circ}\text{C}$ ) : A platinum resistance thermometer with a pure platinum wire having a diameter lying between 0.05 and 0.02 mm is recommended. The formula to be used for the determination of temperature is

$$R_t = R_0 [1 + \alpha t + \beta t^2 + \gamma (t - 100)t^3]$$

where  $R_0$  is the resistance at  $0^{\circ}\text{C}$ . The constants  $\alpha$ ,  $\beta$  and  $\gamma$  are determined by determining the resistance,  $R_t$ , at oxygen point, the triple point of water, the steam point and the zinc point.

(2) *From the triple point of water to antimony point* ( $0^{\circ}\text{C}$  to  $660^{\circ}\text{C}$ ) : The same platinum resistance thermometer is used. The recommended formula is

$$R_t = R_0 (1 + \alpha t + \beta t^2)$$

where  $R_0$ ,  $\alpha$  and  $\beta$  can be determined in a manner similar to that followed in (1)

(3) *From antimony point to the gold point* ( $660^{\circ}\text{C}$  to  $1063^{\circ}\text{C}$ ) : A thermo-electric thermometer (thermocouple made of platinum and platinum-rhodium alloy (90% Pt; 10% Rh) is used. The temperature is computed by using the formula

$$E = a + bt + ct^2$$

where  $E$  is the thermo-electromotive force developed when one junction of the thermo-couple is at the triple point of water and the other at the unknown temperature  $t$  to be determined. The constants  $a$ ,  $b$  and  $c$  are determined from measured values of  $E$  at antimony point, silver point and gold point.



(4) *Above the gold point* (beyond 1063°C) : An *optical pyrometer* is used for measuring temperature in this range. The temperature  $t$  is defined by the following formula based on Planck's law.

$$\frac{J_t}{J_{Au}} = \frac{e^{C_2/\lambda(t_{Au}+T_0)} - 1}{e^{C_2/\lambda(t+T_0)} - 1}$$

where  $J_t$  and  $J_{Au}$  are the radiant powers per unit area and per unit wavelength at wavelength  $\lambda$  emitted by a black body at a temperature  $t$  and at gold point respectively.  $T_0$  is the temperature of the ice point in Kelvin (273.15). The constant  $C_2 = 1.438 \text{ cm-deg K}$ .

### High Temperature Measurement

According to the International scale of temperature *high temperature* means the range whose lower limit is the temperature corresponding to normal melting point of antimony, that is, 630.50°C. This is called the *antimony point*. For temperature above this point the following thermometers may be employed.

(1) **Liquid thermometers** : Gallium melts at 29.7°C and boils at 1700°C at standard atmospheric pressure. Thus a thermometer made of molten gallium in a quartz tube can be used with moderate accuracy up to a temperature of 1000°C.

(2) **Resistance thermometers** : A resistance thermometer made of pure platinum can be used with sufficient accuracy up to the gold point, that is 1064.5°C. The outer protective of the thermometer should be made of glazed porcelain. However, the accuracy of the thermometer falls off above 630.5°C. Moreover, it is a slow measuring instrument.

(3) **Gas thermometers** : Constant volume hydrogen thermometer having a platinum bulb can be used upto 500°C. It cannot be used at higher temperature as hydrogen diffuses through the platinum bulb. But by using a porcelain bulb, the thermometer may be used upto 1100°C. Beyond 1100°C, using helium (or nitrogen) in place of hydrogen, the thermometer may be used to measure temperature upto 1600°C.

(4) **Thermo-couple thermometers** : The most reliable thermo-couple for measuring temperatures from 630.5°C to 1064.5°C is a



thermo-couple made of platinum and an alloy of platinum and rhodium (10 – 13% rhodium). With a lesser degree of accuracy, it can be used up to  $1600^{\circ}\text{C}$ . For measuring still higher temperature, say up to  $2100^{\circ}\text{C}$ , a thermo-couple made of irridium and irridium-rubidium alloy (10% rubidium) is recommended. Thermo-couple of tungsten and molybdenum can be used up to even  $3000^{\circ}\text{C}$ .

(5) **Pyrometers** : The International Committee recommends the use of optical and total radiation pyrometers for measuring temperatures above gold point. Above  $3000^{\circ}\text{C}$ , total radiation pyrometers are generally employed.

### Measurement of Low Temperature

Low temperature means the range of temperature whose *upper limit* is  $0^{\circ}\text{C}$ . The following thermometers are usually recommended for measuring temperature in this region.

1. **Liquid thermometers** : Mercury thermometers can be used to measure temperature upto  $-30^{\circ}\text{C}$ . With alcohol thermometers, temperatures upto  $-100^{\circ}\text{C}$  can be measured. A special thermometer with fractionally distilled petroleum ether can be used upto  $-190^{\circ}\text{C}$ , with liquid pentane down to even  $-200^{\circ}\text{C}$ .

Liquid thermometers are convenient and their response is quick. The accuracy is about  $0.1^{\circ}\text{C}$ . The main difficulty with liquid thermometers is that liquid solidifies at low temperatures.

2. **Gas thermometers** : Constant volume hydrogen thermometer is the best correct thermometer in the moderately low temperature region and can be used to measure upto  $-250^{\circ}\text{C}$ . Helium may bring the scale down to its boiling point (4.2K). These thermometers are quite accurate; but they are quite bulky.

3. **Resistance thermometers** : Platinum resistance thermometer may be used to measure temperatures upto  $-190^{\circ}\text{C}$  with an accuracy of  $0.01^{\circ}\text{C}$ . Although quite accurate, its response is quite slow.

4. **Thermo-couple thermometers** : Copper-constantan or Iron-constantan develops fairly large *e.m.f.* and so can be used up to  $-200^{\circ}\text{C}$ . Thermo-couples of gold and silver or platinum and gold

are suggested for temperatures below  $-200^{\circ}\text{C}$ . The galvanometers to be used in the circuit must be highly sensitive.

5. **Vapour pressure thermometers** : These can be used to measure upto  $-268^{\circ}\text{C}$ . Below  $-268^{\circ}\text{C}$ , helium vapour pressure thermometer can be used upto  $-272^{\circ}\text{C}$ .

6. **Magnetic thermometers** : According to Curie's law, the susceptibility of a paramagnetic substance is inversely proportional to the absolute temperature  $T$ . This is utilised in measuring temperature below  $-272^{\circ}\text{C}$  – near about the absolute zero.

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### EXERCISES

- [1] Name three physical properties which are used for the measurement of temperature.

State briefly the principles underlying the working of the following thermometers :

- (i) Constant volume hydrogen thermometer
  - (ii) Resistance thermometer
  - (iii) Thermo-electric thermometer
- [2] Describe a constant volume hydrogen thermometer. How would you use it to measure temperature? Discuss the various sources of error that are involved in the measurement of temperature by this thermometer and the conditions necessary.
- [3] Describe the principle, construction and working of a platinum resistance thermometer. Discuss its advantages and disadvantages.
- [4] Describe a platinum resistance thermometer. Explain how it works with the help of Callendar and Griffith's bridge. How does the platinum temperature of a body differ from its true temperature?
- [5] What is Seebeck effect? How can it be used for the measurement of high temperature?

Explain the terms thermo-couple, neutral temperature and temperature of inversion.



- [6] Describe the principle, construction and working of a thermo-electric thermometer. State its advantages and disadvantages.

What thermo-couples would you use for measuring a temperature between (i)  $0^{\circ}\text{C} - 400^{\circ}\text{C}$ , (ii)  $400^{\circ}\text{C} - 800^{\circ}\text{C}$ , (iii)  $600^{\circ}\text{C} - 1200^{\circ}\text{C}$ , (iv)  $1200^{\circ}\text{C} - 1600^{\circ}\text{C}$ , (v)  $1600^{\circ}\text{C} - 2000^{\circ}\text{C}$  and (vi)  $2000^{\circ}\text{C} - 3000^{\circ}\text{C}$  and why?

- [7] Describe two methods of measuring high temperatures. State clearly the principles underlying them and the range and sensitivity of each.

- [8] Give the construction of a thermo-electric thermometer and compare its performance with a platinum resistance thermometer. What precautions should be taken in measuring temperatures with a thermo-electric thermometer.

- [9] Describe the relative merits of thermoelectric and platinum resistance methods of measuring temperatures.

- [10] What is a pyrometer? Describe the construction and working of a total radiation pyrometer. How could the range of temperature measured by it be extended?

- [11] Describe the principle, construction and working of a disappearing filament pyrometer. How would you extend the range of temperature measured by it.

- [12] On what principle is the working of a platinum resistance thermometer based? Describe Callendar and Griffith's bridge for the accurate measurement of resistance. How is the true temperature deduced from the measured platinum temperature?

- [13] Write short notes on :

(i) Constant volume hydrogen thermometer

(ii) Platinum resistance thermometer

(iii) Thermo-electric thermometer

(iv) Total radiation pyrometer

(v) Optical pyrometer.

- [14] How would you define high temperature and low temperature? Mention the different methods employed in high and low temperature thermometry.

- [15] The pressures indicated by a gas in a constant volume gas thermometer are 80.0 cms and 109.3 cms at  $0^{\circ}\text{C}$  and  $100^{\circ}\text{C}$  respectively. The pressure indicated by the gas is 100 cms when the bulb of the thermometer is placed in hot water. Find the temperature of hot water. ( $68.25^{\circ}\text{C}$ ).



- [16] The pressures indicated by a constant volume hydrogen thermometer are 23.5 cms, 75.0 cms and 102.4 cms of mercury when the bulb is immersed in liquid air, ice and steam respectively. What is the temperature of the liquid air on the constant volume hydrogen scale? ( $-187.95^{\circ}\text{C}$ ).
- [17] When the bulb of a constant volume gas thermometer is placed in melting ice, the level of mercury in the open tube is 5 cms below the level in the closed limb. When the bulb is at  $273^{\circ}\text{C}$ , the level in the open limb is 65 cms above. Calculate the barometric pressure. (75 cms).
- [18] The resistance of a platinum thermometer at  $0^{\circ}\text{C}$ ,  $100^{\circ}\text{C}$  and boiling point of sulphur are 2.56, 3.56 and 6.78 ohms respectively. Calculate the boiling point of sulphur on (i) the platinum resistance scale and (ii) the celsius scale correct up to the first order of approximation. Assume  $\delta = 1.5$  for platinum. ( $422^{\circ}$ ,  $442.3^{\circ}\text{C}$ ).
- [19] If 2.57 ohms, 3.53 ohms and 2.89 ohms be the resistances of a platinum thermometer at  $0^{\circ}\text{C}$ ,  $100^{\circ}\text{C}$  and in a bath at  $t^{\circ}\text{C}$ , calculate the temperature of the bath (i) on the platinum scale and (ii) on the celsius scale, assuming  $\delta = 1.5$  for platinum. ( $33.3^{\circ}$ ,  $33^{\circ}\text{C}$ ).
- [20] The resistance of a platinum resistance thermometer at the ice point is 5 ohms and at the steam point is 6.93 ohms. The pressure exerted by the gas in a constant volume gas thermometer at the ice point is 100 cms of mercury and that at the steam point is 136.6 cms of mercury. When both the thermometers are placed in an unknown temperature bath, the resistance of the platinum thermometer is 5.795 ohms and the pressure of the gas is 114.9 cms of mercury. Calculate (i) on the platinum scale and (ii) on the gas scale, the temperature of the unknown bath. ( $41.19^{\circ}$ ,  $40.71^{\circ}\text{C}$ ).
- [21] A thermo-couple having a mean thermoelectric power of  $50\mu\text{V}/^{\circ}\text{C}$  is used to measure an unknown temperature. The potentiometer wire is 10 metres long and is used in series with a cell of *e.m.f.* 2 volts and a resistance of 2550 ohms. The balance point is obtained with a length of 520 cms of the wire between the thermocouple terminals. If the resistance per cm of the wire is 0.05 ohms, calculate the value of the unknown temperature. The cold junction is at  $0^{\circ}\text{C}$ . ( $400^{\circ}\text{C}$ ).

## CHAPTER XV

## CALORIMETRY

*Newton's law of cooling – Specific heat of a liquid – Specific heats of gases – Relation between the two specific heats – Ratio of the two specific heats – Change in internal energy of a gas – Regnault's method for determination of  $C_p$  – Determination of  $C_v$  by Joly's differential steam calorimeter – Determination of  $\gamma = C_p/C_v$  – General heat energy equation of a gas – Isothermal change – Adiabatic change – Isochoric process – Isobaric process – Adiabatic equation of a perfect gas – Relation between volume and temperature in an adiabatic change – Relation between pressure and temperature in an adiabatic change – Adiabatic curves are steeper than isothermal curves – Isothermal and adiabatic elasticities – Work done during expansion and compression of a gas – Solved problems – Exercises.*

## 15.1 Newton's law of cooling

When a body is at higher temperature than its surroundings, it will go on losing heat until its temperature becomes equal to that of its surroundings. This loss of heat from the body is due to the process of conduction, convection and radiation. If conduction can be neglected by thermally lagging the body by an enclosed air layer or non-conducting materials like wood, cork, etc., the cooling process will be partly due to convection and partly due to radiation. The rate of loss of heat depends upon the following factors.

- (i) *the difference in the temperature of the body and that of the surrounding medium*
- (ii) *the area of the surface exposed to the medium (the larger the area exposed to air, more rapid is the cooling)*
- (iii) *the nature of the exposed surface or its emissive power. A dull black surface loses heat more rapidly than a polished one.*

The emissive power of a surface is defined as the quantity of heat radiated from the surface per second per unit area per unit temperature difference with the surroundings.

If  $-\frac{dQ}{dt}$  is the rate of loss of heat, then



$$-\frac{dQ}{dt} = A.E.f(\theta - \theta_0)$$

where

A = area of the exposed surface

E = emissive power of the surface

$f(\theta - \theta_0)$  = some function of the difference in temperature between the body and the surrounding,  $\theta$  and  $\theta_0$  being the respective temperatures of the body and the surrounding.

When the surrounding medium is undisturbed, the loss of heat *in case of liquids* is due to natural convection. The rate of loss of heat is then given by

$$-\frac{dQ}{dt} = k(\theta - \theta_0)^{5/4}$$

If the calorimeter containing the liquid is cooled *not in still air but in a uniform current of air* or under the fan, the loss of heat is due to forced convection. Then the rate of loss of heat is

$$-\frac{dQ}{dt} = k(\theta - \theta_0)$$

$$\text{or, } -\frac{dQ}{dt} \propto (\theta - \theta_0)$$

This is the mathematical expression for Newton's law of cooling which may be stated as follows :

*when a liquid is cooled by a draught of air, the rate of loss of heat is proportional to the difference between the temperature of the liquid and that of the surrounding.*

Let a body of mass  $m$  and specific heat  $s$  be heated to a temperature  $\theta$ . If the body is now placed in a surrounding whose temperature is  $\theta_0$  then, due to convection and radiation, the temperature of the body will gradually fall. If  $-d\theta$  be the fall in temperature in an extremely small interval of time  $dt$ , then the heat lost in time  $dt$  will evidently be  $-m.s.d\theta$ . So the rate of loss of heat is



$$-\frac{dQ}{dt} = -m.s. \frac{d\theta}{dt} \quad (15.1)$$

According to Newton's law of cooling, the rate of loss of heat is proportional to  $\theta - \theta_0$ , i.e.,

$$-\frac{dQ}{dt} = k(\theta - \theta_0) \quad (15.2)$$

where  $k$  is a constant of proportionality and depends only upon the area and nature of the surface exposed.

Combining eqns. (15.2) and (15.3), we get

$$\begin{aligned} -m.s. \frac{d\theta}{dt} &= k(\theta - \theta_0) \\ \text{or, } -\frac{d\theta}{dt} &= \frac{k}{m.s.}(\theta - \theta_0) \\ &= c(\theta - \theta_0) \end{aligned} \quad (15.3)$$

where  $c = k/ms$  is another constant known as Newton's cooling constant.

It, therefore, appears from Newton's law of cooling that the rate of fall in temperature is proportional to the difference in temperature between the body and its surroundings.

Rearranging eqn. (15.4), we get

$$\frac{d\theta}{\theta - \theta_0} = -c \cdot dt$$

Integrating,

$$\int \frac{d\theta}{\theta - \theta_0} = -c \int dt$$

$$\text{or, } \log_e (\theta - \theta_0) = -ct + K \quad (15.4)$$

where  $K$  is a constant of integration.

If a graph is plotted with  $t$  along the x-axis and  $\log_e (\theta - \theta_0)$  along the y-axis, the resulting graph will be a straight line (Fig. 15.1).

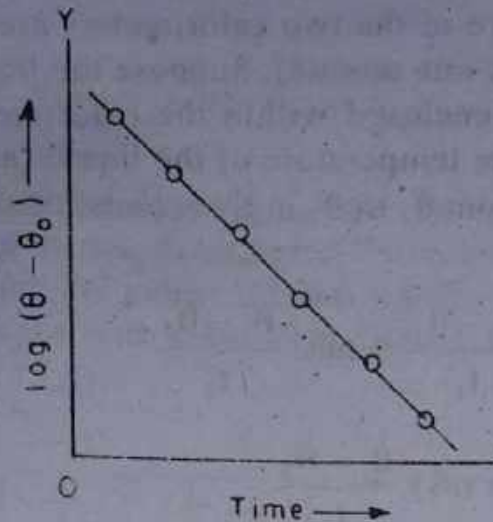


Fig. 15.1

Newton's law of cooling strictly holds good only for small difference of temperature  $(\theta - \theta_0)$  – about  $10^\circ\text{C}$  between the hot body and the surrounding but may be applied upto a difference of  $30^\circ\text{C}$ . However, the law is true even for large difference in temperature provided the cooling takes place in a uniform current in air, as in forced convection.

## 15.2 Specific heat of a liquid

Newton's law of cooling may be applied to determine the specific heat of a liquid by observing the time taken by the liquid in cooling from one temperature to another.

Let A and B are two identical calorimeter containing equal volumes of a hot liquid and hot water respectively. The two calorimeters are made of the same material, about an inch in diameter and three inches high. Their outer surface are equally polished. They are placed on non-conducting supports inside a double – walled metal vessel which, in practice, consists of two tin boxes placed one inside the other. The space between the boxes may be filled with water at room temperature to ensure a fixed surrounding for the calorimeters. The calorimeters are fitted with rubber lids having a central and a side hole. A thermometer can pass through the central hole and, if necessary, a rotary type of stirrer may be introduced in the liquid whose stem can project out of the liquid through its side hole. The thermometers  $T_1$  and  $T_2$  measure the temperature of water and liquid (Fig. 15.2).

The temperature of the two calorimeters are noted after regular interval of time (say one minute). Suppose the liquid of mass  $M_1$  and specific heat  $S_1$  is enclosed within the calorimeter of mass  $m$  and specific heat  $s$ . If the temperature of the liquid (as well as that of the calorimeter) falls from  $\theta_1$  to  $\theta_2$  in  $t_1$  seconds, then the average rate of loss of heat is

$$M_1 \cdot S_1 \cdot \frac{\theta_1 - \theta_2}{t_1} + m \cdot s \cdot \frac{\theta_1 - \theta_2}{t_1}$$

$$= (M_1 S_1 + ms) \frac{\theta_1 - \theta_2}{t_1}$$

Constant temperature enclosure

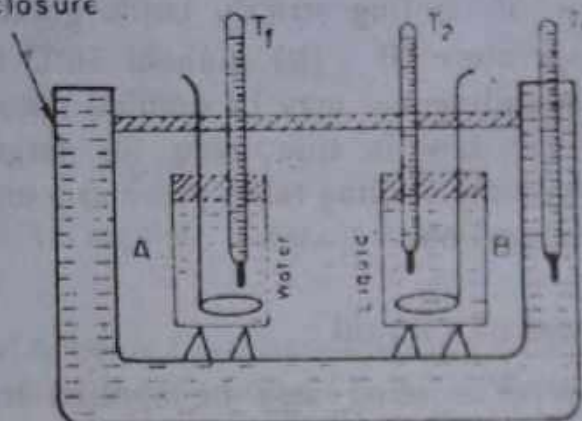


Fig. 15.2

Let the mass of water be  $M_2$  and its specific heat be  $S_2$  and the mass and specific heat of the calorimeter in which it is contained be again  $m$  and  $s$  respectively. If the time taken by water to cool through the same range of temperature ( $\theta_1$  to  $\theta_2$ ) under similar conditions be  $t_2$  seconds, then the average rate of loss of heat is

$$(M_2 S_2 + ms) \frac{\theta_1 - \theta_2}{t_2}$$

since the conditions are similar, these two rates are equal.

$$\therefore (M_1 S_1 + ms) \frac{\theta_1 - \theta_2}{t_1} = (M_2 S_2 + ms) \frac{\theta_1 - \theta_2}{t_2}$$



$$\text{or, } S_1 = \frac{M_2 S_2 t_1 + m s (t_1 - t_2)}{M_1 t_2} \quad (15.5)$$

Other things being known,  $S_1$  can be found from the above expression.  $t_1$  and  $t_2$  are determined from the two cooling curves (one for water and the other for the liquid), plotted on the same piece of graph paper with time as the abscissa and temperature as the ordinate (Fig. 15.3).

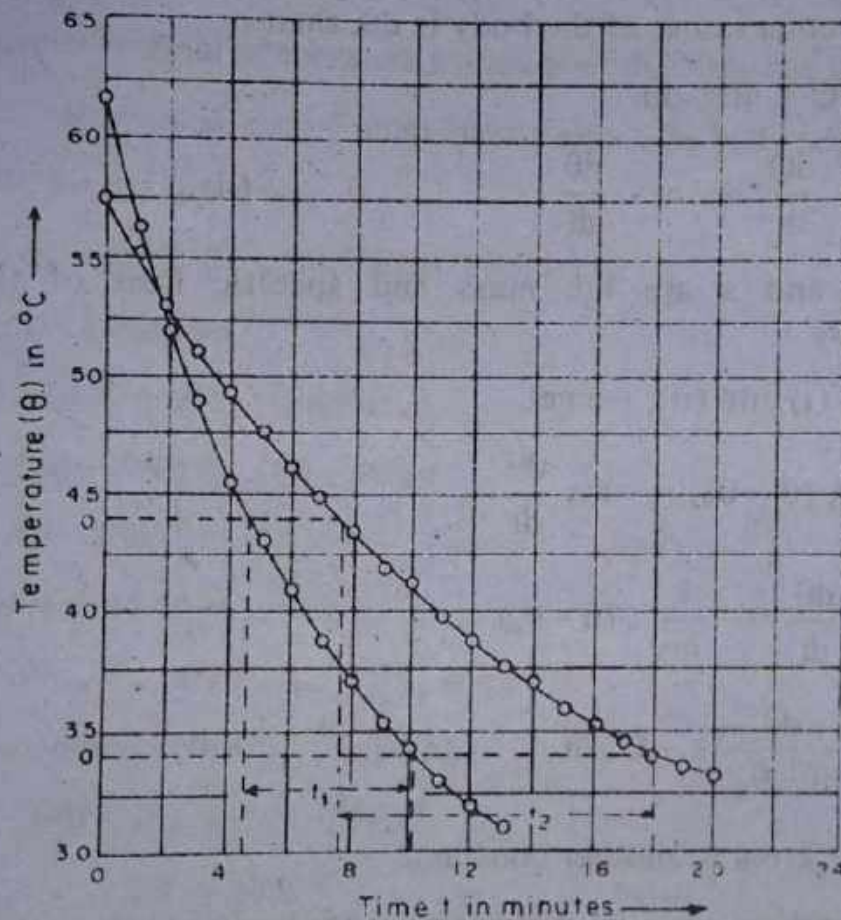


Fig. 15.3

**Example 15.1.** State Newton's law of cooling and express it as a mathematical equation.

By applying this equation, find the time taken by a body to cool from 60°C to 40°C in surroundings at a temperature of 20°C. The body took 5 minutes to cool from 100°C to 60°C.

**Soln.**

Newton's law of cooling, can be expressed mathematically as

$$-\frac{dQ}{dt} \propto (\theta - \theta_0)$$

$$\text{or, } -\frac{dQ}{dt} = k \cdot (\theta - \theta_0) \quad \text{.(i)}$$

where  $-dQ$  represents the loss of heat and  $k$  represents a constant. If the fall in temperature of the body is  $d\theta$ , then

$$dQ = m.s. d\theta$$

$$\text{or, } -\frac{dQ}{dt} = -m.s. \frac{d\theta}{dt} \quad \text{(ii)}$$

where  $m$  and  $s$  are the mass and specific heat of the body respectively.

From (i) and (ii), we get

$$k (\theta - \theta_0) = -ms \frac{d\theta}{dt}$$

$$\text{or, } \frac{d\theta}{dt} = -\frac{k}{ms} \cdot (\theta - \theta_0)$$

$$\text{or, } \frac{d\theta}{\theta - \theta_0} = -C \cdot dt \quad \text{(iii)}$$

where  $C = k/ms$  is another constant.

Integrating eqn. (iii) between proper limits, we have

$$\int_{\theta_1}^{\theta_2} \frac{d\theta}{(\theta - \theta_0)} = -C \int_0^t dt$$

$$\text{or, } [\log_e (\theta - \theta_0)]_{\theta_1}^{\theta_2} = -Ct$$

$$\therefore \log_e (\theta_2 - \theta_0) - \log_e (\theta_1 - \theta_0) = -Ct.$$

$$\text{or, } \log_e \frac{\theta_2 - \theta_0}{\theta_1 - \theta_0} = -Ct$$

$$\text{or, } \frac{\theta_2 - \theta_0}{\theta_1 - \theta_0} = e^{-Ct}$$

$$\text{or, } \theta_2 - \theta_0 = (\theta_1 - \theta_0) e^{-Ct} \quad (\text{iv})$$

$$\text{or, } \theta'' = \theta' e^{-Ct} \quad (\text{v})$$

where  $\theta''$  = final temperature excess after the body has cooled down

$\theta'$  = initial temperature excess before the body starts cooling.

Now, for the first case,

$$\theta_2 = 60^\circ\text{C}, \theta_1 = 100^\circ\text{C} \text{ and } \theta_0 = 20^\circ\text{C};$$

$$t = 5 \text{ minutes}$$

So, from  $\theta'' = \theta' e^{-Ct}$ , we have

$$(60 - 20) = (100 - 20) e^{-5C}$$

$$\text{or, } 40 = 80e^{-5C} \quad (\text{vi})$$

In the second case,

$$\theta_2 = 40^\circ\text{C}, \theta_1 = 60^\circ\text{C}, \theta_0 = 20^\circ\text{C} \text{ and } t = ?$$

so from,  $\theta'' = \theta' e^{-Ct}$  we have

$$(40 - 20) = (60 - 20) e^{-Ct}$$

$$\text{or, } 20 = 40 e^{-Ct} \quad (\text{vii})$$

From eqns. (vi) and (vii) we get

$$\frac{20}{40} = \frac{40 e^{-Ct}}{80 e^{-5C}} ; \text{ or, } 1 = \frac{e^{-Ct}}{e^{-5C}}$$

$$\therefore e^{-Ct} = e^{-5C}$$

$$\text{or, } t = 5 \text{ minutes.}$$



**Example 15.2.** A liquid takes 5 minutes to cool from  $80^{\circ}\text{C}$  to  $50^{\circ}\text{C}$ . How much time will it take to cool from  $60^{\circ}\text{C}$  to  $30^{\circ}\text{C}$ ? The temperature of the surroundings is  $20^{\circ}\text{C}$ .

**Soln.**

Integrating  $\frac{d\theta}{\theta - \theta_0} = -c dt$  between proper limits we have,

$$\int_{\theta_1}^{\theta_2} \frac{d\theta}{(\theta - \theta_0)} = -c \int_0^t dt$$

In the first case,

$$\int_{80}^{50} \frac{d\theta}{\theta - \theta_0} = -c \times 5$$

where  $\theta_1 = 80^{\circ}\text{C}$ ,  $\theta_2 = 50^{\circ}\text{C}$ ,  $\theta_0 = 20^{\circ}\text{C}$  and  $t = 5$  minutes.

$$\text{or, } [\log_e(\theta - \theta_0)]_{80}^{50} = -5c$$

$$\text{or, } \log_e \frac{50 - 20}{80 - 20} = -5c$$

$$\text{or, } \log_e \frac{30}{60} = -5c$$

$$\text{or, } \log_e \frac{30}{60} = 5c. \quad (i)$$

In the second case,

$\theta_1 = 60^{\circ}\text{C}$ ,  $\theta_2 = 30^{\circ}\text{C}$ ,  $\theta_0 = 20^{\circ}\text{C}$  and  $t = ?$

$$\int_{60}^{30} \frac{d\theta}{\theta - \theta_0} = t.c.$$

$$\text{or, } \log_e \frac{50 - 20}{80 - 20} = -t.c.$$

$$\text{or, } \log_e \frac{10}{40} = -t.c$$

$$\text{or, } \log_e \frac{40}{10} = t.c. \quad (\text{ii})$$

Dividing (ii) by (i), we get

$$\begin{aligned} \frac{t}{5} &= \frac{\log_e 4}{\log_e 2} = \frac{\log_e 2^2}{\log_e 2} \\ &= \frac{2 \log_e 2}{\log_e 2} = 2 \end{aligned}$$

$$\text{or, } t = 5 \times 2 = 10 \text{ minutes.}$$

**Example 15.3.** A liquid takes 4 minutes to cool from  $65^\circ\text{C}$  to  $50^\circ\text{C}$ . What will be its temperature after the next 10 minutes? The temperature of the surroundings is  $35^\circ\text{C}$ . Assume that Newton's law of cooling holds good throughout the process.

**Soln.**

$$\int_{\theta_1}^{\theta_2} \frac{d\theta}{(\theta - \theta_0)} = -c \int_0^t dt$$

$$\text{or, } \log_e \frac{\theta_2 - \theta_0}{\theta_1 - \theta_0} = -ct$$

In the first case,  $\theta_1 = 65^\circ\text{C}$ ,  $\theta_2 = 50^\circ\text{C}$ ,  $\theta_0 = 35^\circ\text{C}$  and  $t = 4$  mts.

$$\therefore \log_e \frac{50 - 35}{65 - 35} = -4c$$

$$\log_e \frac{15}{30} = -4c$$

$$\text{or, } \log_e (0.5) = -4c \quad (\text{i})$$

In the second case,

$$\theta_1 = 50^\circ\text{C}, \quad \theta_2 = x, \quad \theta_0 = 35^\circ\text{C} \text{ and } t = 10 \text{ mts.}$$

$$\log_e \frac{x - 35}{50 - 35} = -10c \quad (\text{ii})$$

Dividing (ii) by (i)

$$\log_e \frac{x - 35}{50 - 35} = \frac{10}{4}$$

$$\log_e \frac{x - 35}{50 - 35} = \frac{5}{2} \log_e (0.5)$$

$$\log_e \frac{x - 35}{50 - 35} = \log_e (0.5)^{5/2}$$

$$\frac{x - 35}{50 - 35} = (0.5)^{5/2}$$

or,  $x = 37.65$

**Example 15.4.** A body initially at  $80^\circ\text{C}$  cools to  $64^\circ\text{C}$  in 5 minutes and to  $52^\circ\text{C}$  in 10 minutes. What will be its temperature after 15 minutes and what is the temperature of the surroundings?

**Soln.**

$$\int_{\theta_1}^{\theta_2} \frac{d\theta}{(\theta - \theta_0)} = -c \int_0^t dt = -c.t$$

In the first case;  $\theta_1 = 80^\circ\text{C}$ ,  $\theta_2 = 64^\circ\text{C}$  and  $t = 5$  mts.

$$\int_{80}^{64} \frac{d\theta}{\theta - \theta_0} = -c.5$$

or,  $\log_e \frac{64 - \theta_0}{80 - \theta_0} = -5c$  (i)

In the second case,  $\theta_1 = 64^\circ\text{C}$ ,  $\theta_2 = 52^\circ\text{C}$ ,  $t = 10 - 5 = 5$  mts.

$$\therefore \int_{64}^{52} \frac{d\theta}{\theta - \theta_0} = -5c$$

or,  $\log_e \frac{52 - \theta_0}{64 - \theta_0} = -5c$  (ii)



From (i) and (ii)

$$\frac{64 - \theta_0}{80 - \theta_0} = \frac{52 - \theta_0}{64 - \theta_0}$$

$$\text{or, } \theta_0 = 16^\circ\text{C}$$

Let the temperature after 15 minutes be  $\theta_2$ . Then  $\theta_1 = 52^\circ\text{C}$ ,  $\theta_2 = ?$ ,  $\theta_0 = 16^\circ\text{C}$ ,  $t = 15 - 10 = 5$  mts.

$$\int_{\theta_1}^{\theta_2} \frac{d\theta}{(\theta - \theta_0)} = -5c$$

$$\text{or, } \log_e \frac{\theta_2 - \theta_0}{\theta_1 - \theta_0} = -5c$$

$$\text{or, } \log_e \frac{\theta_2 - 16}{52 - 16} = -5c \quad (\text{iii})$$

Combining (iii) with either (i) or (ii), we get

$$\frac{\theta_2 - 16}{52 - 16} = \frac{52 - 16}{64 - 16} ; \text{ or, } \theta_2 = 43^\circ\text{C}.$$

**Example 15.5.** 250 gms. of water and an equal volume of liquid of mass 200 gms. are placed successively in the same calorimeter, and cools from  $60^\circ\text{C}$  to  $55^\circ\text{C}$  in 130 secs, and 60 secs, respectively. Find the specific heat of the liquid. The water equivalent of the calorimeter is 10 gms.

**Soln.**

Let  $M_1$ ,  $s_1$  and  $t_1$  be the mass and specific heat of the liquid and the time required by it to cool from  $60^\circ\text{C}$  to  $55^\circ\text{C}$  respectively.  $M_2$ ,  $s_2$  and  $t_2$  are the corresponding quantities for water.  $m$  and  $s$  are the mass and specific heat of the calorimeter.

Since the conditions are similar, the rate of loss of heat from liquid and calorimeter should be equal to that from water and calorimeter.

$$(M_1 s_1 + ms) \frac{\theta_1 - \theta_2}{t_1} = (M_2 s_2 + ms) \frac{\theta_1 - \theta_2}{t_2}$$

$$\text{or, } s_1 = \frac{M_2 s_2 t_1 + ms (t_1 - t_2)}{M_1 t_2}$$

$$M_1 = 200 \text{ gms}$$

$$M_2 = 250 \text{ gms}$$

$$ms = 10 \text{ gms}$$

$$s_2 = 1 \text{ gm/cm}^3$$

$$t_1 = 60 \text{ secs}$$

$$t_2 = 130 \text{ secs}$$

$$\therefore s_1 = 0.55 \text{ cal. gm}^{-1} \text{ } ^\circ\text{C}^{-1}.$$

**Example 15.6.** Equal volumes of water (density  $1 \text{ gm/cm}^3$ ) and of a liquid (density  $0.8 \text{ gm/cm}^3$ ) are put into similar calorimeters. They are allowed to cool under precisely similar conditions, from  $50^\circ\text{C}$  to  $40^\circ\text{C}$  and are observed to take 10 minutes and 6 minutes respectively. Calculate the specific heat of the liquid. Thermal capacity of each calorimeter is numerically equal to the volume of either liquid.

**Soln.**

Let the volume of either liquid be  $V$ .

Mass of water,  $M_2 = V \times 1 = V$

Mass of liquid,  $M_1 = V \times 0.8 = 0.8V$

Water equivalent of each calorimeter =  $ms = V$

$$t_1 = 6 \times 60 = 360 \text{ secs.},$$

$$t_2 = 10 \times 60 = 600 \text{ secs.}$$

$$s_2 = 1 \text{ and } s_1 = ?$$

$$s_1 = \frac{M_2 s_2 t_1 + ms (t_1 - t_2)}{M_1 t_2}$$

$$= \frac{V \times 1 \times 600 - V (360 - 600)}{0.8V \times 600}$$



$$\begin{aligned} &= \frac{600\text{V} - 240\text{V}}{480\text{V}} \\ &= \frac{360\text{V}}{480\text{V}} \\ &= 0.75 \text{ cal. gm}^{-1} \cdot ^\circ\text{C}^{-1}. \end{aligned}$$

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### 15.3 Specific heats of gases

The specific heat of a substance is defined as the ratio of the heat supplied to the rise in temperature per unit mass *i.e.*, the heat required per unit mass per unit temperature change (one degree celsius or one degree kelvin since the kelvin and the celsius degree are of the same size). The definition is alright for a solid or a liquid; but when applied to a gas, the definition becomes vague. For example, let a certain quantity of gas be suddenly compressed when work has to be done against the pressure or opposing force of the gas. This results in an increase of temperature. Thus even without any supply of heat from external source, there may be a rise in temperature. The ratio, *heat added / rise in temperature* ( $dQ/dT$ ) becomes equal to zero, *i.e.*, the *specific heat* = 0. If the compressed gas is now allowed to expand suddenly, cooling takes place. If this cooling is prevented by supplying the required quantity of heat from outside so that the temperature remains constant, the ratio  $dQ/dT$  or the *specific heat becomes infinite*. Thus the specific heat of a gas can have values lying between zero and infinity, which is absurd. Therefore, the specific heat of a gas should be more precisely defined.

The behaviour of a gas is completely defined by its three parameters, namely pressure, volume and temperature ( $P, V, T$ ). If the temperature  $T$  is changed, the other two factors tend to change. Thus two specific heats are of great importance. One is the specific heat of a gas at constant volume, denoted by  $c_v$ , and is defined as the amount of heat required to raise the temperature of one gramme of the gas through one degree Celsius (or Kelvin), the volume being kept constant during heating. The other is the specific heat at constant pressure, denoted by  $c_p$ , and is defined as the amount of heat required to raise the temperature of one gramme of the gas through one degree Celsius, the pressure being kept constant during heating.



A convenient unit of mass for a gas is the mole or a gramme molecule. When one gm-molecule of the gas is used instead of one gramme, the corresponding specific heat is known as the *gm-molecular* or *molar specific heat* and is denoted by either  $C_v$  or  $C_p$ . If  $M$  is the molecular weight of the gas expressed in grammes, then

$$C_p = c_p \times M \quad \text{and} \quad C_v = c_v \times M$$

**$C_p$  is greater than  $C_v$**

(i) *Specific heat at constant volume ( $c_v$ )*

Let  $m$  gm. of a gas be enclosed in a container with its lid fixed, i.e., the volume is kept constant. When heated, the volume of the gas cannot increase, hence both its *pressure and temperature* increase. Since there is no net change in volume ( $dV = 0$ ), the work done by the gas is zero. According to kinetic theory the internal energy of an ideal gas is proportional to the temperature and depends only on the temperature. Hence, all the heat supplied remains within the gas and in fact, represents the *increase in the internal energy of the gas*.

If the gas is heated at constant volume from an initial temperature  $T_1^\circ\text{C}$ , to the final temperature  $T_2^\circ\text{C}$ , then the quantity of heat  $Q$  supplied to the gas is

$Q = \text{mass} \times \text{specific heat (at constant volume)} \times \text{rise in temperature}$

$$= m \times c_v \times (T_2 - T_1)$$

$$= mc_v dT$$

(15.6)

= increase in internal energy ( $dU$ ).

Thus the increase in internal energy of a gas = specific heat at constant volume  $\times$  mass of the gas  $\times$  increase in temperature =  $c_v \cdot m \cdot dT$ .

(ii) *Specific heat at constant pressure ( $c_p$ )*

Let the same  $m$  gm of the gas be now confined in a cylinder fitted with a *weightless, free-moving and frictionless piston* (Fig. 15.4). When the gas is heated, its temperature and volume increase but due to the movement of the piston the pressure remains constant (equal to the external pressure  $P$ ). Heating is, therefore, done at constant pressure and required for two purposes :

(a) *for doing external work*

Let the cross-section of the piston be  $A$  and let it move up through a distance  $x$  during heating. Since  $P$  is the pressure, the force on the piston

$$F = P \times A$$

So,

Work done = force  $\times$  distance

$$= P \cdot A \times x = PA \times$$

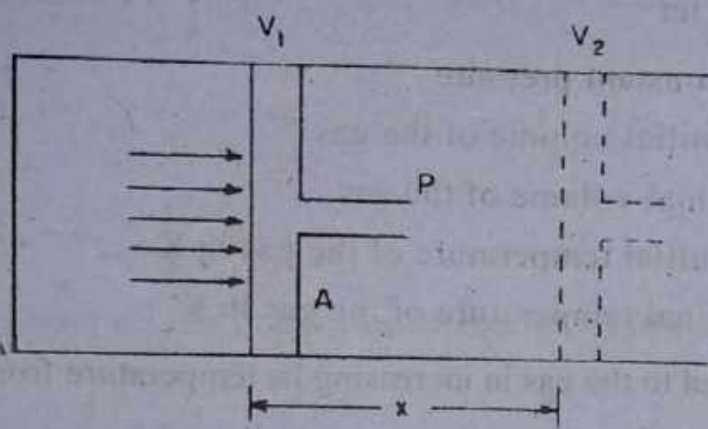


Fig. 15.4

Now  $A \times x$  = increase in volume =  $dV$ .

$$\therefore \text{Work done} = P \times dV = P (V_2 - V_1)$$

where  $V_1$  and  $V_2$  are the initial and final volumes.

(b) *for raising the temperature or internal energy of the gas*

It has been found experimentally that when an ideal gas is heated through the same range of temperature whether at constant pressure or at constant volume, the increase in internal energy is the same in both the cases as it should be according to kinetic theory. Thus the increase in internal energy in this case is also  $c_v \cdot m \cdot dT$ .

If  $c_p$  is the specific heat at constant pressure then the heat  $Q$  supplied to increase the temperature of  $m$  gm of the gas from  $T_1^\circ\text{C}$  to  $T_2^\circ\text{C}$  is

$$Q = c_p \cdot m \cdot (T_2 - T_1) = c_p \cdot m \cdot dT$$



$$\text{or, } c_p \cdot m \cdot dT = c_v \cdot m \cdot dT + P \cdot dV \quad (15.8)$$

From eqn. (15.7) it is quite obvious that during heating at constant pressure, an extra amount of work has to be done for the same rise in temperature.  $c_p$  is, therefore, greater than  $c_v$  by that much amount.

#### 15.4 Relation between the two specific heats

Let  $m$  gm of a perfect gas be heated at constant pressure in a container fitted with a weightless, freemoving and frictionless piston (Fig. 15.4). And let

$P$  = constant pressure

$V_1$  = initial volume of the gas

$V_2$  = final volume of the gas

$T_1$  = initial temperature of the gas in K

$T_2$  = final temperature of the gas in K

Heat supplied to the gas in increasing its temperature from  $T_1$  to  $T_2$   
 $= c_p \cdot m \cdot (T_2 - T_1)$  cal.

where  $c_p$  is the specific heat at constant pressure (for 1 gm).

This heat is partly utilized in increasing the internal energy of the gas and partly in doing external work.

Heat utilized in increasing internal energy

$$= c_v \cdot m \cdot (T_2 - T_1) \text{ cal.}$$

where  $c_v$  is the specific heat at constant volume (for 1 gm)

Work done against the external pressure  $P$  when the volume increases from  $V_1$  to  $V_2$

$$= P \cdot (V_2 - V_1) \text{ ergs}$$

Thermal equivalent or heat required for this work

$$= \frac{P(V_2 - V_1)}{J} \text{ cal}$$

where  $J$  is the mechanical equivalent of heat.



$$\therefore c_p \cdot m \cdot (T_2 - T_1) = c_v \cdot m \cdot (T_2 - T_1) + \frac{P \cdot (V_2 - V_1)}{J} \quad (15.8)$$

Using the general gas equation  $PV = mrT$  where  $r$  is the gas constant for 1 gm of the gas, we have

$$PV_1 = mrT_1 \text{ and } PV_2 = mrT_2$$

$$\text{Hence } P(V_2 - V_1) = mr(T_2 - T_1)$$

Eqn. (15.9), therefore, becomes

$$c_p \cdot m \cdot (T_2 - T_1) = c_v \cdot m \cdot (T_2 - T_1) + \frac{rm(T_2 - T_1)}{J}$$

$$\text{or, } c_p = c_v + \frac{r}{J}$$

$$\text{or, } c_p - c_v = \frac{r}{J} \text{ cal} \quad (15.9)$$

Expressing all the quantities of eqn. (15.9) in ergs or joules, we have

$$c_p - c_v = r \quad (15.11)$$

Multiplying both sides of eqn. (15.10) by  $M$ , the molecular weight of the gas, we get

$$c_p \times M - c_v \times M = r \times M$$

$$\text{or, } C_p - C_v = R \quad (15.12)$$

where  $R$  is now the molar (gm - molecular) or universal gas constant.

The result proves that *the difference between the specific heat at constant pressure and the specific heat at constant volume of a gas is equal to the characteristic gas constant of the gas.*

### 15.5 Ratio of the two specific heats – adiabatic index

The ratio of the specific heat at constant pressure to the specific heat at constant volume is represented by  $\gamma$  and is a very important constant in thermodynamics. This is also referred to as the *adiabatic index*.

$$\gamma = \frac{C_p}{C_v}$$

Since  $C_p > C_v$ ,  $\gamma$  is always greater than unity.

Now

$$C_p = C_v + \frac{R}{J} = C_v + AR$$

where  $A = \frac{1}{J} = 0.24 \text{ cal / joule (approx.)}$

$$\therefore \gamma = \frac{C_p}{C_v} = \frac{C_v + AR}{C_v} = 1 + \frac{AR}{C_v}$$

$$\text{or, } \gamma - 1 = \frac{AR}{C_v}; \quad \therefore C_v = \frac{AR}{\gamma - 1}$$

$$\text{Also } C_p = \gamma \cdot C_v = \frac{\gamma \cdot AR}{\gamma - 1}$$

### 15.6 Change in internal energy of a gas

Let  $n$  gm - mol. of a gas be heated from a temperature  $T_1^\circ\text{C}$  to  $T_2^\circ\text{C}$ . Then the increase in the internal energy,

$$\begin{aligned} dU &= n \cdot C_v \cdot (T_2 - T_1) \\ &= \frac{n \cdot AR (T_2 - T_1)}{\gamma - 1} \quad \text{since } C_v = \frac{AR}{\gamma - 1} \end{aligned}$$

Also  $PV_1 = nRT_1$  and  $PV_2 = nRT_2$

$$\text{or, } P(V_2 - V_1) = nR(T_2 - T_1)$$

$$\text{or, } T_2 - T_1 = \frac{P(V_2 - V_1)}{nR}$$

Substituting this value of  $(T_2 - T_1)$ , we get

$$dU = \frac{P(V_2 - V_1)}{nR} \text{ in cal.}$$

$$= \frac{J \cdot A \cdot P(V_2 - V_1)}{\gamma - 1} \text{ in ergs}$$

$$= \frac{P(V_2 - V_1)}{\gamma - 1} \text{ in ergs}$$



If we consider 1 gm-mol. of a gas, then

$$dU = \frac{RA(T_2 - T_1)}{\gamma - 1} \text{ in calcs.}$$

$$= \frac{R(T_2 - T_1)}{\gamma - 1} \text{ in ergs.}$$

### 15.7 Regnault's method for determination of $c_p$ of a gas

Regnault determined the specific heat of a gas at constant pressure. His experimental arrangement is shown in Fig. 15.5.

R represents a reservoir of pure and dry compressed gas whose specific heat is to be determined. The reservoir is kept immersed in a constant temperature water bath whose temperature is recorded by the thermometer  $T_1$ . The flow of the gas from the reservoir is controlled by the regulatory value V. A mercury manometer M records the pressure of the gas.

$H_1$  is a long spiral immersed in a hot oil bath which can be maintained at any constant temperature. The gas flows through this coiled pipe and is heated to the temperature  $t_2^\circ\text{C}$  of the oil-bath. Next, it enters another coil placed in a calorimeter C of known water equivalent and containing a liquid of known specific heat. While passing through the spiral tube in the calorimeter C, the gas heats up the liquid and the calorimeter and finally escapes to the atmosphere at a temperature  $t_3^\circ\text{C}$  which is also the final temperature of the calorimeter

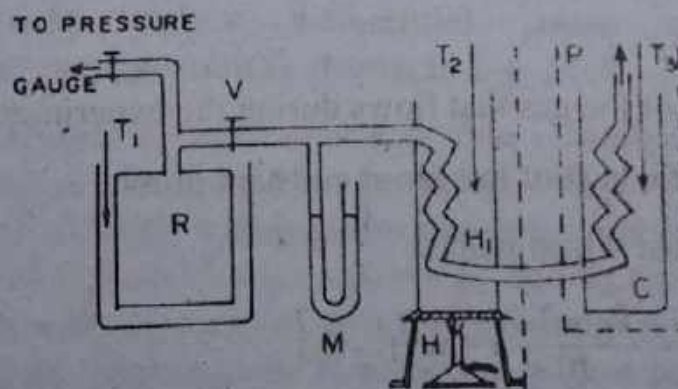


Fig. 15.5



and its contents. The flow of gas entering and leaving the coil in the calorimeter is maintained steady and at constant pressure by continuous adjustment of the regulating valve V. As the gas pressure inside R falls, the regulating valve is opened more and more. This ensures that the pressure of the gas as it enters the oil bath and the calorimeter, is always kept constant, although the pressure of the gas in R falls.

In order that the heated gas may not lose heat while passing from the hot bath to the calorimeter, the passing tube is covered by an insulating covering. To minimize the direct transmission of heat from the hot oil bath to the calorimeter, a non-conducting partition (dotted lines) is placed between the calorimeter and the hot oil bath. The calorimeter itself is kept inside a suitable enclosure (dotted lines) to minimize loss of heat due to conduction, convection and radiation.

Let  $t_2$  and  $t_3$  denote the initial and final temperature of the calorimeter and  $w$  the water equivalent of the calorimeter and its contents. Then the heat gained by the system during the flow of the gas is given by

$$\text{Heat gained} = w (t_3 - t_2)$$

Now the average temperature of the calorimeter during the flow of the gas is  $\frac{T_2 + T_3}{2}$  and if  $t_1$  denotes the temperature of the oil bath, we have

$$\text{Heat lost} = mc_p \left( t_1 - \frac{T_2 + T_3}{2} \right)$$

where

$m$  = mass of the gas that flows during the experiment

$c_p$  = specific heat of the gas at constant pressure.

Since, heat lost = heat gained

$$mc_p \left( t_1 - \frac{T_2 + T_3}{2} \right) = w (t_3 - t_2) \quad (15.13)$$

If the mass of the gas that flows out during the experiment can be determined,  $c_p$  can be found out.

The mass of the gas flowing out can be estimated in the following manner.

Let  $P_1$ ,  $\rho_1$  denote the initial pressure and density of the gas in the reservoir and  $P_2$ ,  $\rho_2$  the final pressure and density. If  $T$  denotes the absolute temperature of the gas in the reservoir and  $\rho_0$  the density of the gas at  $0^\circ\text{C}$ , we have, from the gas laws

$$\frac{\rho_1 T}{P_1} = \frac{\rho_2 T}{P_2} = \frac{\rho_0 \cdot 273}{76}$$

$$\text{or, } \frac{(\rho_1 - \rho_2) T}{P_1 - P_2} = \frac{\rho_0 \cdot 273}{76}$$

$$\text{or, } (\rho_1 - \rho_2) = \frac{\rho_0 \cdot 273}{76} \cdot \frac{(P_1 - P_2)}{T}$$

$\therefore m = \text{mass of the gas flowing out}$

$$= (\rho_1 - \rho_2) v = \frac{\rho_0 \cdot 273 \cdot (P_1 - P_2)}{76 \cdot T} \quad (15.14)$$

where  $v$  is the volume of the reservoir.  $(P_1 - P_2)$  can be determined from the initial and final reading of the manometer.

The mass of the gas flowing out being thus known,  $c_p$  can be determined from eqn. (15.13) since,  $t_1$ ,  $t_2$ ,  $t_3$  and  $w$  are all experimentally determined.

### 15.8 Determination of $c_v$ by Joly's differential steam calorimeter.

The specific heat of a gas at constant volume can be accurately determined by Joly's differential steam calorimeter. The experimental arrangement is shown in Fig. 15.6.

In the figure  $V$  represents a double-walled metal enclosure, called the steam chamber. It is provided with an inlet pipe  $I$  and an outlet pipe  $O$  through which steam can enter and leave the chamber. Two identical hollow copper spheres  $S_1$  and  $S_2$  are suspended inside the chamber with the help of two thin platinum wires from the pans of a balance  $B$ . Each sphere is provided with a pair of small metal plates at the bottom and the top, called *catch* and *shelter* respectively. The *catch* holds the condensed steam and the *shelter*







heat the gas also. The equilibrium of the balance is disturbed which is restored by placing additional weights on the scale pan. This gives the excess of steam condensed.

Let

$m$  = mass of the gas enclosed in sphere  $S_1$ .

$c_v$  = specific heat of the gas at constant volume

$w$  = excess of steam condensed

$t_1$  = initial temperature of the steam-chamber

$t_2$  = final temperature of the steam-chamber

$L$  = latent heat of steam.

Then,

heat gained by the gas =  $mc_v(t_1 - t_2)$

heat lost by steam =  $wL$

So, from heat lost = heat gained,

we have  $mc_v(t_2 - t_1) = wL$

$$\text{or, } c_v = \frac{wL}{m(t_2 - t_1)} \quad (15.15)$$

In the above relation all quantities excepting  $c_v$  are known. Hence  $c_v$  can be determined.

The heat required by the sphere  $S_1$  itself does not appear in the expression for the simple reason that it is compensated for by that taken by the sphere  $S_2$ .

To get accurate results the following corrections will be necessary :

(i) The weighings at  $t_1^0\text{C}$  and  $t_2^0\text{C}$  are made in air and steam respectively. These should be corrected for buoyancy and reduced to the weights in vacuum.

(ii) In practice, it is difficult to get two spheres of exactly equal thermal capacities. The error due to this is removed by repeating the experiment by evacuating and filling the sphere  $S_2$  while keeping  $S_1$  empty. The mean value of  $c_v$  is then taken.

(iii) The volume of the gas inside  $S_1$  does not remain constant due to expansion of the sphere. If  $h$  is the amount of heat necessary to do work during this expansion of the gas, then the eqn. 15.15 is modified to

$$wL - h = mc_v (t_2 - t_1).$$

### 15.9 Determination of the ratio of the specific heats ( $\gamma = \frac{C_p}{C_v}$ ).

The ratio of the specific heats of a gas at constant pressure and at constant volume, denoted by  $\gamma$ , can be experimentally determined by various methods. For common laboratory work the method due to Clement and Desormes is generally followed and is described below.

Clement and Desormes apparatus consists of a glass reservoir A with three side tubes. One of the tubes is connected to a liquid manometer M whose liquid has low density and low vapour pressure. The two other tubes are provided with stopcocks T and D. In this method it is supposed that the air occupying the volume below the dotted line (Fig. 15.7) remain in the flask all the time.

To start with, the stop-cock T is kept closed, D is opened and some dry air is compressed into the reservoir A. The gas inside is allowed to acquire the temperature of the surroundings. The pressure is calculated from the difference in the levels of the liquid in the manometer.

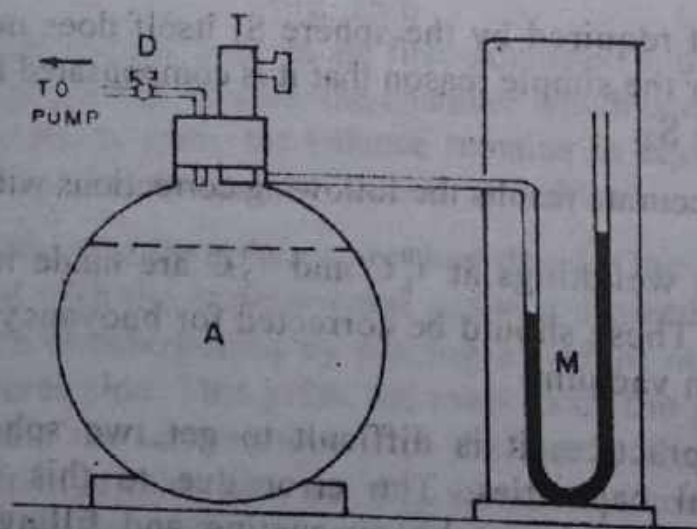


Fig. 15.7



Next the stop-cock T is opened for a very short time and then quickly closed after giving it one half-turn. During the short time for which the stop-cock remains open, the air from the reservoir rushes out to equalise the pressure with the outside atmosphere. The whole process takes place in so short a time that it may be supposed that there is no exchange of heat between the reservoir and its surroundings. The condition of the expansion may, therefore, be regarded as adiabatic and so the temperature of the gas inside falls. The inside gas is allowed to warm up and return to the temperature at the beginning of the experiment (room temperature). During the process, the pressure in A increases slightly as shown by the difference in the levels of the liquid in the manometer limbs. It does not quite recover its original value.

Let the volume of the entire flask be  $V$ , while that below the dotted line be  $V_0$ . If the initial pressure be  $P_0$  and that *immediately* after the adiabatic expansion be  $B$ , then

$$P_0 V_0^\gamma = BV^\gamma \quad (i)$$

After the adiabatic expansion, the gas inside A is allowed to return to the room temperature. When the temperature has become steady, the pressure will be different from  $B$ . Let this new pressure be  $P$ . Thus the gas has passed from volume  $V_0$  and pressure  $P_0$  by an isothermal process to volume  $V$  and pressure  $P$ . Then

$$P_0 V_0 = PV$$

$$\text{or, } P_0^\gamma V_0^\gamma = P^\gamma V^\gamma \quad (ii)$$

Dividing eqn. (ii) by (i), we have

$$\frac{P^\gamma}{B} = P_0^{\gamma-1}$$

$$\text{or, } \left(\frac{P}{P_0}\right)^\gamma = \frac{B}{P_0}$$

$$\text{or, } \gamma = \frac{\log \frac{B}{P_0}}{\log \frac{P}{P_0}}$$



Let the initial and final difference in heights of the manometer liquid level be  $h_0$  and  $h$  respectively. Then, if pressure be measured in terms of the heights of the liquid columns

$$P_0 = B + h_0, \quad P = B + h$$

where  $B$  denotes the atmospheric pressure as indicated by the barometer reading. Then

$$\gamma = \frac{\log \left(1 - \frac{h_0}{P_0}\right)}{\log \left(1 - \frac{h_0 - h}{P_0}\right)}$$

$$= \frac{\frac{h_0}{P_0}}{\frac{h_0 - h}{P_0}} \text{ approximately.}$$

by expansion in logarithmic series and neglecting higher powers of  $\frac{h_0}{P_0}$  and  $\frac{h_0 - h}{P_0}$  than the first. This is permissible since the values of  $h$  involved are only a few centimeters.

$$\text{Hence, } \gamma = \frac{h_0}{h_0 - h} \quad (15.16)$$

Clement and Desormes method commonly used in the laboratory does not give accurate results. To get accurate result  $\gamma$  is determined by Partington and Shilling's method which involves the determination of the velocity of the sound in the gas by Kundt's tube.  $\gamma$  is then determined from the formula  $v = \sqrt{\frac{\gamma P}{\rho}}$  where  $v$  denotes the velocity of sound,  $P$  the pressure of the gas and  $\rho$  the density of the gas.

**Example 15.7.** Find the value of the universal gas constant  $R$  for one gram-molecule of a gas.

**Soln.**

Let  $P_0$ ,  $V_0$  and  $T_0$  be the pressure, volume and temperature of 1 gm-molecule of any gas at N.T.P. Then we have

$$P_0 V_0 = RT_0; \quad \text{or, } R = \frac{P_0 V_0}{T_0}$$

where  $R$  is the gas constant for 1 gm-molecule of a gas (gm-molecular constant or molar gas constant of the gas).

Now one gm-molecule of a gas at N.T.P. occupies a volume of 22.4 litres. Hence

$$V_0 = 22.4 \text{ litres} = 22,400 \text{ c.c.}$$

$$P_0 = 76 \times 13.6 \times 981 \text{ dynes / cm}^2$$

$$T_0 = 273^\circ\text{K}$$

$$R = \frac{76 \times 13.6 \times 981 \times 22,400}{273}$$

$$= 8.31 \times 10^7 \text{ ergs per degree absolute per gm - molecule (ergs/mole - K)}$$

$$= \frac{8.31 \times 10^7 \text{ ergs per degree absolute per gm - molecule}}{4.17 \times 10^7 \text{ ergs per calorie}}$$

$$= 1.98 \text{ cals per degree per gm - molecule (cals/mole - K)}$$

For rough calculations the value may be taken to be 2 calories.

Again, if

$$P_0 = 1 \text{ atmosphere}$$

$$V_0 = 22.4 \text{ litres}$$

$$R = \frac{P_0 V_0}{T_0} = \frac{1 \times 22.4}{273}$$

$$= 0.082 \text{ litre atmos per degree per gm - molecule (litre - atmos/mole - K)}$$

**Example 15.8.** Calculate the gas constant for one gram of a gas.

**Soln.**

It is found that the volume occupied by one gramme of different gases under the same conditions of temperature and pressure are different. However, the value of the gas constant for one gm-molecule will be same for all gases under the same conditions of temperature and pressure.



Let the gas constant for one gram of a gas be denoted by  $r$ ,

(i) For one gram of hydrogen, let

$$PV = rT; \text{ or, } r = \frac{PV}{T}$$

where  $V$  is the volume occupied by 1 gm. of hydrogen. If the density of hydrogen at N.T.P. is 0.00009 gm/c.c., then

$$V = \frac{1}{0.00009} \text{ c.c.}$$

(since density = mass / volume

$$\text{or, volume} = \frac{\text{mass}}{\text{density}} = \frac{1}{0.00009} \text{ c.c.}).$$

$$\therefore r = \frac{76 \times 13.6 \times 981}{273 \times 0.00009}$$

$$= 4.13 \times 10^7 \text{ ergs/degree/gm.}$$

Since 1 gm molecule of hydrogen contains 2 gm, the gas constant for 1 gm-molecule of hydrogen

$$R = 2 \times 4.13 \times 10^7$$

$$= 8.26 \times 10^7 \text{ ergs/degree/gm-molecule.}$$

(ii) For 1 gm of oxygen (density = 0.0014286 gm/c.c. at N.T.P.)

$$\therefore r = \frac{PV}{T} = \frac{76 \times 13.6 \times 981}{273 \times 0.0014286}$$

$$= 2.60 \times 10^6 \text{ ergs/degree/gm}$$

Since 1 gm-molecule of oxygen contains 32 gms of oxygen,

$$R = 2.60 \times 10^6 \times 32 = 8.32 \times 10^7 \text{ ergs/degree/gm-molecule}$$

**Example 15.9.** Calculate the specific heat of air at constant volume, given that specific heat at constant pressure is 0.23, density of air N.T.P. 1.293 gram/litre and  $J = 4.2 \times 10^7$  ergs/cal.

**Solu.**



$$c_p = 0.23, J = 4.2 \times 10^7 \text{ ergs/cal}, c_v = ?$$

Density of air at N.T.P. = 1.293 gm/litre

$$= \frac{1.293}{1000} \text{ gm / c.c.}$$

$\therefore$  volume of 1 gm of air at N.T.P.

$$= \frac{1000}{1.293} \text{ c.c.}$$

From  $PV = rT$

$$r = \frac{PV}{T}$$

where  $r$  is the gas constant for 1 gm of air

$$r = \frac{76 \times 13.6 \times 980 \times 1000}{273 \times 1.293}$$

$$c_p - c_v = \frac{r}{J}$$

$$\text{or, } c_v = c_p - \frac{r}{J}$$

$$= 0.23 - \frac{76 \times 13.6 \times 980 \times 1000}{273 \times 1.293 \times 4.2 \times 10^7}$$

$$= 0.23 - 0.0693$$

$$= 0.1617$$

**Example 15.10.** Calculate the difference in the two specific heats of one gram of helium, given that the molecular weight of helium = 4 and the gram-molecular volume of helium at N.T.P. = 22.4 litres.

**Soln.**

$$c_p - c_v = \frac{r}{J} \text{ for 1 gm of a gas}$$

volume occupied by 1 gm-molecule i.e., 4 gm of helium at N.T.P. is 22.4 litres. Hence the volume occupied by 1 gm of helium.

$$V = \frac{22.400}{4} \text{ c.c.} = 5600 \text{ c.c.}$$

From  $PV = rT$

$$r = \frac{PV}{T} = \frac{76 \times 13.6 \times 980 \times 5600}{273}$$

$$c_p - c_v = \frac{r}{J}$$

$$= \frac{76 \times 13.6 \times 980 \times 5600}{273 \times 4.2 \times 10^7}$$

$$= 0.4946$$

**Example 15.11.** One kg-molecule of helium occupies 22.42 m<sup>3</sup> at N.T.P. If the value of molar specific heat at constant volume of helium is 3.0 Kcal per kg-mole per K, find the value of molar specific heat at constant pressure of helium. Given  $g = 9.8 \text{ m/sec}^2$ , density of mercury =  $13.6 \times 10^3 \text{ kg/m}^3$  and  $J = 4200 \text{ joules per kcal}$ .

**Soln.**

$$C_p - C_v = \frac{R}{J}$$

The molar gas constant  $R$  for helium can be obtained from the equation

$$PV = RT$$

$$\text{or, } R = \frac{PV}{T}$$

$$= \frac{0.76 \times 13.6 \times 10^3 \times 9.8 \times 22.42}{273}$$

$$= 8320 \text{ joules / kg - mole - K}$$

$$\therefore C_p - C_v = \frac{R}{J} = \frac{8320}{4200} = 1.981$$

$$\text{or, } C_p = C_v + 1.981 = 3.0 + 1.981$$

$$= 4.981 \text{ kcal/kg-mole-K.}$$

**Example 15.12.** Find the value of the mechanical equivalent of heat, given the following data for hydrogen : specific heats at constant pressure and constant volume 3.42 and 2.42 cal/gm-K respectively, density of hydrogen at N.T.P. is 0.0887 gm/litre.

**Soln.**

From  $PV = m r T$ , we get

$$P = \frac{m}{V} r T = \rho r T$$

$$\begin{aligned} \rho &= \text{density of the gas} = 0.0887 \text{ gm / litre} \\ &= 0.0887 \times 10^{-3} \text{ gm/c.c.} \end{aligned}$$

$$P = 76 \times 13.6 \times 981 \approx 10^6 \text{ dynes/cm}^2$$

$$T = 273 \text{ K}$$

So, the gas constant,  $r$ , for one gram of hydrogen is

$$\begin{aligned} r &= \frac{P}{\rho T} = \frac{10^6}{0.0887 \times 10^{-3} \times 273} \\ &= 4.13 \times 10^7 \text{ ergs/gm - K.} \end{aligned}$$

From

$$c_p - c_v = \frac{r}{J}$$

we have

$$\begin{aligned} J &= \frac{r}{c_p - c_v} = \frac{4.13 \times 10^7}{3.42 - 2.42} \\ &= 4.13 \times 10^7 \text{ ergs/cal.} \\ &= 4.13 \text{ joules/cal.} \end{aligned}$$

**Example 15.13.** The density of oxygen at normal pressure and a temperature of  $27^\circ\text{C}$  is  $1.28 \text{ kg/m}^3$ . Its specific heat at constant pressure is 1050 joules per kg per K.

Calculate



- (a) the gas constant per kg of oxygen  
 (b) the specific heat of oxygen at constant volume in kcal per kg per K.

Given  $g = 9.8 \text{ m/sec}^2$ , density of mercury  $= 13.6 \times 10^3 \text{ kg/m}^3$ .

$$J = 4186 \text{ joules/kcal.}$$

**Soln.**

$$(a) \quad PV = mRT;$$

$$\text{or, } P = \frac{m}{V} RT = \rho RT$$

where  $\rho$  is the density of oxygen.

$$P = 0.76 \times 9.8 \times 13.6 \times 10^3$$

$$= 1.013 \times 10^5 \text{ N/m}^2$$

$$T = (27 + 273) \text{ K} = 300 \text{ K}$$

$$r = 1.28 \text{ Kg/m}^3.$$

$$\therefore R = \frac{P}{\rho T} = \frac{1.013 \times 10^5}{1.28 \times 300}$$

$$= 360 \text{ joules /Kg - K.}$$

$$(b) \quad C_p - C_v = R$$

$$\text{or, } C_v = C_p - R$$

$$= 1050 - 260$$

$$= 790 \text{ joules per Kg per K}$$

$$= \frac{790}{4186} = 0.19 \text{ kcal/Kg - K.}$$

**Example 15.14.** The specific heat of hydrogen at constant pressure is  $3.41 \text{ cal/gm-K}$  and the ratio of the specific heat at constant pressure to that at constant volume is  $1.41$ . Obtain a value for the mechanical equivalent of heat in joules/cal. Assume that the density of hydrogen at N.T.P. is  $0.09 \text{ gm/litre}$ .

**Soln.**

$$PV = mrT; \quad \text{or, } P = \frac{m}{V} rT = \rho rT$$

where  $r$  is the density of hydrogen.

At N.T.P.

$$P = 76 \times 13.6 \times 981 \text{ dynes/cm}^2$$

$$= 1.013 \times 10^6 \text{ dynes/cm}^2$$

$$T = (0 + 273) \text{ K} = 273 \text{ K}$$

$$r = 0.09 \text{ gm/litre}$$

$$= 0.09 \times 10^{-3} \text{ gm/c.c.}$$

$$\therefore r = \frac{P}{\rho T} = \frac{1.013 \times 10^6}{0.09 \times 10^{-3} \times 273}$$

$$= 0.412 \times 10^8 \text{ ergs/gm.K.}$$

$$\text{Now } c_p = 3.41 \text{ and } \frac{c_p}{c_v} = 1.41$$

$$\therefore c_v = \frac{3.41}{1.41} = 2.42$$

$$\text{From } c_p - c_v = \frac{r}{J}, \text{ we get}$$

$$J = \frac{r}{c_p - c_v} = \frac{0.412 \times 10^8}{3.41 - 2.42}$$

$$= \frac{0.412 \times 10^8}{0.99}$$

$$= 0.416 \times 10^8 \text{ ergs/gm-K}$$

$$= 4.16 \times 10^7 \text{ ergs/gm-K}$$

$$= 4.16 \text{ joules/gm-K.}$$

**Example 15.15.** The specific heat of hydrogen at constant pressure is 3.405 cal/gm-K. The volume of 1 gram of hydrogen at N.T.P. is 11.2 litres. If the mechanical equivalent of heat is 4.2 joules/cal, find the ratio of specific heats of hydrogen.

**Soln.**

$$PV = mrT; \text{ or, } P = \frac{m}{V} rT$$

$$= \rho rT$$

$$\text{or, } r = \frac{P}{\rho T}$$

where  $r$  is the gas constant for 1 gram of the gas. At N.T.P.

$$P = 76 \times 13.6 \times 981$$

$$= 1.013 \times 10^6 \text{ dynes/cm}^2.$$

$$T = 273 \text{ K}$$

$$\rho = \frac{1}{11.2} = 0.0893 \text{ gm/litre}$$

$$= 0.893 \times 10^{-3} \text{ gm/c.c.}$$

$$\therefore r = \frac{P}{\rho T} = \frac{1.013 \times 10^6}{0.0893 \times 10^{-3} \times 273}$$

$$= 4.16 \times 10^7 \text{ ergs/gm - K.}$$

$$\text{Now } c_p - c_v = \frac{r}{J} = \frac{4.16 \times 10^7}{4.2 \times 10^7} = 0.99$$

$$\therefore c_v = c_p - 0.99$$

$$= 3.405 - 0.99$$

$$= 2.415 \text{ cal/gm - K.}$$

Ratio of specific heats,

$$\gamma = \frac{c_p}{c_v} = \frac{3.405}{2.415} = 1.41.$$



### 15.10 General heat-energy equation of a gas – first law of thermodynamics

In general, when heat is supplied to a gas, a portion of it increases the internal energy of the gas and the rest is utilized in the external work done by the gas. If

$dQ$  = the heat supplied to the gas

$dU$  = the increase in internal energy of the gas

$dW$  = the work done by the gas,

then, according to law of conservation of energy,

$$dQ = dU + \frac{dW}{J} \quad (15.17)$$

where  $dW/J$  is the heat equivalent of the work done,  $J$  being the mechanical equivalent of heat.

Eqn. (15.17) represents the *first law of thermodynamics* and states that *in all transformations the amount of heat given to a system must be balanced by the sum of the gain in the internal energy of the system due to the rise in temperature and the external work done.*

### 15.11 Isothermal change

Isothermal changes are those which take place at some constant temperature. A gas is heated when compressed and is cooled when allowed to expand. If the compression is to take place isothermally, the heat generated due to compression must come out of the gas by radiation or otherwise to maintain the temperature constant. Again when the gas expands, it cools and to keep the temperature constant, the gas must absorb heat from the surroundings. Thus to secure this *isothermal* (or constant temperature) condition, the compression or expansion must take place in a vessel which must be a good conductor of heat *i.e.*, the vessel must be in perfect thermal contact with its surroundings. However, since no vessel can be regarded as a perfect conductor of heat, the compression or expansion must take place very slowly so that the gas gets sufficient time to lose or gain heat, as the case may be, by conduction and radiation. Therefore, for a process to be isothermal, it

(i) must take place in a vessel which is a good conductor of heat and

(ii) should be an extremely slow process.

Since in an isothermal change there is no change in temperature, i.e.,  $dT = 0$ , there is no change in internal energy of the gas, i.e.,  $dU = 0$ .

The general heat-energy equation of the gas then becomes

$$dQ = 0 + dW/J = dW/J \quad (15.18)$$

or, heat added = work done by the gas

conversely,

heat subtracted = work done on the gas.

### relation between pressure and volume in an isothermal change

Boyle's law which states that the volume of a gas varies inversely as the pressure to which it is subjected, provided temperature remains constant, represents the isothermal change of a gas. The isothermal equation of a perfect gas is, therefore,

$$PV = \text{constant}.$$

### 15.12 Adiabatic change

Adiabatic changes are those in which there is no exchange of heat between the gas and its surroundings i.e., the gas neither receives nor gives out heat to its surroundings. Thus, when a gas is compressed adiabatically, it gets heated and when it is allowed to expand adiabatically it gets cooled. For adiabatic change to take place, the vessel containing the gas must be a perfect non-conductor of heat. In practice, however, there is no such substance, which is a perfect non-conductor of heat. Therefore, in order that the change should be adiabatic, the operation should be so quick that the system does not get time for the exchange of heat with its surroundings. Thus, for a change to be adiabatic, it

(i) should take place in a vessel which is a perfect non-conductor of heat and

(ii) should be an extremely quick process.



Since there is no exchange of heat in an adiabatic process,  $dQ = 0$ . But since there is an increase or decrease of temperature, there will be an increase or decrease in internal energy of the system. The general heat-energy equation then becomes,

$$0 = dU + dW/J \quad (15.19)$$

which means that the increase or decrease in the internal energy of the gas is exactly equal to the work done on or by the gas.

### 15.13 Isochoric and isobaric processes

If the gas is heated in a non-expanding chamber, the heat supplied will increase the pressure and temperature but the volume will remain constant. Such a process is called an *isochoric process*. The work done in this case is zero since there is no increase in volume. The general heat-energy equation then reduces to

$$dQ = dU \quad \text{since } dW = 0.$$

Hence the heat transferred in such a process is

$$dQ = dU = nC_v dT \quad (15.20)$$

where  $n$  is the number of gam-mol. of the gas and  $C_v$  is the molar specific heat of the gas at constant volume.

A process taking place at constant pressure is called an *isobaric process*.

The general heat-energy equation for such a process is

$$dQ = dU + dW/J$$

### 15.14 Adiabatic equation of a perfect gas

Let us consider 1 gm-molecule of a perfect gas undergoing an adiabatic change. Since no heat either goes into or comes out of the vessel  $dQ = 0$ . The general heat-energy equation

$$dQ = dU + dW/J$$

then becomes

$$0 = dU + dW/J$$



Let us now find the values of  $dU$  and  $dW$ . When the gas, expands by an small amount  $dV$ , then the work done by the gas,  $dW = P.dV$ . If the temperature of the gas falls by  $dT$  as a result of this expansion, then the change (decrease) in internal energy is  $dU = nC_v.dT = C_v.dT$  ( $n = 1$ )

$$\therefore C_v.dT + \frac{P.dV}{J} = 0$$

$$\text{or, } -C_v.dT = \frac{P.dV}{J} \quad (15.21)$$

Differentiating the general gas equation  $PV = RT$ , we get

$$P.dV + V.dP = R.dT$$

But  $C_p - C_v = R/J$ ; hence  $R = (C_p - C_v) J$

Substituting this value of  $R$  in eqn. (15.22), we have

$$(C_p - C_v) J.dT = P.dV + V.dP \quad (15.23)$$

Dividing eqn. (15.23) by eqn. (15.21), we get

$$-\frac{(C_p - C_v) J.dT}{C_v.dT} = \frac{(P.dV + V.dP) J}{P.dV}$$

$$\text{or, } -\frac{C_p - C_v}{C_v} = \frac{P.dV + V.dP}{P.dV}$$

$$\text{or, } -\frac{C_p}{C_v} + 1 = 1 + \frac{V}{P} \cdot \frac{dP}{dV}$$

$$\text{or, } -\gamma = \frac{V}{P} \cdot \frac{dP}{dV}$$

where  $C_p/C_v = \gamma$  = ratio of the specific heat at constant pressure of the gas to its specific heat at constant volume.

$$\text{or, } -\gamma \cdot \frac{dV}{V} = \frac{dP}{P}$$

$$\text{or, } \gamma \frac{dV}{V} + \frac{dP}{P} = 0$$

Integrating both sides, we get

$$\gamma \cdot \log_e V + \log_e P = \log_e K$$

where  $K$  is a constant.

$$\text{or, } \log_e PV^\gamma = \log_e K$$

$$\text{or, } PV^\gamma = K = \text{constant} \quad (15.24)$$

Eqn. (15.24) is known as the *adiabatic equation of a perfect gas*.

If  $P_1, V_1$  be pressure and volume respectively of a quantity of a gas, and  $P_2, V_2$  their corresponding values after an adiabatic change, then

$$P_1 V_1^\gamma = P_2 V_2^\gamma = \text{constant.}$$

$$\text{or, } P_1 V_1^\gamma = P_2 V_2^\gamma = \dots = \text{a constant}$$

### 15.15 Relation between volume and temperature in an adiabatic change

Proceeding from the beginning exactly in the manner as followed in the deduction of  $PV^\gamma = \text{const.}$ , we have

$$C_v dT + \frac{PdV}{J} = 0 \quad (15.25)$$

From gas equation  $PV = RT$ , we have  $P = \frac{RT}{V}$ . Substituting for  $P$  in eqn. (15.25), we get

$$C_v dT + \frac{RT}{J} \cdot \frac{dV}{V} = 0$$

$$\text{or, } C_v \frac{dT}{T} + \frac{R}{J} \frac{dV}{V} = 0$$

$$\text{or, } C_v \frac{dT}{T} + (C_p - C_v) \frac{dV}{V} = 0 \quad (\because \frac{R}{J} = C_p - C_v)$$

Dividing by  $C_v$ ,

$$\frac{dT}{T} + \left( \frac{c_p}{c_v} - 1 \right) \frac{dV}{V} = 0$$

$$\text{or, } \frac{dT}{T} + (\gamma - 1) \frac{dV}{V} = 0$$

Integrating

$$\log T + (\gamma - 1) \log V = \text{a constant} = \log K \text{ (say)}$$

$$\text{or, } \log TV^{\gamma-1} = \log K$$

$$\text{or, } TV^{\gamma-1} = K = \text{a constant} \quad (15.26)$$

The more useful form of eqn. (15.26) is

$$T_1 V_1^{\gamma-1} = T_2 V_2^{\gamma-1} = \dots = \text{a constant.}$$

Alternately let us eliminate P from equation (15.24) with the help of the general gas equation  $PV = RT$  from which we get  $P = \frac{RT}{V}$ . Substituting this value of P in eqn. (15.24), we get

$$\left( \frac{RT}{V} \right) V^{\gamma} = K = \text{constant}$$

$$\text{or, } TV^{\gamma-1} = \frac{K}{R} = K_1 = \text{another constant}$$

$$\text{or, } T_1 V_1^{\gamma-1} = T_2 V_2^{\gamma-1} = \dots = \text{constant.}$$

### 15.16 Relation between pressure and temperature in an adiabatic change

Proceeding as in the deduction of  $PV^{\gamma} = K$  we have

$$C_v dT + \frac{PdV}{J} = 0 \quad (15.27)$$

From gas equation  $PV = RT$ , we have  $V = \frac{RT}{P}$ . Again differentiating  $PV = RT$ , we get



$$PdV + VdP = RdT$$

$$\text{or, } PdV = RdT - VdP$$

Substituting these in eqn. (15.27) we get

$$C_v dT + (RdT - \frac{RTdP}{P}) / J = 0$$

$$\text{or, } C_v dT + \frac{R}{J} (dT - \frac{TdP}{P}) = 0$$

$$\text{or, } C_v dT + (C_p - C_v) (dT - \frac{TdP}{P}) = 0$$

$$(\because \frac{R}{J} = C_p - C_v)$$

$$\text{or, } C_p dT - (C_p - C_v) \frac{TdP}{P} = 0$$

$$\text{or, } \frac{dT}{T} - (1 - \frac{1}{\gamma}) \frac{dP}{P} = 0$$

Integrating

$$\log T - (\frac{\gamma-1}{\gamma}) \log P = \log K, \text{ a constant.}$$

$$\text{or, } \frac{T}{P^{\frac{\gamma-1}{\gamma}}} = K$$

$$\text{or, } TP^{1-\gamma/\gamma} = \text{constant} \quad (15.28)$$

$$\text{or, } T^\gamma \cdot P^{1-\gamma} = K^\gamma = \text{a constant} \quad (15.29)$$

The more useful form of eqn. (15.28) is

$$T_1 P_1^{(1-\gamma)/\gamma} = T_2 P_2^{(1-\gamma)/\gamma} = \dots = \text{constant}$$

Alternately let us now eliminate V from eqn. (15.24). From eqn. PV = RT we get  $V = \frac{RT}{P}$ . Substituting this value of V in eqn. (15.24)

$$P. \left( \frac{RT}{P} \right)^\gamma = K = \text{Constant}$$

$$\text{or, } \frac{(RT)^\gamma}{P^\gamma P^{-1}} = K ;$$

$$\text{or, } \frac{T^\gamma}{P^{\gamma-1}} = \frac{K}{R^\gamma} = \text{constant}$$

$$\text{or, } T^\gamma \cdot P^{1-\gamma} = \frac{K}{R^\gamma} = \text{constant}$$

$$\text{or, } TP^{(1-\gamma)/\gamma} = \frac{K^{1/\gamma}}{R^\gamma} = \text{constant}$$

**Example 15.17.** A quantity of dry air at  $27^\circ\text{C}$  is compressed (i) slowly and (ii) suddenly to  $\frac{1}{3}$  rd of its volume. Find the change in temperature in each case, assuming  $\gamma$  to be 1.4 for dry air.

**Soln.**

(i) when the process is slow, the temperature of the system remains constant. Therefore, there is no change in temperature,

$$(ii) \text{ sudden means adiabatic, } \therefore T_1 V_1^{\gamma-1} = T_2 V_2^{\gamma-1}$$

$$\text{or, } T_2 = \left( \frac{V_1}{V_2} \right)^{\gamma-1} \times T_1$$

$$T_1 = (27 + 273)^\circ\text{K} = 300 \text{ K} ; \gamma = 1.4$$

$$V_1 = V \text{ (say)}$$

$$V_2 = \frac{V}{3}, T_2 = ?$$

$$\therefore T_2 = \left( \frac{V}{V/3} \right)^{1.4-1} \times 300$$

$$= 300 \times (3)^{0.4} = 465.5 \text{ K} = 192.5^\circ\text{C}.$$

$$\therefore \text{Increase in temperature} \\ = 192.5 - 27 = 165.6^\circ\text{C}.$$

**Example 15.18.** A motor tyre is pumped to a pressure of 2 atmospheres at  $15^\circ\text{C}$ , when it suddenly bursts. Calculate the resulting drop in temperature, given  $\gamma = 1.4$ .

**Soln.**

$$\frac{T_1^\gamma}{P_1^{\gamma-1}} = \frac{T_2^\gamma}{P_2^{\gamma-1}}$$

$$\text{or, } \left(\frac{T_2}{T_1}\right)^\gamma = \left(\frac{P_2}{P_1}\right)^{\gamma-1}$$

$$\text{or, } T_2 = T_1 \times \left(\frac{P_2}{P_1}\right)^{(\gamma-1)/\gamma}$$

$$T_1 = 15 + 273 = 288 \text{ K}$$

$$P_1 = 2 \text{ atmospheres}$$

$$P_2 = 1 \text{ atmosphere, } T_2 = ?$$

$$T_2 = 288 \times \left(\frac{1}{2}\right)^{1.4-1/1.4}$$

$$= 288 \times (0.5)^{0.4/1.4}$$

$$= 236.2 \text{ K}$$

$$= (236.2 - 273)^\circ\text{C} = -36.8^\circ\text{C}$$

$$\therefore \text{resulting drop in temperature} \\ = 15 - (-36.8) = 51.8^\circ\text{C}.$$

**Example 15.19.** A quantity of dry air at  $27^\circ\text{C}$  and atmospheric pressure is suddenly compressed to half its original volume. Find (i) the final pressure and (ii) the final temperature.  $\gamma = 1.4$ .

**Soln.**

$$(i) \quad P_1 V_1^\gamma = P_2 V_2^\gamma$$



$$\text{or, } P_2 = P_1 \times \left(\frac{V_1}{V_2}\right)^\gamma$$

$$P_1 = 1 \text{ atmosphere, } V_1 = V \text{ (say),}$$

$$V_2 = \frac{V}{2}, P_2 = ?$$

$$\begin{aligned} \therefore P_2 &= 1 \times \left(\frac{V}{V/2}\right)^{1.4} \\ &= 1 \times 2^{1.4} = 2.636 \text{ atmosphere.} \end{aligned}$$

$$(ii) T_1 V_1^{\gamma-1} = T_2 V_2^{\gamma-1};$$

$$\text{or, } T_2 = T_1 \left(\frac{V_1}{V_2}\right)^{\gamma-1}$$

$$T_1 = 27 + 273 = 300 \text{ K,}$$

$$V_1 = V, V_2 = V/2, T_2 = ?$$

$$\begin{aligned} T_2 &= 300 \times \left(\frac{V}{V/2}\right)^{1.4-1} = 300 \times 2^{0.4} \\ &= 395.9 \text{ K} = (395.9 - 273)^\circ\text{C} \\ &= 122.9^\circ\text{C.} \end{aligned}$$

**Example 15.20.** A mass of gas at  $80^\circ\text{C}$  and 50 atmospheric pressure is adiabatically expanded to ten times its initial volume. If the ratio of specific heats is 1.4, calculate the resulting pressure and temperature.

If the total mass of the gas is a gramme-molecule, calculate the work done in the process. The universal gas constant is 2 calories.

**Soln.**

$$P_1 V_1^\gamma = P_2 V_2^\gamma \quad ; \text{ or, } \frac{P_2}{P_1} = \left(\frac{V_1}{V_2}\right)^\gamma$$

$$\text{or, } P_2 = P_1 \left(\frac{V_1}{V_2}\right)^\gamma$$

$$P_1 = 50 \text{ atmos., } V_1 = V \text{ (say),}$$

$$V_2 = 10V, P_2 = ?$$

$$\therefore P_2 = 50 \times \left(\frac{V}{10V}\right)^{1.4}$$

$$= 50 \times (0.1)^{1.4} = 1.99 \text{ atmospheres}$$

Again,

$$T_1 V_1^{\gamma-1} = T_2 V_2^{\gamma-1}$$

$$\text{or, } \left(\frac{T_2}{T_1}\right) = \left(\frac{V_1}{V_2}\right)^{\gamma-1}$$

$$\text{or, } T_2 = T_1 \times \left(\frac{V_1}{V_2}\right)^{\gamma-1}$$

$$T_1 = 80 + 273 = 353 \text{ K,}$$

$$V_1 = V \text{ (say),}$$

$$V_2 = 10V, T_2 = ?$$

$$\therefore T_2 = 353 \times \left(\frac{V}{10V}\right)^{1.4-1}$$

$$= 353 \times (0.1)^{0.4} = 140.6 \text{ K}$$

$$= -132.4^\circ\text{C,}$$

Since it is one gramme-molecule of the gas, the work done during the (adiabatic) process (Art. 15.22),

$$W = \frac{R(T_1 - T_2)}{\gamma - 1} \text{ calories,}$$

$$R = 2 \text{ calories, } T_1 = 353 \text{ K,}$$

$$T_2 = 140.6 \text{ K, } W = ?$$

$$\therefore W = \frac{2(353 - 140.6)}{1.4 - 1} = 1062 \text{ cal.}$$

**Example 15.16.** A motor car tyre has a pressure of 2 atmospheres at the room temperature of  $27^\circ\text{C}$ . If the tyre suddenly bursts, find the resulting temperature.  $\gamma = 1.4$ .

**Soln.**

The relation connecting pressure and temperature for an adiabatic process is

$$= \frac{T_1^\gamma}{P_1^{\gamma-1}} = \frac{T_2^\gamma}{P_2^{\gamma-1}} ; \quad \text{or, } \left( \frac{T_2}{T_1} \right)^\gamma = \left( \frac{P_2}{P_1} \right)^{\gamma-1}$$

$$\text{or, } T_2 = T_1 \times \left( \frac{P_2}{P_1} \right)^{\gamma-1/\gamma}$$

$$T_1 = (27 + 273) \text{ K} = 300 \text{ K}$$

$$P_1 = 2 \text{ atmospheres}$$

$P_2 = 1$  atmosphere (after the tyre bursts, the pressure inside is the same as the pressure outside *i.e.*, atmospheric pressure).

$$T_2 = ?$$

$$T_2 = 300 \times \left( \frac{1}{2} \right)^{1.4-1/1.4}$$

$$= 300 \times (0.5)^{0.4/1.4}$$

$$= 246.1 \text{ K} = -26.9 \text{ K.}$$

### 15.17 Adiabatic curves are steeper than isothermal curves

The isothermal equation  $PV = \text{constant}$  and the adiabatic equation  $PV^\gamma = \text{constant}$  are plotted in Fig. 15.8. The slope of each curve is given by  $dP/dV$ .

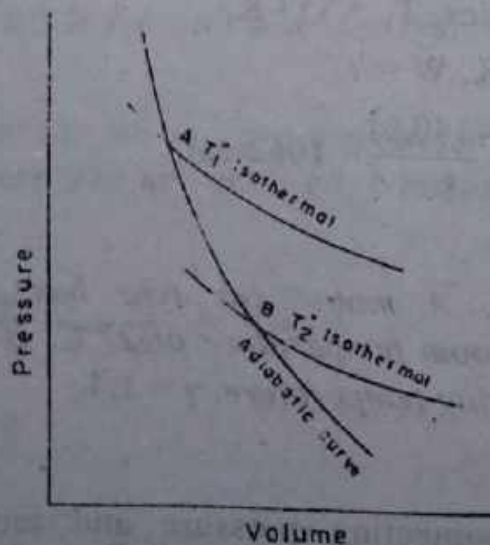


Fig. 15.8



The equation of an isothermal curve is given by  $PV = \text{constant}$ . Differentiating with respect to  $V$ , we get

$$V \frac{dP}{dV} + P = 0$$

$$\text{or, } \frac{dP}{dV} = - \frac{P}{V} \quad (15.30)$$

The equation of an adiabatic curve is given by  $PV^\gamma = \text{constant}$ . Differentiating with respect to  $V$ , we get

$$V^\gamma \frac{dP}{dV} + P \cdot \gamma \cdot V^{\gamma-1} = 0$$

Dividing by  $V^{\gamma-1}$ ,

$$V \cdot \frac{dP}{dV} + P \cdot \gamma = 0$$

$$\text{or, } \frac{dP}{dV} = - \gamma \cdot \frac{P}{V} \quad (15.31)$$

From equations (15.30) and (15.31) we see that the slope of an isothermal curve is given by  $-\frac{P}{V}$  whereas the slope of an adiabatic curve is given by  $-\gamma \cdot \frac{P}{V}$ . Thus, the slope of an adiabatic curve =  $\gamma \times$  slope of an isothermal curve.

Since  $\gamma$  is always numerically greater than one, *adiabatic curves are steeper than isothermal curves*. The gradient of an adiabatic at any point is thus  $\gamma$  times the gradient of the isothermal through the same point.

### 15.13 Isothermal and adiabatic elasticities

All matter possess volume elasticity which is defined as the ratio of the volume stress to the volume strain. Let us suppose that under the action of a small change in pressure,  $dP$ , the original volume  $V$  of a gas changes by an amount  $dV$ . Then

$$\text{volume elasticity} = E = \frac{\text{volume stress}}{\text{volume strain}}$$

$$\begin{aligned} \text{or, } E &= \frac{\text{force/area}}{\text{change in volume/original volume}} \\ &= \frac{\text{pressure}}{\text{change in volume/original volume}} \end{aligned}$$

$$\text{i.e., } E = - \frac{dP}{dV/V} = - V \left( \frac{dP}{dV} \right)$$

Minus (-) sign arises because elasticities are always expressed as positive quantities and an increase in pressure is always accompanied by a decrease in volume.

If the increase in pressure takes place very slowly, the heat generated goes out. On the contrary, when the increase in pressure is very sudden, the heat generated has very little time to go out. Thus the value of  $\frac{dP}{dV}$  will depend upon the conditions under which the measurements are made and will in general be different for an isothermal change and an adiabatic change. Let the isothermal and adiabatic elasticities be represented by  $E_T$  and  $E_Q$  respectively.

Or,

$$E_T = - V \left( \frac{dP}{dV} \right)_T$$

$$\text{and } E_Q = - V \left( \frac{dP}{dV} \right)_Q \quad (15.32)$$

**Isothermal elasticity :** An isothermal change is represented by the equation  $PV = \text{constant}$ . Differentiating we get

$$PdV + VdP = 0$$

$$\text{or, } P = - V \left( \frac{dP}{dV} \right)_T$$

So, from eqn. (14.32)

$$E_T = - V \left( \frac{dP}{dV} \right)_T = P \quad (15.33)$$



i.e., the isothermal elasticity of a gas is numerically equal to the pressure of the gas.

**Adiabatic elasticity:** For an adiabatic change,

$PV^\gamma = \text{constant}$ . Differentiating

$$V^\gamma \cdot dP + \gamma PV^{\gamma-1} dV = 0$$

$$\begin{aligned} \text{or, } \gamma \cdot P &= - \frac{V^\gamma \cdot dP}{V^{\gamma-1} dV} \\ &= - V \left( \frac{dP}{dV} \right)_Q \end{aligned}$$

(15.34)

From eqn. (14.34)

$$E_Q = - V \left( \frac{dP}{dV} \right)_Q = \gamma \cdot P = \gamma \cdot E_T$$

$$\text{or, } \frac{E_Q}{E_T} = \frac{\gamma \cdot P}{P} = \gamma = \frac{C_p}{C_v}$$

(15.34)

Thus, in the case of a perfect gas, the ratio of the adiabatic and isothermal elasticities is equal to the ratio of the specific heat at constant pressure to the specific heat at constant volume. Alternately, adiabatic elasticity is  $\gamma$  times the isothermal elasticity.

#### 14.19 Work done during expansion or compression of a gas

If a gas expands by an amount  $dV$  against an external pressure  $P$ , the work done by the gas in expanding against the pressure  $P$  is  $P \cdot dV$ . Total work done by the gas in expanding from an initial volume  $V_1$  to a final volume  $V_2$  will then be given by

$$\int_{V_1}^{V_2} P \cdot dV$$

Again, when the gas is compressed from an initial volume  $V_1$  to a final volume  $V_2$ , work is done on the gas and is given by

$$- \int_{V_1}^{V_2} P \cdot dV$$



where the minus (-) sign indicates that the increase in pressure is accompanied by a decrease in volume.

*It may be noted that when a gas is compressed, work is done on the gas by an external agency. If the gas expands, work is done by the gas.*

### work done by a gas (expansion)

*Isothermal change* : Let us consider 1 gm-molecule of a gas. For an isothermal change  $PV = RT$ . Hence, the work done when the gas expands isothermally from a volume  $V_1$  to a volume  $V_2$  is

$$\begin{aligned} W_1 &= \int_{V_1}^{V_2} P.dV = RT \int_{V_1}^{V_2} \frac{dV}{V} \\ &= RT \log_e \frac{V_2}{V_1} \\ &= 2.303 RT \log_{10} \frac{V_2}{V_1} \quad (15.36) \end{aligned}$$

*Adiabatic change* : For an adiabatic change  $PV^\gamma = K$ . Considering 1 gm-molecule of a gas, the work done when the gas expands adiabatically from a volume  $V_1$  to a volume  $V_2$  is

$$\begin{aligned} W_2 &= \int_{V_1}^{V_2} P.dV = K \int_{V_1}^{V_2} \frac{dV}{V^\gamma} \\ &= K \left[ \frac{V^{-\gamma+1}}{-\gamma+1} \right]_{V_1}^{V_2} \\ &= \frac{K}{-\gamma+1} [V_2^{-\gamma+1} - V_1^{-\gamma+1}] \\ &= - \frac{K}{-\gamma+1} [V_1^{-\gamma+1} - V_2^{-\gamma+1}] \end{aligned}$$

$$= \frac{K}{\gamma-1} [V_1^{-\gamma+1} - V_2^{-\gamma+1}]$$

$$\text{Now } K = P_1 V_1^\gamma = P_2 V_2^\gamma$$

Hence,

$$\begin{aligned} W_2 &= \frac{P_1 V_1^\gamma \cdot V_1^{-\gamma+1} - P_2 V_2^\gamma \cdot V_2^{-\gamma+1}}{\gamma-1} \\ &= \frac{P_1 V_1 - P_2 V_2}{\gamma-1} \end{aligned} \quad (15.37)$$

If the initial temperature of the gas is  $T_1$  and  $T_2$  be the final temperature after the adiabatic expansion, then

$$P_1 V_1 = RT_1 \text{ and } P_2 V_2 = RT_2$$

$$\begin{aligned} \therefore W_2 &= \frac{R}{\gamma-1} (T_1 - T_2) \\ &= \frac{C_p - C_v}{\frac{C_p}{C_v} - 1} (T_1 - T_2) \\ &= \frac{C_p - C_v}{\frac{C_p - C_v}{C_v}} (T_1 - T_2) \\ &= C_v (T_1 - T_2) \end{aligned} \quad (15.38)$$

**work done on the gas (compression)**

*Isothermal change* : When the gas is compressed isothermally from an initial volume  $V_1$  to a final volume  $V_2$  the work done on the gas is

$$\begin{aligned} W_3 &= - \int_{V_1}^{V_2} P \cdot dV = - RT \int_{V_1}^{V_2} \frac{dV}{V} \\ &= - RT \log_e \frac{V_2}{V_1} \end{aligned}$$

$$\begin{aligned}
 &= RT \log_e \frac{V_1}{V_2} \\
 &= 2.303 RT \log_{10} \frac{V_1}{V_2} \quad (15.39)
 \end{aligned}$$

*Adiabatic change* : Similarly the work done on the gas during adiabatic compression from a volume  $V_1$  to a volume  $V_2$  is

$$\begin{aligned}
 W_4 &= - \int_{V_1}^{V_2} P dV = -K \int_{V_1}^{V_2} \frac{dV}{V^\gamma} \\
 &= \frac{P_2 V_2 - P_1 V_1}{\gamma - 1} \quad (15.40)
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{R}{\gamma - 1} (T_2 - T_1) \\
 &= C_v (T_2 - T_1) \quad (15.41)
 \end{aligned}$$

**Example 14.21.** 40 gms of hydrogen gas at  $27^\circ\text{C}$  are isothermally compressed to one fourth of the original volume. Find the value of work done during the process.  $R = 8.4$  joules / gm-molecule.

**Soln.**

Gramme – molecular weight of hydrogen is 2 gms. Hence 40 gms of hydrogen contains 20 gm-molecule of the gas.

Work done during isothermal compression is given by

$$\begin{aligned}
 W &= nRT \log_e \frac{V_1}{V_2} \\
 n &= 20, R = 8.4 \text{ joules / gm-molecule,} \\
 V_1 &= V \text{ (say), } V_2 = \frac{V}{4}, T = (273 + 27) \text{ K} = 300 \text{ K}
 \end{aligned}$$

$$\therefore W = 20 \times 8.4 \times 300 \times \log_e \frac{V}{V/4}$$



$$= 20 \times 8.4 \times 300 \times \log_e 4$$

$$= 20 \times 8.4 \times 300 \times 1.3863$$

$$= 7 \times 10^4 \text{ joules.}$$

**Example 14.22.** One gramme-molecule of oxygen is heated from a temperature of  $20^\circ\text{C}$  and a pressure of 1 atmosphere to a temperature of  $100^\circ\text{C}$ . (i) How much heat must be supplied if the volume is kept constant during the heating? (ii) How much heat must be supplied if the pressure is kept constant? (iii) How much work is done if the pressure is constant?

**Soln.**

(i) The specific heat at constant volume is

$$C_v = \frac{5}{2} nR = \frac{5}{2} \times 1 \times 2 \text{ cal/K (see Art. 14.11)}$$

where  $n = 1$  and  $R = 2 \text{ cal.gm-molecule}^{-1} \text{K}^{-1}$ . Hence heat added to raise the temperature from  $(20 + 273) = 293 \text{ K}$  to  $(100 + 273) = 373 \text{ K}$  is

$$Q = C_v \cdot dT = 5 \times (373 - 293)$$

$$= 5 \text{ cal/K} \times 80 \text{ K}$$

$$= 400 \text{ calories.}$$

Since the volume remains constant, no work is done during the heating. Hence, from

$$dQ = dE + dW$$

$$\text{we have } dE = dQ = 400 \text{ cal. (} dW = 0 \text{).}$$

So the change in internal energy ( $dE$ ) when the gas temperature change from  $293\text{K}$  to  $373\text{K}$  is  $400 \text{ cal}$ . Since internal energy depends only on temperature, this is true for both isothermal and adiabatic processes.

(ii) Heat that must be added when the pressure is kept constant

$$= C_p \cdot dT = \frac{7}{2} nR \cdot dT = \frac{7}{2} \times 1 \times 2 \times 80$$

$$= 560 \text{ cal.}$$

where  $C_p$  is the specific heat at constant pressure. This is greater because although the change in internal energy is again 400 cal., work is now done during the process as volume is not constant and an expansion of volume takes place.

(iii) From

$$dQ = dE + dW, \text{ we have}$$

$$560 = 400 + dW$$

$$\text{or, } dW = 560 - 400 = 160 \text{ calories.}$$

**Example 14.23.** Air is compressed adiabatically to half its volume. Calculate the change in its temperature.

**Soln.**

Let the initial temperature be  $T_1$  and the final temperature be  $T_2$ .

$$\text{Let } V_1 = V \text{ (say), then } V_2 = \frac{V}{2}.$$

$$T_1 V_1^{\gamma-1} = T_2 V_2^{\gamma-1}$$

$$\text{or, } T_2 = T_1 \times \left( \frac{V_1}{V_2} \right)^{\gamma-1}$$

$$= T_1 \times \left( \frac{V}{V/2} \right)^{\gamma-1} = T_1 \times (2)^{1.4-1}$$

$$= 1.319 T_1.$$

$$\therefore \text{ Change in temperature} = T_2 - T_1$$

$$= 1.319 T_1 - T_1 = 0.319 T_1 \text{ K.}$$



## EXERCISES

- [1] State Newton's law of cooling, mentioning its limitations. How will you determine the specific heat of a liquid by Newton's law of cooling?
- [2] State and explain Newton's law of cooling. How would you determine the specific heat of kerosene by the method of cooling?
- [3] Define the specific heat of a gas at constant pressure and at constant volume. Why is the specific heat of a gas at constant pressure always greater than that at constant volume?
- [4] Describe an accurate method for finding the specific heat of gas at constant pressure deducing the formula you would use.
- [5] Explain why a gas has two specific heats and why one is greater than the other. Show that the difference between the two specific heats of a perfect gas is equal to the gas constant.
- [6] Describe a method of measuring the specific heat of a gas at constant volume. Indicate the sources of error, if any, and the necessary corrections.
- [7] Prove that for an ideal gas  $C_p - C_v = R$  where the symbols have their usual meanings.
- [8] Obtain an expression for the difference between two values of the specific heat for a perfect gas.
- [9] Describe an accurate method of measuring the specific heat of a gas at constant pressure.
- [10] Distinguish between isothermal, adiabatic and isochoric processes. Show that for an adiabatic change in a perfect gas  $PV^\gamma = \text{constant}$ , where  $\gamma$  is the ratio of the specific heats at constant pressure and constant volume respectively.
- [11] Derive the formulae connecting volume, pressure and temperature of a perfect gas undergoing adiabatic transformation.
- [12] Explain what is meant by isothermal and adiabatic transformations.
- [13] Prove the following :

$$PV^\gamma = \text{constant}$$

$$TV^{\gamma-1} = \text{constant}$$

$$TP^{\frac{1-\gamma}{\gamma}} = \text{constant}$$



- [14] Deduce from the first principle the adiabatic equation of a perfect gas.
- [15] Describe a method for determining the ratio of the two specific heats of a gas.
- [16] Distinguish between isothermal and adiabatic changes. Show that isothermal elasticity of a perfect gas is numerically equal to its pressure.
- [17] Distinguish between isothermal and adiabatic changes. Prove that an adiabatic curve is steeper than the isothermal curve.
- [18] Show that the adiabatic elasticity of a perfect gas is  $\gamma$  times its isothermal elasticity.
- [19] State the first law of thermodynamics and apply it to isothermal, adiabatic and isochoric processes.
- [20] Obtain an expression for the work done by a perfect gas in expanding (i) isothermally and (ii) adiabatically from a volume  $V_1$  to a volume  $V_2$ .
- [21] Distinguish between isothermal and adiabatic processes. Show that for an adiabatic process

$$P_1 V_1^\gamma = P_2 V_2^\gamma$$

$$\text{and } T_1 V_1^{\gamma-1} = T_2 V_2^{\gamma-1}.$$

- [22] An ideal gas is compressed (i) isothermally and (ii) adiabatically from a volume  $V_1$  to a volume  $V_2$ . Obtain expression for work done for each process.
- [23] An ideal gas expands adiabatically from an initial temperature  $T_1$  to a final temperature  $T_2$ . Prove that the work done by the gas is  $C_v (T_1 - T_2)$ .
- [24] A body cools in 5 minutes from  $60^\circ\text{C}$  to  $40^\circ\text{C}$ . What will be its temperature after a further 5 minutes? Assume Newton's law of cooling to hold. ( $28^\circ\text{C}$ ).
- [25] It was observed that a heated piece of iron was cooled from  $100^\circ\text{C}$  to  $90^\circ\text{C}$  in 1 min. 38 secs., and from  $50^\circ\text{C}$  to  $45^\circ\text{C}$  in 2 min. 24 secs. Assuming Newton's law of cooling to hold throughout the range, estimate the temperature of the surrounding atmosphere to which heat was being radiated. ( $23^\circ\text{C}$ ).
- [26] A liquid cools in 6 minutes from  $80^\circ\text{C}$  to  $60^\circ\text{C}$ . What will be its temperature after the next 10 minutes? Temperature of the

surroundings is  $30^{\circ}\text{C}$ . Assume that Newton's law of cooling is applicable throughout the process. ( $42.8^{\circ}\text{C}$ ).

- [27] If a body takes 3 minutes to cool from  $100^{\circ}\text{C}$  to  $60^{\circ}\text{C}$ , how long will it take to cool from  $60^{\circ}\text{C}$  to  $20^{\circ}\text{C}$ , assuming that the temperature of the surroundings is  $10^{\circ}\text{C}$  and that Newton's law of cooling is obeyed? (8.22 mins. App.)
- [28] A liquid takes 10 minutes to cool from  $90^{\circ}\text{C}$  to  $60^{\circ}\text{C}$ . How much time will it take to cool from  $70^{\circ}\text{C}$  to  $40^{\circ}\text{C}$ ? The temperature of the surroundings is  $30^{\circ}\text{C}$ . (20 minutes).
- [29] A liquid takes 5 minutes to cool from  $70^{\circ}\text{C}$  to  $50^{\circ}\text{C}$ . How much time will it take to cool from  $50^{\circ}\text{C}$  to  $35^{\circ}\text{C}$ ? The temperature of the surroundings is  $20^{\circ}\text{C}$ . (6.785 minutes).
- [30] Equal volume of water (density  $1\text{ gm/cm}^3$ ) and alcohol (density  $0.8\text{ gm/cm}^3$ ) when put in identical calorimeters take 100 seconds and 74 seconds respectively to cool from  $50^{\circ}\text{C}$  to  $40^{\circ}\text{C}$ . Calculate the specific heat of alcohol. Thermal capacity of each calorimeter is numerically equal to the volume of either liquid. ( $0.6\text{ cal gm}^{-1}^{\circ}\text{C}^{-1}$ ).
- [31] Find the specific heat of a liquid which takes 2 minutes in cooling from  $50^{\circ}\text{C}$  to  $40^{\circ}\text{C}$  in a vessel in which the same volume of water takes 5 minutes in cooling through the same range of temperature. Mass of water = 100 gm, mass of liquid = 85 gm, water equivalent of the vessel = 10 gm. ( $0.4\text{ calories gm}^{-1}^{\circ}\text{C}^{-1}$ ).
- [32] Two identical copper calorimeters weighing 100 gms contain respectively 50 c.c. of water and 50 c.c. of alcohol. They are allowed to cool from  $60^{\circ}\text{C}$  to  $40^{\circ}\text{C}$ . The times occupied respectively for the calorimeters containing water and alcohol are 10 min and 5.5 min. Calculate the specific heat of alcohol. (sp. ht. of copper =  $0.093\text{ cal gm}^{-1}^{\circ}\text{C}^{-1}$ ; density of alcohol =  $0.81\text{ gm/cm}^3$ ) ( $0.58\text{ cal gm}^{-1}^{\circ}\text{C}^{-1}$ ).
- [33] A calorimeter of water equivalent 2 gms is first filled with 10 gms of water and then with 15 gms of a liquid. The time taken in the two cases to cool from  $65^{\circ}\text{C}$  to  $60^{\circ}\text{C}$  are 200 secs. and 150 secs. respectively. Find the specific heat of the liquid. ( $0.466\text{ cal. gm}^{-1}^{\circ}\text{C}^{-1}$ ).
- [34] One litre of hydrogen at N.T.P. weighs 0.0896 gramme. Find the value of R for one gramme-molecule of the gas. Molecular weight of hydrogen = 2. ( $8.3 \times 10^7\text{ ergs/degree/gm-molecule}$ ).
- [35] One litre of air at N.T.P. weighs 1.293 grammes. Calculate the value of the gas constant for one gramme of air. ( $2.871 \times 10^8\text{ ergs/gm-K}$ ).



- [36] Calculate the value of the gas constant for  $\text{CO}_2$  given that 22.4 litres of the gas at N.T.P. weigh 44 gms. ( $8.32 \times 10^7$  ergs/gm-molecule K).
- [37] If the density of hydrogen at N.T.P. is  $8.99 \times 10^{-2}$  gm/litre and its specific heat at constant pressure is 3.41 cal/gm-K, find the value of specific heat at constant volume. Given  $J = 4.18$  joules/cal. ( $2.42$  cal/gm-K).
- [38] The specific heats of nitrogen at constant pressure and constant volume are 0.245 and 0.175 respectively. If the density of nitrogen at N.T.P. is 1.25 gm per litre, calculate the value of mechanical equivalent of heat. Density of mercury =  $13.6$  gm/cm<sup>3</sup>,  $g = 980$  cm/sec<sup>2</sup>. ( $4.25 \times 10^7$  ergs/cal.).
- [39] The density of hydrogen at N.T.P. is  $0.09 \times 10^{-3}$  gm/cm<sup>3</sup>. If the specific heat of hydrogen at constant volume is 2.4 cal/gm-K, find its specific heat at constant pressure.  $J = 4.2$  joules/cal.,  $g = 980$  cm/sec<sup>2</sup> density of mercury =  $13.6$ /cm<sup>3</sup>. ( $3.38$  cal/gm-K).
- [40] Calculate the difference between the specific heats of air at constant pressure and at constant volume if the density of air at a pressure of 76 cms of mercury and at  $0^\circ\text{C}$  is  $0.00129$  gm/cm<sup>3</sup>. Establish the equation you would use, given :  $J = 4.18$  joules/cal,  $g = 980$  cm/sec<sup>2</sup>, density of mercury =  $13.6$  gm/cm<sup>3</sup>. ( $0.069$  cal/gm-K)
- [41] The specific heats at constant pressure and constant volume of helium gas are 5.0 kcal/kg-mole-K and 3.0 kcal/kg-mole-K respectively. Knowing that one kilogramme-mole of a gas occupies a volume of  $22.4$  m<sup>3</sup>, calculate the value of mechanical equivalent of heat. ( $4160$  joules/kcal).
- [42] The specific heat of argon at constant pressure is 0.127 and the ratio of its specific heats is 1.667. Calculate the mechanical equivalent of heat. One cubic metre of argon weighs 1.786 kilogramme at N.T.P. ( $4160$  joules / kcal).
- [43] Assuming that the density of a gas at a pressure of  $10^5$  N/m<sup>2</sup> and at a temperature of  $27^\circ\text{C}$  is  $1.775$  kg/m<sup>3</sup> and that its specific heat at constant pressure is 846 joules/kg-K, find the ratio of its specific heats. (1.29).
- [44] Calculate the value of  $J$  given that one m<sup>3</sup> of hydrogen at  $0^\circ\text{C}$  and a pressure of 760 mm. of mercury weighs 0.0896 kg.  $C_p = 3.409$  and  $C_v = 2.411$  ( $4150$  joules/kcal).
- [45] A mass of dry air at  $15^\circ\text{C}$  is expanded adiabatically to double its volume. Calculate approximately its new temperature, given  $\gamma = 1.40$  ( $-54.7^\circ\text{C}$ ).



- [46] Dry air enclosed at  $25^{\circ}\text{C}$  and at atmospheric pressure is suddenly compressed to half its volume. Find the resulting pressure. (2.64 atmos).
- [47] Air is compressed adiabatically to one-fourth of its volume. Calculate the change in temperature and pressure.
- [48] 1 gm-molecule of a monatomic perfect gas ( $\gamma = 5/3$ ) at  $27^{\circ}\text{C}$  is adiabatically compressed in a reversible process from an initial pressure of 1 atmosphere to a final pressure of 50 atmospheres. Calculate the resulting change in temperature. (temperature increases by  $1134^{\circ}\text{C}$ ).
- [49] A certain mass of gas at N.T.P. is expanded to three times its volume under adiabatic conditions. Calculate the resulting temperature and pressure.  $\gamma = 1.40$ . ( $-97^{\circ}\text{C}$ , 0.2148 atmos.).
- [50] A certain quantity of air ( $\gamma = 1.40$ ) at N.T.P. is adiabatically compressed to  $\frac{1}{4}$  of its original volume. Find the resulting temperature and pressure. ( $202^{\circ}\text{C}$ , 529.3 cm of Hg).
- [51] A certain volume of dry air at N.T.P. is expanded to three times its volume under (a) isothermal condition and (b) adiabatic condition. Calculate in each case the final pressure and volume. Given,  $\gamma =$  for air = 1.40. ([a] isothermal : 25.33 cm of mercury ; no change in temperature. [b] adiabatic : 16.32 cm of mercury,  $-97^{\circ}\text{C}$ ).
- [52] Air at  $17^{\circ}\text{C}$  and a pressure of 76 cm of mercury is allowed to expand isothermally until its volume is doubled and then adiabatically until it is redoubled. Find the final temperature and pressure of the gas. The ratio of the two specific heats is 1.4. ( $-53.2^{\circ}\text{C}$ , 14.4 cm of mercury).
- [53] A quantity of air at  $20^{\circ}\text{C}$  and 76 cm of mercury is suddenly expanded to twice its volume under adiabatic conditions. What will then be its temperature? If its temperature is allowed to rise again to  $20^{\circ}\text{C}$ , what will be the pressure.  $\gamma = 1.40$ . ( $-52.5^{\circ}\text{C}$ , 38 cm of mercury).
- [54] 100 c.c. of air at  $0^{\circ}\text{C}$  and 76 cm of pressure are compressed adiabatically to 20 c.c.. Find (a) the new pressure (b) the new temperature and (c) the work done in compression. Assume  $\gamma = 1.4$ . ([a] 723 cm. [b]  $247^{\circ}\text{C}$  [c] 22.9 joules).
- [55] A gramme of air at  $275^{\circ}\text{C}$  expands adiabatically to 5 times its initial volume. Find the work done in its expansion, assuming that the air behaves as an ideal gas. Ratio of the specific heats of air = 1.40. Gas

constant for 1 gramme of air =  $2.88 \times 10^6$  ergs per degree. ( $1.87 \times 10^9$  ergs).

- [56] Find the work necessary to compress one litre of oxygen to one-tenth of its volume, the process being conducted isothermally at  $0^\circ\text{C}$  from an initial pressure of one atmosphere. What work would be done by the gas if it is expanded adiabatically to its original volume? Take the ratio of the specific heats as 1.4. ( $2.334 \times 10^6$  ergs;  $1.525 \times 10^9$  ergs).
- [57] 27.3 gm of air at  $15^\circ\text{C}$  and 76 cm pressure expands adiabatically until the volume has increased by 50 per cent. What will be the final pressure and temperature? Find, by considering the internal energy of the gas, the heat equivalent of the work it has done against the external pressure during the expansion. Take  $C_v = 0.17$ ,  $C_p = 0.24$ . (42.9 cm of mercury,  $-29.1^\circ\text{C}$ , 204.6 cal).
- [58] Calculate the work done in compressing isothermally a litre of air at standard temperature and pressure to one-tenth of its initial volume. (Ans.  $6.376 \times 10^{10}$  ergs).
- [59] If 1 gramme of oxygen at 76 cm pressure and  $0^\circ\text{C}$  has its volume doubled by an adiabatic change, calculate the change in internal energy. Universal gas constant = 2 cal per gm-mole per K,  $J = 4.2$  joules/cal and ratio of specific heats for oxygen = 1.4. (internal energy decreases by 433.9 joules).



## CHAPTER – XVI

## KINETIC THEORY OF GASES

*Introduction-Fundamental assumptions of the kinetic theory-Expression for pressure exerted by a gas-The root mean square speed-Kinetic interpretation of temperature – Derivation of the gas equation – Deduction of gas laws – Degrees of freedom – Equipartition of energy – Ratio of specific heats of gases – Mean free path – Viscosity of gases – Solved problems – Exercises.*

**16.1 Introduction**

At very low pressures (or densities), gases obey some simple laws with regard to their physical properties in addition to their chemical properties. It was thought that since gases, in general, obey such simple laws as Boyle's law, Charles' law, Avogadro's law, Dalton's law of partial pressures, etc., they should have a common and simple structure. The many accurate experiments, mostly by James P. Joule, on the conversion of work into heat established that heat is a form of energy which is connected with the motion of molecules of which matter is made. Attempts were then made to explain the behaviour of gases from a simple mechanical structure. The scientists who contributed to this structure are, among others, J.C. Maxwell, L. Boltzmann and R. Clausius. The outcome of their labour is known as the *kinetic theory of matter (gases)*.

**16.2 Fundamental assumptions of the kinetic theory of gases**

The kinetic theory of gases is based upon the following simplifying *assumptions or postulates*, first stated by Clausius in 1860.

(i) *The kinetic theory assumes that a gas consists of an exceedingly large number of minute particles. These particles are referred to as the molecules in the atomic theory. For any one chemical species, all molecules are identical.*

(ii) *The molecules are supposed to resemble perfectly rigid spheres like a steel ball-bearing. They obey Newton's laws of motion (conservation of momentum and energy).*



(iii) The molecules are continuously moving at random with all possible velocities in all possible directions, colliding with each other and against the wall of the containing vessel. The collisions between molecules and the walls as well as molecules themselves are assumed to be perfectly elastic; in other words, there is no loss in Kinetic energy when the collisions occur. As the chance of collisions in all directions is the same, the collisions do not affect the molecular density.

(iv) The pressure exerted by a gas is due to the continuous bombardment on the walls of the containing vessel by the gas molecules as a result of their unordered motion. Obviously, the collisions among molecules themselves and against the walls will be greater in number when the volume is decreased because there is now less room for motion. This explains why pressure gets increased when the volume is decreased.

(v) The molecules are themselves of very small dimensions, their actual volume being negligible in comparison with the space at their disposal, i.e., the volume of the vessel.

(vi) The molecules of the gas are sufficiently far apart most of the time so that they do not exert any force of attraction or repulsion on one another except when they collide with each other and with the wall. Between two collisions, the molecules, therefore, move in a straight line with uniform velocity i.e., they obey Newton's law.

(vii) Because the molecules do not exert any force of attraction or repulsion on one another, they cannot have any potential energy (other than gravitational) but have only kinetic energy.

(viii) When there is no external field or force, the molecules are distributed uniformly throughout a container; on the average the number of molecules present in a small element of volume remains constant.

(ix) The size of the molecules is infinitesimally small compared to the distance traversed by a molecule between any two consecutive collisions, called its free path.

(x) The time during which a collision lasts is negligible in comparison to the time taken to traverse the mean free path.

### 15.3 Expression for pressure exerted by a gas

Let us imagine a certain quantity of a perfect gas to be confined in a cube the length of whose side is  $l$ . Let the total number of molecules be  $n$  and let each of them have a mass equal to  $m$ . The molecules are moving at random with different speeds the average of which is  $c$  [ $c$  is really not the *usual* average speed but is another type of average speed called the *root mean square* speed].

Since the average speed of the molecules is  $c$ , we may regard that all molecules are moving with the same velocity  $c$  in all possible directions. Let us calculate the effect produced on each wall as a result of the collisions of the molecules with the wall. Concentrating our attention upon a particular molecule, we can resolve

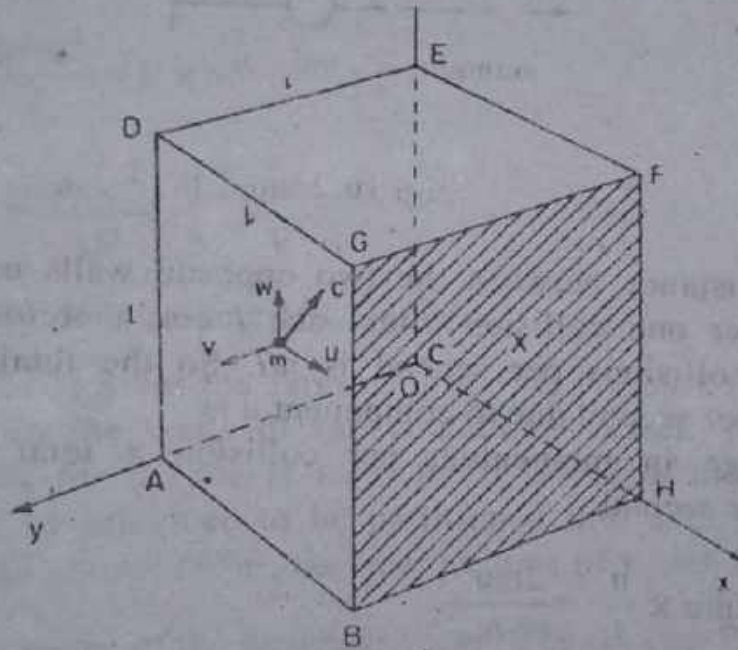


Fig. 16.1

$c$  into three components,  $u$ ,  $v$  and  $w$  at right angles to one another and parallel to the sides AB, AC and AD of the containing vessel (Fig. 16.1).

The component  $u$  parallel to AB, has effect only on the wall ACED and the wall opposite to it; it has no effect on the other walls. The molecule strikes with the velocity  $u$  and rebounds with the same velocity, since the walls and the molecules behave like perfectly



elastic bodies. So the momentum before collision in the direction  $AB$  is  $mu$  and after collision in the same direction is  $-mu$  (Fig. 16.2).

Hence the change of momentum per collision

= original momentum - final momentum

$$= +mu - (-mu) = 2mu$$

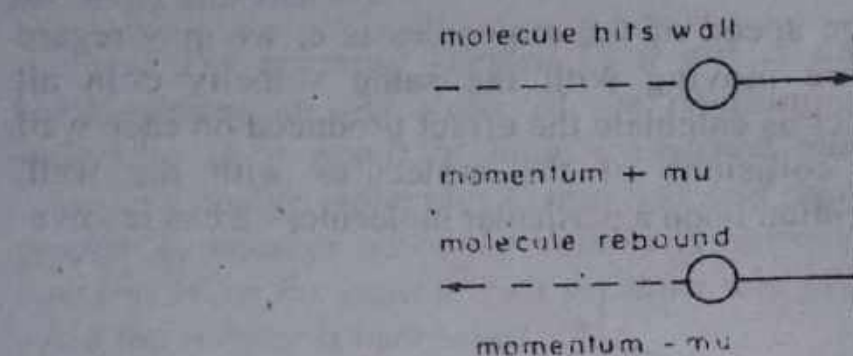


Fig. 16.2

If the distance between the two opposite walls be  $l$  cm, the particle suffers one collision after each  $l$  cm; therefore the total number of collisions per second is  $u/l$ . So the total change of momentum per second due to component  $u$  is

= change in momentum per collision  $\times$  total number of collisions per second

$$= 2mu \times \frac{u}{l} = \frac{2mu^2}{l}$$

Similarly the change in momentum per second for the other components  $v$  and  $w$  are  $\frac{2mv^2}{l}$  and  $\frac{2mw^2}{l}$  respectively. Therefore the total change of momentum per molecule per second

$$= \frac{2mu^2}{l} + \frac{2mv^2}{l} + \frac{2mw^2}{l}$$

$$= \frac{2m}{l} (u^2 + v^2 + w^2) = \frac{2m}{l} c^2$$



since by dynamics  $c^2 = u^2 + v^2 + w^2$ .

If there are  $n$  molecules present, the total effect on the walls will be multiplied  $n$  times. Therefore, the total change in momentum for all the molecules per second is

$$\frac{2mc^2}{l} \times n = \frac{2mnc^2}{l}$$

According to second law of motion, the impressed force is numerically equal to the rate of change of momentum or momentum generated per second (see note below). Now the impressed force *i.e.*, the total pressure on the six walls of the cube is  $6l^2 \times P$  where  $P$  is the force per unit area of the wall,  $l^2$  is the area of each wall (face of the cube) and there are six faces in a cube. Therefore,

$$\frac{2mnc^2}{l} = P \times 6l^2$$

$$\text{or, } P = \frac{mnc^2}{3l^3} = \frac{1}{3} \frac{mnc^2}{V} \quad (16.1)$$

since  $l^3 = V = \text{volume of the cube.}$

Eqn. (16.1) gives the expression for the pressure exerted by gas molecules on the walls of the containing vessel. *The deduction though made for a cube is valid for a vessel of any shape.* Any vessel may be imagined to be partitioned into a large number of infinitesimally small cubes, the total volume of which is equal in the

limit to the volume of the particular vessel. The relation  $P = \frac{1}{3} \frac{mnc^2}{V}$  is

then valid for each cube; but the pressure on opposite sides of the common wall neutralize each other. Then the only surface left on which the gas pressure acts is the outward surface of the exterior cubes, which in the limit is equal to the surface of the containing vessel.

Equatin (16.1) may be rearranged as

$$P = \frac{1}{3} \frac{mnc^2}{V} = \frac{2.n \cdot \frac{1}{2} mc^2}{3V}$$

The numerator in the equation is twice total kinetic energy of motion of all the molecules in the gas, while the denominator is three times the volume of the gas. Thus we have

$$\text{pressure} = \frac{2}{3} \frac{\text{total kinetic energy of translation}}{\text{volume}}$$

The pressure is, therefore, two-thirds of the total kinetic energy of translation of the molecules in unit volume.

**Note :** According to second law of motion, impressed force is proportional (not equal) to the rate of change of momentum, the mathematical expression for which is  $F = ma$  where  $F$  is the impressed force which produces an acceleration  $a$ . For a body initially at rest, acquiring a velocity  $v$  in time  $t$  due to the action of force  $F$ ,

$$F = ma$$

$$= m \frac{v}{t}$$

$$= \frac{mv}{t}$$

$$= \text{momentum generated per second}$$

$$= \text{rate of change of momentum}$$

Thus, it can be said that the applied force is numerically equal to the rate of change of momentum or to the momentum generated per second.

#### 16.4 The root mean square speed

In the deduction of the pressure exerted by a gas, the speed of the gas molecules has been for simplicity called the average speed but in reality it is not the average speed  $c_{av}$  but a speed called the root mean square speed  $c$ . The root mean square (r.m.s.) speed may be defined as the hypothetical velocity which all the molecules of a gas would possess if the total kinetic energy was equally divided among them. It is given by the square Root of the Mean value of the Squares of the velocities of all molecules. For example, if there are



two molecules moving with velocities 2 and 14 cm per second respectively, their average speed is only 8 cm per second but their

r.m.s. speed is the square root of  $\frac{1}{2} (2^2 + 14^2)$  i.e., 10 cm per sec.

The reason that the r.m.s. speed  $c$  is to be used rather than the average speed  $c_{av}$ , is owing to the fact that the total *kinetic energy* is an invariable quantity at constant temperature (Art. 16.5) and this

total kinetic energy is  $\frac{1}{2} Mc^2$ , and not  $\frac{1}{2} M c_{av}^2$

Thus in the above example of the two molecules, the total kinetic energy is

$$= \frac{1}{2} m \cdot 2^2 + \frac{1}{2} m \cdot 14^2$$

$$= 2 \times \frac{1}{2} m \cdot 10^2$$

$$= \text{number of molecules} \times \frac{1}{2} m (\text{r.m.s. speed})^2$$

If  $n$  molecules have speeds  $c_1, c_2, c_3, \dots, c_n$  respectively, the average speed,  $c_{av}$  and the r.m.s. speed,  $c$  are given by

$$\text{Average speed, } c_{av} = \frac{c_1 + c_2 + c_3 + \dots + c_n}{n}$$

$$\text{r.m.s. speed, } c = \sqrt{\frac{c_1^2 + c_2^2 + c_3^2 + \dots + c_n^2}{n}}$$

### calculation of the root mean square speed

The expression for pressure exerted by a gas is

$$p = \frac{1}{3} \frac{mnc^2}{V}$$

$$= \frac{1}{3} \frac{M}{V} c^2 \text{ where } M \text{ is the mass of the gas of volume } V$$

$$= \frac{1}{3} \rho c^2$$



where  $\rho = \frac{M}{V}$  is the density of the gas

$$\text{or, } c^2 = \frac{3P}{\rho}$$

$$\text{or, } c = \sqrt{\frac{3P}{\rho}} \quad (16.2)$$

Equation (16.2) enables us to determine the value of  $c$  for any particular gas at a given temperature, if  $P$  and  $\rho$  for it be known.

The root mean square speeds of some common gas at  $0^\circ\text{C}$  are given in table - 1 :

TABLE - 1

Gas	Molecular weight	Root Mean Square speed in cm/sec
Hydrogen	2.016	$18.4 \times 10^4$
Helium	4	$13.1 \times 10^4$
Nitrogen	28	$4.95 \times 10^4$
Oxygen	32	$4.16 \times 10^4$
Argon	40	$4.14 \times 10^4$
Carbon dioxide	44	$3.95 \times 10^4$
Chlorine	71	$3.11 \times 10^4$

### 16.5 Kinetic interpretation of temperature

Let us consider 1 gm-molecule of a gas occupying a volume  $V$ . From eqn. (16.1), we have

$$\begin{aligned}
 PV &= \frac{1}{3} mnc^2 \\
 &= \frac{1}{3} Mc^2 \quad (16.3)
 \end{aligned}$$

where  $M = mn =$  total mass of all the molecules present in 1 gm - molecule

= molecular weight

From molar gas equation, we have

$$PV = RT \quad (16.4)$$

where  $R$  is the molar gas constant.

Combining eqns. (16.4) and (16.3), we get

$$\frac{1}{3} MC^2 = RT$$

$$\text{or, } \frac{1}{2} MC^2 = \frac{3}{2} RT \quad (16.5)$$

Thus the total kinetic energy of a gm-molecule of a gas is equal to  $\frac{3}{2} RT$ .

$$\text{If } T = 1K, R = \frac{2}{3} \left( \frac{1}{2} MC^2 \right) \quad (16.6)$$

Equation (16.6) above gives a physical meaning to the gas constant  $R$  and shows that it is equal to two-third of the total translational energy of the molecules in one gm-molecule of a gas at a temperature of one degree absolute. The root mean square speed of the molecules of a gas at a given temperature can also be calculated with the help of eqn. (16.6) merely from a knowledge of the value of the gas constant and of the absolute temperature. It can further be seen that at a given temperature  $TK$  the mean kinetic energy of translation of one gm-molecule of any gas has the same value which is  $\frac{3}{2} RT$ .

Now  $n$ , the total number of molecules in one gm-molecule of a gas is the Avogadro's number and is denoted by  $N$ . Dividing both sides of eqn. (16.5) by  $N$  we have

$$\frac{1}{2} \frac{M}{N} c^2 = \frac{3}{2} \frac{R}{N} T$$



But  $\frac{M}{N} = m$ , mass of a single molecule

$$\therefore \frac{1}{2} mC^2 = \frac{3}{2} kT \quad (16.7)$$

where  $k = \frac{R}{N}$  is the gas constant per molecule and is called the *Boltzmann's constant*.

Thus we see that *the mean kinetic energy of translation of one molecule of any gas at a temperature  $TK$  is  $\frac{3}{2}kT$* . It can be seen both from eqns. (16.5) and (16.7) that whether it is one gm-molecule or one molecule of a gas, the mean kinetic energy of translation is a function of temperature only. In other words, temperature is a measure of kinetic energy of translation of the gas molecules. This is referred to as the *kinetic interpretation of temperature*.

From the above discussion it becomes obvious that the absolute temperature is proportional to the mean kinetic energy of translation of the molecules. When the temperature of a gas is increased, the mean kinetic energy of the molecules increase. When heat is withdrawn from a gas, the mean kinetic energy of the molecules decreases. Thus, at absolute zero temperature, the kinetic energy should be zero. Since the mass of the molecules cannot be zero, the velocities of the molecules should be zero. Hence, from the stand point of kinetic theory of gases, *absolute zero of temperature is the temperature at which the molecules will be devoid of all motion*. But Kelvin's thermodynamic interpretation of the absolute zero does not require that the molecular motion should cease at that temperature. Before the absolute zero temperature is reached, all gases change their state to liquids and solids.

## 16.6 Derivation of the gas equation

For 1 gm-molecule of a gas, we have

$$PV = \frac{1}{3} mnc^2$$



where  $m$  is the mass of a molecule and  $n$  is the total number of molecules. But the total number of molecules in 1 gm-molecule of the gas is the Avogadro's number,  $N$ . Therefore,

$$PV = \frac{1}{3} mNc^2$$

The above equation can be rearranged as  $PV = \frac{2}{3} N \cdot \frac{1}{2} mc^2$

But  $\frac{1}{2} mc^2$ , i.e., kinetic energy of one molecule of a gas is equal to  $\frac{3}{2} kT$  where  $k (= \frac{R}{N})$  is the Boltzmann's constant.

Hence

$$\begin{aligned} PV &= \frac{2}{3} N \cdot \frac{3}{2} kT \\ &= \frac{2}{3} N \cdot \frac{3}{2} \frac{R}{N} T \\ &= RT \end{aligned} \quad (16.8)$$

where  $R$  is the molar gas constant. Equation (16.8) is the gas equation.

### 16.7 Evaluation of some important constants

(i) The gramme-molecular (molar) gas constant,  $R$  : The gm-molecular gas constant is given by

$$R = \frac{PV}{T}$$

where  $V$  = molar volume = 22.4 litres =  $22.4 \times 10^3$  c.c.

$P$  = normal atmospheric pressure

$$= 76 \times 13.6 \times 981$$

$$= 1.013 \times 10^6 \text{ dyne/cm}^2$$

$$T = 273.2 \text{ K.}$$

$$\therefore R = \frac{1.013 \times 10^6 \times 22.4 \times 10^3}{273.2}$$

$$= 8.31 \times 10^7 \text{ ergs/mole/K}$$

$$= 8.31 \text{ J/mole/K}$$

$$= \frac{8.31 \times 10^7}{4.18 \times 10^7} = 1.99 \text{ cal/degree.}$$

$$= \frac{1 \times 22.4}{273.2} = 0.082 \text{ lit-atmos/K/mole}$$

(ii) Avogadro's number,  $N$  :

$$N = \frac{M}{m} = \frac{\text{molecular weight in gms}}{\text{weight of a molecule}}$$

For hydrogen,  $M = 2$  gms,

$$m = 3.32 \times 10^{-24} \text{ gms}$$

$$\therefore N = \frac{2}{3.32 \times 10^{-24}}$$

$$= 6.02 \times 10^{23} \text{ per gm-molecule.}$$

(iii) Boltzmann's constant,  $k$  (gas constant per molecule) :

$$k = \frac{R}{N} = \frac{\text{molar gas constant}}{\text{Avogadro's number}}$$

$$= \frac{8.31 \times 10^7}{6.02 \times 10^{23}} = 1.38 \times 10^{-16} \text{ ergs/K}$$

$$= 1.38 \times 10^{-23} \text{ J/K.}$$

## 16.8 Deduction of gas laws

(i) *Boyle's law* : Temperature remaining constant, the volume of a given mass of a gas is inversely proportional to the pressure. Or

$$V \propto \frac{1}{P}, \text{ when } T \text{ is constant,}$$

or,  $PV = \text{constant}$ .

From kinetic theory, we have

$$PV = \frac{1}{3} mnc^2$$

For a given amount of any gas,  $m$  and  $n$  are obviously constant.  $c$  is also constant as long as the temperature remains constant, since an increase in  $c$  means an increase in kinetic energy of the particles, which is not possible unless heat is supplied from outside. Thus at constant temperature, the right hand side of the above equation is constant.

or,  $PV = \text{constant}$

which is Boyle's law.

(ii) *Charles' law* : At constant pressure, the volume of a given mass of a gas varies directly as the absolute temperature of the gas, i.e.,  $V \propto T$ , when  $P$  is constant. Or,  $\frac{V}{T} = \text{constant}$ .

This is also known as Gay-Lussac's law.

Let us consider one gm-molecule of a gas. If the volume occupied by one gm-molecule of the gas be  $V$  then from kinetic theory, we have

$$PV = \frac{1}{3} mnc^2 = \frac{1}{3} Mc^2 \quad (16.9)$$

where  $M$  is the molecular weight of the gas. From the above equation we get  $V \propto c^2$  when

$$P \text{ is constant.} \quad (16.10)$$

But for a perfect gas,

$$PV = RT \quad (16.11)$$

Comparing eqns. (16.8) and (16.11), we get

$$\frac{1}{3} Mc^2 = RT$$



or,  $c^2 \propto T$

$$PV = \frac{1}{3} mnc^2 \quad (16.12)$$

Combining eqns. (16.10) and (16.12)

$V \propto T$  when  $P$  is constant,

which is Charles' law

(iii) *Avogadro's law* : Equal volumes of all gases under like conditions of temperature and pressure contain the same number of molecules.

Let  $m_1$ ,  $n_1$  and  $c_1$  be the mass of a molecule, total number of molecules and r.m.s. speed of the molecules of the first gas and  $m_2$ ,  $n_2$  and  $c_2$  be the corresponding quantities of the second gas.

From kinetic theory, we have

$$PV = \frac{1}{3} m_1 n_1 c_1^2 = \frac{1}{3} m_2 n_2 c_2^2 \quad (16.13)$$

since pressure and volume are the same for the two gases.

Again, since the two gases are at the same temperature

$$\frac{1}{2} m_1 c_1^2 = \frac{1}{2} m_2 c_2^2 \quad (16.14)$$

both being equal to  $\frac{3}{2} kT$

Dividing (16.13) by (16.14), we get

$$n_1 = n_2$$

which is Avogadro's law.

(iv) *Dalton's law of partial pressures* : The pressure exerted by a mixture of gases on the walls of the containing vessel is equal to the sum of the pressures that each gas would exert separately.

Let us suppose that a vessel contains a mixture of gases of densities  $\rho_1, \rho_2, \rho_3, \dots$  having r.m.s. velocities  $c_1, c_2, c_3, \dots$ . Then from kinetic theory, we get,

$$PV = \frac{1}{3} mnc^2,$$

$$\text{or, } P = \frac{1}{3} \frac{M}{V} c^2$$

$$= \frac{1}{3} \rho c^2 = \frac{2}{3} E$$

where  $E$  is the total kinetic energy per unit volume (Art. 16.3). Since kinetic energies are additive it follows from the above equation that pressures must also be additive.

$$\therefore P = \frac{1}{3} \rho_1 c_1^2 + \frac{1}{3} \rho_2 c_2^2 + \frac{1}{3} \rho_3 c_3^2 + \dots$$

$$= P_1 + P_2 + P_3 + \dots$$

where  $P_1, P_2, P_3, \dots$  are the pressures exerted by the first, second, third, . . . gas respectively. This is Dalton's law of partial pressures.

(v) Graham's law of diffusion : If two gases at the same temperature and pressure be allowed to diffuse into one another through fine pores, the rate of diffusion will be inversely proportional to the square root of their densities.

It is logical to assume that the rate of diffusion of a gas through a porous vessel will obviously be proportional to the r.m.s. speed of the molecules. Or

Rate of diffusion  $\propto$  r.m.s. speed  $c$

But from  $PV = \frac{1}{3} mnc^2$ , we have

$$c^2 = \frac{3PV}{mn} = \frac{3PV}{M} = \frac{3P}{\rho}$$

$$\text{or, } c = \sqrt{\frac{3P}{\rho}}$$

where  $\rho = \frac{M}{V}$ , is the density of the gas.

$\therefore$  rate of diffusion  $\propto c \propto \frac{1}{\sqrt{\rho}}$  since pressure and temperature are constant.

This is Graham's law of diffusion.

**Example 16.1.** Calculate the root mean square (r.m.s.) velocity of the molecules of hydrogen, oxygen and air at  $0^{\circ}\text{C}$  and atmospheric pressure.

**Soln.**

$$c = \sqrt{\frac{3P}{\rho}}$$

(i) For hydrogen the density at N.T.P.

$$= 0.000089 \text{ gm/c.c.}$$

Atmospheric pressure  $P$

$$= 76 \times 13.6 \times 981 \text{ dynes / cm}^2.$$

$$\therefore c = \sqrt{\frac{3 \times 76 \times 13.6 \times 981}{0.000089}}$$

$$= 1.84 \times 10^5 \text{ cm/sec.}$$

(ii) For oxygen the density at N.T.P.

$$= 16 \times 0.000089 \text{ gm/c.c.}$$

$$\therefore c = \sqrt{\frac{3 \times 76 \times 13.6 \times 981}{16 \times 0.000089}}$$

$$= 4.6 \times 10^4 \text{ cm/sec.}$$

(iii) For air the density at N.T.P.

$$= 0.001293 \text{ gm/c.c.}$$

$$\therefore c = \sqrt{\frac{3 \times 76 \times 13.6 \times 981}{0.001293}}$$

$$= 4.850 \times 10^4 \text{ cm/sec.}$$

**Example 16.2.** Given that a kg-mole of hydrogen occupies a volume of  $22.4 \text{ m}^3$  at N.T.P., calculate the root mean square velocity of the molecules at N.T.P.



**Soln.**

$$PV = \frac{1}{3} mnc^2 = \frac{1}{3} Mc^2$$

for 1 kg-mole of hydrogen

M = Molecular weight of hydrogen in kg.

$$\text{Again } PV = RT$$

R = kg.-molecular gas constant.

Combining the two relations, we get

$$\frac{1}{3} Mc^2 = RT$$

$$\text{or, } c = \sqrt{\frac{3RT}{M}}$$

From  $PV = RT$ , we have

$$R = \frac{PV}{T}$$

$$P = 0.76 \times 13.6 \times 10^3 \times 9.8$$

$$= 1.013 \times 10^5 \text{ N/m}^2$$

$$V = 22.4 \text{ m}^3$$

$$T = 273 \text{ K}$$

$$\therefore R = \frac{1.013 \times 10^5 \times 22.4}{273}$$

$$= 8310 \text{ joules/kg-mole} - \text{K.}$$

$$\therefore c = \sqrt{\frac{3 \times 8310 \times 273}{2}}, \text{ assuming } M = 2.$$

$$= 1845 \text{ m/sec.}$$

**Example 16.3.** Oxygen is confined in a container at  $0^\circ\text{C}$  and normal atmospheric pressure. The temperature of the gas is then

increased until the pressure is doubled. Neglect any change in volume of the container. Find the r.m.s. speed of the molecules at this temperature. [The density of oxygen at N.T.P. = 1.43 gm/litre].

**Soln.**

$$c = \sqrt{\frac{3P}{\rho}}$$

$$P = h \rho g = 76 \times 13.6 \times 981.$$

$$= 1.01 \times 10^6 \text{ dynes/cm}^2$$

$$\rho = 1.43 \text{ gm/litre}$$

$$= 0.00143 \text{ gm/c.c.}$$

$$\therefore c = \sqrt{\frac{3 \times 1.01 \times 10^6}{0.00143}}$$

$$= 4.6 \times 10^4 \text{ cm/sec.}$$

From  $PV = RT$ ,  $P$  will be doubled when  $T$  is doubled, volume remaining constant.

Let the r.m.s. speed of oxygen be  $c_1$  at  $T$  and  $c_2$  at  $2T$ .

$$\text{Now from } PV = \frac{1}{3} Mc^2 = RT.$$

$$c \propto \sqrt{T}$$

$$\therefore \frac{c_1}{c_2} = \sqrt{\frac{T}{2T}} = \sqrt{\frac{1}{2}}$$

$$\text{or, } c_2 = \sqrt{2} \cdot c_1$$

$$= \sqrt{2} \times 4.6 \times 10^4$$

$$= 6.5 \times 10^4 \text{ cm/sec.}$$

**Example 16.4.** Calculate the r.m.s. speed of oxygen molecules at  $27^\circ\text{C}$ .

**Soln.**

The r.m.s. speed of oxygen at  $0^{\circ}\text{C}$  (Example 16.1)

$$c_1 = 4.6 \times 10^4 \text{ cm/sec.}$$

Let the r.m.s. speed of oxygen at  $27^{\circ}\text{C}$  be  $c_2$

$$\text{Then } \frac{c_1}{c_2} = \sqrt{\frac{T_1}{T_2}}$$

$$\text{or, } c_2 = c_1 \times \sqrt{\frac{T_1}{T_2}}$$

where  $T_1 = 0^{\circ}\text{C} = 273 \text{ K}$

$T_2 = 27^{\circ}\text{C} = 300 \text{ K.}$

$$\begin{aligned} \therefore c_2 &= 4.6 \times 10^4 \times \sqrt{\frac{300}{273}} \\ &= 4.84 \times 10^4 \text{ cm/sec.} \end{aligned}$$

**Example 16.5.** At what celsius temperature will oxygen molecules have the same root mean square velocity as that of hydrogen molecules at  $-100^{\circ}\text{C}$ ?

**Soln.**

The energy of a gas molecule at any temperature  $TK$  is given by

$$\frac{1}{2}mc^2 = \frac{3}{2}kT$$

where  $m$  is the mass of the gas molecule,  $T$  is the temperature of the gas on the absolute scale,  $k$  is the Boltzmann's constant and  $c$  the r.m.s. velocity.

For hydrogen molecule

$$\frac{1}{2}m_1 c_1^2 = \frac{3}{2}kT_1 \quad (\text{i})$$

and for oxygen molecule

$$\frac{1}{2}m_2 c_2^2 = \frac{3}{2}kT_2 \quad (\text{ii})$$

Dividing (i) by (ii)



$$\frac{m_1 c_1^2}{m_2 c_2^2} = \frac{T_1}{T_2}$$

Here  $c_1 = c_2$

and  $T_1 = [273 + (-110)] = 173 \text{ K}$

$$\frac{m_1}{m_2} = \frac{1}{16} ; T_2 = ?$$

$$\therefore \frac{m_1}{m_2} = \frac{1}{16} = \frac{T_1}{T_2}$$

$$\begin{aligned} \text{or, } T_2 &= 16.T_1 = 16 \times 173 \\ &= 2768 \text{ K} = 2495^\circ\text{C.} \end{aligned}$$

**Example 16.6.** Calculate the kinetic energy of hydrogen per gram-molecule at  $0^\circ\text{C}$ . [ $R = 8.3 \times 10^7 \text{ ergs/gm-mol.}$ ]

**Soln.**

$$\frac{1}{2} M c^2 = \frac{3}{2} R T \quad \text{for 1 gm - molecule}$$

$$= \frac{3}{2} \times 8.3 \times 10^7 \times 273$$

$$= 3.4 \times 10^{10} \text{ ergs.}$$

**Example 16.7.** Calculate the molecular kinetic energy of 1 gm of hydrogen gas at  $50^\circ\text{C}$ , given that the molecular weight of hydrogen is 2 and  $R = 8.3 \times 10^7 \text{ ergs/gm-molecule.}$

**Soln.**

For 1 gm-molecule.

$$\frac{1}{2} M c^2 = \frac{3}{2} R T$$

$$\text{or, } \frac{1}{2} \cdot 1 \cdot C^2 = \frac{3}{2} \cdot \frac{R}{M} T \quad \text{for 1 gm.}$$

$$M = 2, R = 8.3 \times 10^7.$$

$$T = 273 + 50 = 323 \text{ K.}$$

$$\therefore \frac{1}{2} \cdot 1 \cdot C^2 = \frac{3}{2} \times \frac{8.3 \times 10^7}{2} \times 323$$

$$= 2.01 \times 10^{10} \text{ ergs.}$$

**Note :** Since the molecular weight is different for different gases, the kinetic energy of 1 gm of different gases will be different.

**Example 16.8.** Calculate the average kinetic energy of molecule of a gas at a temperature of 300K.

**Soln.**

The average kinetic energy of a molecule of a gas is a function of temperature only and is given by

$$\frac{1}{2} mc^2 = \frac{3}{2} kT = \frac{3}{2} \cdot \frac{R}{N} T$$

where  $k$  = Boltzmann's constant

$R$  = molar gas constant

$$= 8.3 \times 10^7 \text{ ergs/gm-mole.}$$

$N$  = Avogadro's number

$$= 6.023 \times 10^{23}$$

$T$  = temperature of the gas in K.

$$\therefore \frac{1}{2} mc^2 = \frac{3}{2} \times \frac{8.3 \times 10^7}{6.023 \times 10^{23}} \times 300$$

$$= 6.2 \times 10^{-14} \text{ ergs.}$$

**Note :** The average kinetic energy of a molecule of any gas such as hydrogen, oxygen, helium, nitrogen, air, etc., is the same at the same temperature.

**Example 16.9.** Find the kinetic energy of the molecules of 1 kg of helium at  $0^\circ\text{C}$  if  $R = 8,310 \text{ joules/kg-mole-K}$ . Molecular weight of helium is 4.

**Soln.**

Kinetic energy of one kg-molecule of helium,

$$\frac{1}{2} Mc^2 = \frac{3}{2} RT$$

$$\text{or, } \frac{1}{2} \cdot 1 \cdot c^2 = \frac{3}{2} \cdot \frac{R}{M} T$$

(for 1 kg. of helium.)

$$R = 8310 \text{ joules/kg-mole-K.}$$

$$M = 4 \text{ kg} \quad T = 273 \text{ K.}$$

$$\begin{aligned} \therefore \frac{1}{2} \cdot 1 \cdot c^2 &= \frac{3}{2} \times \frac{8310}{4} \times 273 \\ &= 8.5 \times 10^5 \text{ joules.} \end{aligned}$$

**Example 16.10.** The kinetic energy of a molecule of hydrogen at  $0^\circ\text{C}$  is  $5.64 \times 10^{-14}$  ergs and the molecular gas constant ( $R$ ) =  $8.32 \times 10^7$  ergs/gm-mole-K. Calculate Avogadro's number  $N$ .

**Soln.**

$$\frac{1}{2} mc^2 = \frac{3}{2} kT = \frac{3}{2} \cdot \frac{R}{N} T.$$

$$\text{or, } N = \frac{\frac{3}{2} RT}{\frac{1}{2} mc^2}$$

$$\begin{aligned} &= \frac{\frac{3}{2} \times 8.32 \times 10^7 \times 273}{5.64 \times 10^{-14}} \\ &= 6.08 \times 10^{23}. \end{aligned}$$

**Example 16.11.** Show that  $n$ , the number of molecules per unit volume of an ideal gas is given by



$$n = \frac{PN}{RT}$$

where  $N$  is the Avogadro's number.

**Soln.**

For one gm-molecule of an ideal gas,

$$PV = RT$$

$$= kNT$$

since  $\frac{R}{N} = k = \text{Boltzmann's constant}$

Let  $n$  be the number of molecules per c.c. i.e.,  $N = n$  when  $V = 1$  c.c.

Then

$$P = knT$$

$$\text{or, } n = \frac{P}{kT} = \frac{P}{\frac{R}{N}T} = \frac{PN}{RT}$$

**Example 16.12.** Calculate the number of molecules in one cubic metre of an ideal gas at N.T.P.

**Soln.**

Let the number of molecules per c.c. of the gas be  $n$ . Since 1 cubic metre =  $10^6$  c.c., the number of molecules in 1 cubic metre,  $x = n \times 10^6$

$$\text{Now } n = \frac{PN}{RT}$$

$$\therefore x = n \times 10^6 = \frac{PN}{RT} \times 10^6$$

$$\text{Here } P = 76 \times 13.6 \times 981 \text{ dynes / cm}^2$$

$$N = 6.023 \times 10^{23}$$

$$R = 8.31 \times 10^7 \text{ ergs/gm-mole-K}$$

$$T = 273 \text{ K.}$$

$$\therefore x = \frac{76 \times 13.6 \times 981 \times 6.023 \times 10^{23}}{8.31 \times 10^7 \times 273} \times 10^6$$

$$= 2.688 \times 10^{25}.$$

**Example 16.13.** Calculate the number of molecules in one litre of an ideal gas at  $127^\circ\text{C}$  and 2 atmospheric pressure.

**Soln.**

Let the number of molecules per c.c. of the gas =  $n$ .

$$n = \frac{PN}{RT}$$

No. of molecules per litre ( $10^3$  c.c.),

$$x = n \times 10^3$$

$$= \frac{PN}{RT} \times 10^3$$

$$N = 6.023 \times 10^{23}$$

$$P = 2 \times 76 \times 13.6 \times 981 \text{ dynes/cm}^2$$

$$R = 8.31 \times 10^7 \text{ ergs/gm-mole-K}$$

$$T = (273 + 127) = 400 \text{ K.}$$

$$= \frac{2 \times 76 \times 13.6 \times 981 \times 6.023 \times 10^{23}}{8.31 \times 10^7 \times 400} \times 10^3$$

$$= 36.71 \times 10^{21},$$

**Example 16.14.** At what temperature, pressure remaining constant, will the r.m.s. velocity of a gas molecule be half its value at  $0^\circ\text{C}$ ?

**Soln.** For a gas molecule,

$$\frac{1}{2} mc^2 = \frac{3}{2} kT$$

$$\text{or, } c = \sqrt{\frac{3kT}{m}}$$

If  $c_1$  and  $c_2$  are the velocities of the gas molecule at  $T_1^\circ\text{C}$  and  $T_2^\circ\text{C}$  respectively, then from the above relation,

$$\frac{c_1}{c_2} = \sqrt{\frac{T_1}{T_2}}$$

Here  $c_2 = \frac{1}{2}c_1$  and  $T_1 = 0^\circ\text{C} = 273\text{K}$

$$\therefore 2 = \sqrt{\frac{T_1}{T_2}} = \sqrt{\frac{273}{T_2}}$$

$$\text{or, } T_2 = \frac{273}{4} = 68.25\text{ K} = -204.75^\circ\text{C}.$$

**Example 16.15.** At what temperature is the r.m.s. speed of hydrogen molecules (molecular weight = 2) equal to that of oxygen molecules (molecular weight = 32) at  $47^\circ\text{C}$ ?

**Soln.** Let the required temperature be  $T$ . Then

$$\frac{1}{2} M c_1^2 = \frac{3}{2} RT; \quad \text{or, } c_1^2 = \frac{3RT}{M} = \frac{3RT}{2}$$

r.m.s. speed of oxygen at  $47^\circ\text{C}$  (320K) is

$$\frac{1}{2} M c_2^2 = \frac{3}{2} RT; \quad \text{or, } c_2^2 = \frac{3RT}{M} = \frac{3R(320)}{32}$$

Since  $c_1$  should be equal to  $c_2$ ,

$$\frac{3RT}{2} = \frac{3R(320)}{32}$$

$$\text{or, } T = 2 \times 10 = 20\text{K} = -253^\circ\text{C}.$$

**Example 16.16.** At what temperature is the r.m.s. speed of hydrogen molecules equal to the escape speed from the earth's surface? What is the corresponding temperature for escape of hydrogen from the moon's surface? The escape speed is given by  $\sqrt{2gR}$ .

**Soln.** Escape speed from earth's surface,

$$V_c = \sqrt{(2)(9.8)(6367 \times 10^3)} = 11.2 \times 10^3 \text{ m/s}.$$



We have  $\frac{1}{2}Mc^2 = \frac{3}{2}RT$ ; or,  $T = \frac{Mc^2}{3R}$

$M = 2\text{ kg/kmol}$ ,  $c = 11.2 \times 10^3 \text{ m/s}$ ,  $R = 8314 \text{ J/kmol}$

$\therefore T = \frac{(2)(11.2 \times 10^3)^2}{(3)(8314)} = 10059 \text{ K.}$

For the moon's surface,  $v_M = \sqrt{(2)(1.6)(1750 \times 10^3)} = 2.37 \times 10^3 \text{ m/s}$

Now

$\frac{v_m^2}{v_E^2} = \frac{T_m}{T_E}$ ; or,  $T_m = \frac{v_m^2}{v_E^2} \cdot T_E$

$\therefore T_m = \left( \frac{2.37 \times 10^3}{11.2 \times 10^3} \right)^2 (10059) = 450 \text{ K.}$

### 16.9 Maxwell's law of distribution of molecular velocities

At a given temperature, a gas molecule has a fixed mean kinetic energy. The root mean square velocity of the molecule can be computed from the temperature of the gas. However, it does not mean that the molecule is moving with the same speed throughout its movement. After each encounter the speed of the molecule changes and due to a large number of collisions, the speed is different at different instants. But the root mean square velocity  $c$  remains the same at a fixed temperature. But this does not tell the whole story. At any instant, all the molecules are not moving with the same velocity. For many physical and chemical problems, it is important to know the *speed distribution* of the molecules. For instance, we may like to know how many molecules have more than twice the root mean square speed  $c$  or how many have speeds in the interval  $0.79c$  to  $0.80c$ . This can be found from Maxwell's distribution law. We shall now proceed to derive this law.

Let the number of molecules per unit volume having the component of velocities in  $x$  direction in the range between  $u$  and  $u+du$  be  $n_u du$ . The number  $n_u$  must obviously be a function of  $u$ , say  $nf(u)$ , where  $n$  is the number of molecules per unit volume and  $f(u)$  is some function of  $u$  which is to be determined. The probability that a molecule will have velocities lying between  $u$  and  $u+du$  will then be

$f(u) du$ . Let the velocity of a molecule denoted by  $c$  have the components  $u$ ,  $v$ , and  $w$  at right angles to each other. Maxwell assumed that as the velocity components are perpendicular to each other the distribution of one of these components, among the molecules will not depend upon the values of other components. In other words,  $f(u)$  is independent of  $v$  and  $w$ . Considering the other two component of velocities, it can be similarly assumed that the probability that a molecule will have velocities lying between  $v$  and  $v+dv$  is  $f(v)dv$  and that for a molecule having velocities lying between  $w$  and  $w+dw$  is  $f(w) dw$ ,  $f$ , being the same function on account of isotropy as regards the molecular velocity.

Since the probability of each individual component is independent of the probability of other components, according to the theorem of probability, the probability of a composite event is equal to the product of the probability of individual events. Thus, the probability that a molecule may have its velocity components lying between  $u$  and  $u+du$ ,  $v$  and  $v+dv$ , and  $w$  and  $w+dw$  is

$$f(u) f(v) f(w) du dv dw$$

where  $f(u)du$ ,  $f(v)dv$  and  $f(w)dw$  are the probability of the individual components as mentioned above. The number of such molecules per unit volume will be

$$nf(u) f(v) f(w) du dv dw$$

In order to find out the value of  $f(u) f(v) f(w)$  in terms of known quantities, let us represent all the molecules in a velocity diagram with  $OX$ ,  $OY$ ,  $OZ$  as the co-ordinate axes along which the components  $u$ ,  $v$  and  $w$  of the velocity  $c$  are respectively measured.

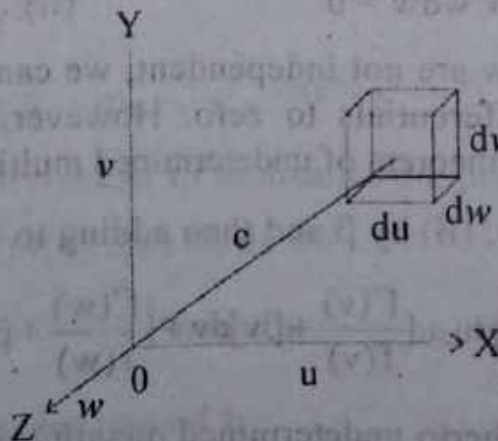


Fig. 16.3



A molecule with component of velocities,  $u$ ,  $v$  and  $w$  will therefore be represented by a point whose co-ordinates are  $u$ ,  $v$  and  $w$ . Such a point is referred to as *velocity point*. All the molecules having velocity component lying between  $u$  and  $u+du$ ,  $v$  and  $v+dv$  and  $w$  and  $w+dw$  will be contained in an element of volume  $du dv dw$  and their numbers will be given by  $nf(u)f(v)f(w)du dv dw$ . Since the resultant velocity of these molecules is given by  $c^2 = u^2 + v^2 + w^2$ , their number must also be given by  $nf(c)du dv dw$  which is the product of some function of  $c$  and the interval  $du$ ,  $dv$ ,  $dw$ . As the directions of the axes are quite arbitrary, there is nothing to distinguish one direction from another in a gas at rest; the number of such molecules cannot depend on the inclination of  $c$  to the axes. We therefore have,

$$nf(u) f(v) f(w) = nF(c) = n \phi(u^2 v^2 w^2)$$

where  $\phi$  is some other function.

We shall now try to solve this equation. It is obvious that for a fixed value of  $c$ ,  $\phi(c^2)$  is constant. Therefore,

$$d[\phi(c^2)] = 0$$

$$\text{or, } d[f(u) f(v) f(w)] = 0$$

$$\therefore f'(u) du f(v) f(w) + f'(v) dv f(u) f(w) + f'(w) dw f(u) f(v) = 0$$

Dividing throughout by  $f(u) f(v) f(w)$ ,

$$\frac{f'(u)}{f(u)} du + \frac{f'(v)}{f(v)} dv + \frac{f'(w)}{f(w)} dw = 0 \quad (i)$$

But  $c^2 = u^2 + v^2 + w^2$ , therefore, for fixed value of  $c$

$$u du + v dv + w dw = 0 \quad (ii)$$

Since  $du$ ,  $dv$ ,  $dw$  are not independent, we cannot equate the co-efficients of the differentials to zero. However, we can *in effect* make them so by the theorem of undetermined multipliers of Lagrange.

Multiplying eqn. (ii) by  $\beta$  and then adding to (i) we get

$$\left[\frac{f'(u)}{f(u)} + \beta u\right] du + \left[\frac{f'(v)}{f(v)} + \beta v\right] dv + \left[\frac{f'(w)}{f(w)} + \beta w\right] dw = 0$$

where  $\beta$  is a hitherto undetermined quantity. As the differentials  $du$ ,  $dv$  and  $dw$  are now in effect independent, the co-efficients of the



of the three differentials in this expression must individually be zero.

$$\left. \begin{aligned} \frac{f'(u)}{f(u)} &= -\beta u \\ \frac{f'(v)}{f(v)} &= -\beta v \\ \frac{f'(w)}{f(w)} &= -\beta w \end{aligned} \right\} \quad \text{(iii)}$$

Integrating the first equation of set (iii) with respect to  $u$ , we get

$$\log f(u) = -\frac{\beta u^2}{2} + a$$

where  $a$  is some constant.

$$\text{or, } \log \frac{f(u)}{a} = -\frac{\beta u^2}{2}$$

$$f(u) = a e^{-\frac{\beta u^2}{2}} = a e^{-bu^2} \quad \text{(iv)}$$

$$\text{where } b = \frac{\beta}{2}$$

Similarly we have

$$f(v) = a e^{-bv^2}$$

$$\text{and } f(w) = a e^{-bw^2}$$

$$\text{Therefore } f(u) f(v) f(w) = a^3 e^{-b(u^2+v^2+w^2)} \quad \text{(v)}$$

Hence the total number of molecules  $dn$ , per unit volume is,

$$dn = n f(u) f(v) f(w) du dv dw$$

$$= na^3 e^{-b(u^2+v^2+w^2)} du dv dw \quad \text{(vi)}$$

### Evaluation of constants $a$ and $b$

To determine the value of  $a$  we integrate the expression for total number of molecules per unit volume having all the velocities from

$-\infty$  to  $+\infty$  and then equate it to  $n$ .

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} n f(u) f(v) f(w) du dv dw = n$$

Substituting the value of the function  $f(u) f(v) f(w)$  from (v) we get

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} a^3 e^{-b(u^2+v^2+w^2)} du dv dw = 1 \quad (\text{vii})$$

The value of definite integral of the form  $\int_{-\infty}^{+\infty} e^{-bu^2} du$  is  $\sqrt{\frac{\pi}{b}}$

Substituting the values of such integral in eqn (vii), we get

$$a^3 \left( \frac{\pi}{b} \right)^{3/2} = 1$$

$$\text{or, } a^3 = \left( \frac{b}{\pi} \right)^{3/2}$$

$$\text{or, } a = \sqrt{\frac{b}{\pi}} \quad (\text{viii})$$

### Evaluation of $b$

In evaluating  $b$  let us consider the pressure  $p$  exerted by the gas which is given by the change in momentum suffered by the molecules striking per unit area of the wall of the containing vessel. If  $m$  is the mass of a molecule moving with a velocity  $u$ , then the change in momentum per collision is (Art. 16.3)

$$mu - (-mu) = 2mu.$$

If  $n_u$  be the number of molecules per unit volume moving with velocity  $u$ , the number striking unit area of the wall in time  $dt$  would be contained in a cylinder of area unity and vertical height  $udt$ .

$$\therefore \text{ volume of the cylinder} = udt$$

$$\therefore \text{ Number of molecules in the cylinder} = n_u udt$$

$\therefore$  Change in momentum per unit area suffered by the above molecules in time  $dt$  is

$$2mu \times n_u u dt$$

Considering all the molecules moving in the positive  $x$ -direction, the above quantity, *i.e.*, the total change in momentum per unit area is

$$2m \sum_0^{\infty} n_u u^2 dt$$

If the above change in momentum results in an average force  $\delta F$ , then,

$$\delta F \cdot dt = 2m \sum_0^{\infty} n_u u^2 dt$$

$$\therefore \delta F = 2m \sum_0^{\infty} n_u u^2$$

Since the area involved is unity  $\delta F$  is the force per unit area which is the pressure  $p$ .

$$\therefore p = 2m \sum_0^{\infty} n_u u^2 \quad \text{(ix)}$$

where  $n_u$  is a function of  $u$ .

The summation sign has been put because the pressure is caused by all the molecules having a positive component of velocity which can have values ranging from 0 to  $\infty$ .

The number of molecules per unit volume having the velocity component lying between  $u$  and  $u+du$  is given by

$$n_u = nf(u) = na e^{-bu^2}$$

$$\text{or, } n_u = n \sqrt{\frac{b}{\pi}} e^{-bu^2} \quad \text{(x)}$$

Substituting the value of  $n_u$  as given by eqn. (x) in eqn (ix) and replacing the summation with integration sign, the pressure  $p$  is given by



$$p = 2m \int_0^{\infty} n \sqrt{\frac{b}{\pi}} e^{-bu^2} u^2 du$$

$$\text{But } \int_0^{\infty} e^{-bu^2} u^2 du = \frac{1}{4} \sqrt{\frac{\pi}{b^3}}$$

$$\therefore p = 2mn \sqrt{\frac{b}{\pi}} \cdot \frac{1}{4} \sqrt{\frac{\pi}{b^3}}$$

$$= \frac{mn}{2b}$$

$p$  is also given by,

$$p = nkT$$

where  $k$  is the Boltzmann constant. Therefore

$$nkT = \frac{mn}{2b}$$

$$\text{or, } b = \frac{m}{2kT} \quad (\text{xix})$$

Substituting this value of  $b$  in eqn. (viii), the constant  $a$  is given by

$$a = \sqrt{\frac{b}{\pi}} = \sqrt{\frac{m}{2k\pi T}} \quad (\text{xii})$$

On substituting the values of  $a$  and  $b$  as given by eqns. (xi) and (xii), in eqn. (vi), the number  $dn$  of the molecules per unit volume having velocity components between  $u$  and  $u+du$ ,  $v$  and  $v+dv$  and  $w$  and  $w+dw$  is given by

$$dn = n \left( \frac{m}{2\pi kT} \right)^{3/2} e^{-\frac{m}{2} \left( \frac{u^2 + v^2 + w^2}{kT} \right)} du dv dw \quad (\text{xiii})$$

Eqn. (xiii) can be used to find the number of molecules whose velocity lie between  $c$  and  $c+dc$ . Let all such molecules lie between two concentric spheres of radii  $c$  and  $c+dc$  with  $O$ , which is the starting point of all the molecules, as the common centre of the spheres (Fig. 16.4).

The volume of the annulus lying between these spheres

$$= \frac{4\pi}{3} \pi [(c+dc)^3 - c^3]$$

$$= 4\pi c^2 dc$$

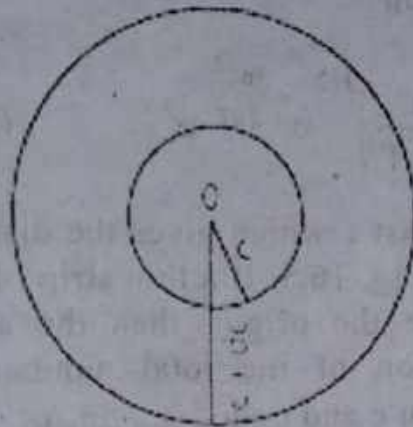


Fig. 16.4

This is the same as the volume given by  $dudvdw$ . Therefore, substituting  $4\pi c^2 dc$  for  $dudvdw$  in eqn. (vi) and putting  $c^2 = u^2 + v^2 + w^2$ , we get

$$dn_c = 4\pi n a^3 e^{-bc^2} c^2 dc$$

$$= 4\pi n \left( \frac{m}{2\pi kT} \right)^{3/2} c^2 e^{-\frac{mc^2}{2kT}} dc \quad \text{(xiv)}$$

where  $dn_c$  is the number of molecules per unit volume having velocities lying between  $c$  and  $c+dc$ . Eqn. (xiv) is known as the Maxwell-Boltzmann law for distribution of velocities of a large number of molecules in a gas. As can be seen from the equation, for a given gas, the velocity distribution depends only on temperature. The total number of molecules in the gas can be obtained by integrating the number present in each differential velocity interval from zero to infinity.

$$n = \int_0^{\infty} 4\pi n \left( \frac{m}{2\pi kT} \right)^{3/2} c^2 e^{-\frac{mc^2}{2kT}} dc \quad \text{(xv)}$$

Eqn. (xiv) can be rearranged as

$$\frac{dn_c}{n} = 4\pi \left( \frac{m}{2\pi kT} \right)^{3/2} c^2 e^{-\frac{mc^2}{2kT}} dc = Fdc$$

where the function

$$F = 4\pi \left( \frac{m}{2\pi kT} \right)^{3/2} e^{-\frac{mc^2}{2kT}} c^2 \quad (\text{xvi})$$

A plot of  $F$  against  $c$  which gives the distribution of molecular velocity is shown in Fig. 16.5. If a thin strip of thickness  $dc$  is taken at a distance  $c$  from the origin, then the area of the strip  $Fdc$  represents the fraction of the total number of molecules with velocity lying between  $c$  and  $c+dc$ , i.e.,  $dn_c/n$ .

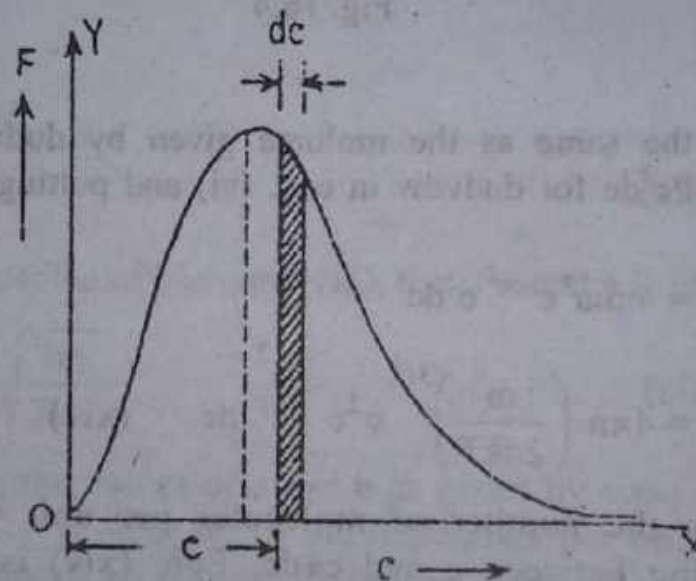


Fig. 16.5

From Maxwell's law, as well as from the nature of the curve, it can be seen that  $dn_c = 0$  when  $c = 0$  and  $c = \infty$ . It means that the number of molecules having velocities tending to zero and also tending to infinity is quite small. Between these two extremes there is a value of  $c$  for which  $F$  is maximum, represented by the dotted ordinate in the figure. Maximum molecules will possess this velocity called the *most probable velocity*.



**Average speed, r.m.s. speed and most probable speed**

Attempt will now be made to obtain, on the basis of Maxwell-Boltzmann velocity distribution law, expressions for the average or mean speed, root mean square speed and the most probable speed.

**Average or mean speed :** Let  $n_1$  molecules have speed  $c_1$ ,  $n_2$  have speed  $c_2$  and so on. Then, by definition, the *average or mean speed*  $\bar{c}$  is given by

$$\bar{c} = \frac{n_1 c_1 + n_2 c_2 + n_3 c_3 + \dots}{n_1 + n_2 + n_3 + \dots}$$

$$\bar{c} = \frac{\sum n_i c_i}{\sum n_i}$$

$$= \frac{1}{n} \int_0^{\infty} c dn_c$$

$$\text{Now } dn_c = 4\pi n a^3 e^{-bc^2} \cdot c^2 dc$$

$$\therefore \bar{c} = \frac{4\pi n a^3}{n} \int_0^{\infty} e^{-bc^2} c^3 dc$$

$$= 4\pi a^3 \int_0^{\infty} e^{-bc^2} c^3 dc$$

$$\text{But } \int_0^{\infty} e^{-bc^2} c^3 dc = \frac{1}{2b^2} \int_0^{\infty} e^{-z} z dz \quad (\text{substituting } bc^2 = z)$$

$$= \frac{1}{2b^2} \tau(2) = \frac{1}{2b^2} \cdot 1 \cdot \tau(1)$$

$$= \frac{1}{2b^2} \quad [\because \tau(1) = 1]$$

$$\therefore \bar{c} = \frac{4\pi a^3}{2b^2} = \frac{2\pi a^3}{b^2}$$

Substituting the values of  $a$  and  $b$

$$\bar{c} = \frac{2\pi(m/2\pi kT)^{3/2}}{(m/2kT)^2}$$

$$= \sqrt{\frac{8kT}{\pi m}}$$

**Root mean square (r.m.s.) speed :** By definition r.m.s. is the square root of the mean value of the squares of the velocities of all molecules.

Thus if  $n_1$  molecules have velocities  $c_1$ ,  $n_2$  molecules have velocities  $c_2$ , and so on, then

$$c_{rms} = \sqrt{\frac{n_1 c_1^2 + n_2 c_2^2 + n_3 c_3^2 + \dots}{n_1 + n_2 + n_3 + \dots}} = \sqrt{\frac{\sum n_i c_i^2}{\sum n_i}}$$

Now,

$$\frac{\sum n_i c_i^2}{\sum n_i} = \frac{1}{n} \int_0^\infty c^2 dn_c \quad (\because \sum n_i = n)$$

$$= \frac{4\pi n a^3}{n} \int_0^\infty c^4 e^{-bc^2} dc \quad (\text{Substituting the value of } dn_c)$$

$$= 4\pi a^3 \int_0^\infty c^4 e^{-bc^2} dc$$

$$= 4\pi a^3 \frac{1}{2b^{5/2}} \tau (5/2) \quad [\because \int_0^\infty c^4 e^{-bc^2} dc = \tau (5/2)]$$

$$= 4\pi a^3 \frac{1}{2b^{5/2}} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}$$

$$= \frac{3}{2} \pi^{3/2} \frac{(m/2\pi kT)^{3/2}}{(m/2kT)^{5/2}} \quad (\text{Substituting the values of } a \text{ and } b)$$

$$= \frac{3kT}{m}$$

$$\therefore \text{r.m.s speed} = \sqrt{\frac{3kT}{m}}$$

**Most probable speed :** From Maxwell's velocity distribution law, we have

$$F = 4\pi a^3 c^2 e^{-bc^2} \quad (1)$$

The value of  $c$  for which  $F$  is maximum is the *most probable velocity*,  $c_m$ .

Differentiating the above equation (1) with respect to  $c$ , we have

$$\frac{dF}{dc} = 8\pi c a^3 e^{-bc^2} - 8\pi b c^3 a^3 e^{-bc^2} \quad (2)$$

For  $F$  to be maximum,  $\frac{dF}{dc} = 0$ ,

$$\therefore 8\pi c_m a^3 e^{-bc^2} - 8\pi b c_m^3 a^3 e^{-bc^2} = 0 \quad (c = c_m)$$

$$\text{or, } 8\pi c_m a^3 e^{-bc^2} = 8\pi b c_m^3 a^3 e^{-bc^2}$$

$$\text{or, } c_m^3 = \frac{1}{b} = \frac{2kT}{m}$$

$$\therefore c_m = \sqrt{\frac{2kT}{m}}$$

From eqn. (2), a second derivative  $\frac{d^2F}{dc^2}$  may be obtained, which

turns out to be negative. So the value of  $c_m$  as given above corresponds to maximum  $F$ .

**Ratio of the three speeds** – The ratio of the average, r.m.s. and most probable velocities at a given temperature is, therefore, given by

$$\begin{aligned} \bar{c} : c_{rms} : c_m &= \sqrt{\frac{8kT}{\pi m}} : \sqrt{\frac{3kT}{m}} : \sqrt{\frac{2kT}{m}} \\ &= \sqrt{8/\pi} : \sqrt{3} : \sqrt{2} \end{aligned}$$

$$\text{or, } \frac{\bar{c}}{c_{rms}} = \sqrt{\frac{8}{3\pi}} = 0.921 \quad \text{and} \quad \frac{c_m}{c_{rms}} = \sqrt{\frac{2}{3}} = 0.817$$



$$\therefore c_m : \bar{c} : c_{rms} = 1 : 1.128 : 1.224$$

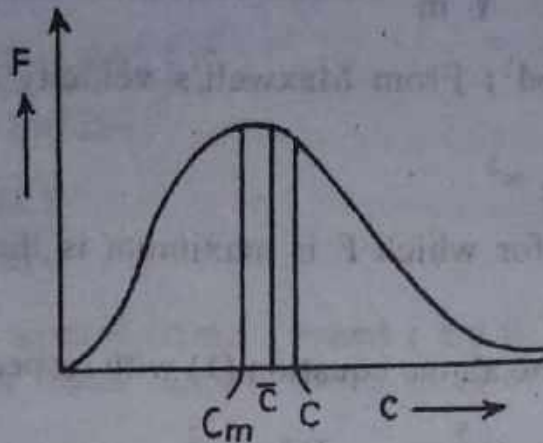


Fig. 16.6

### Energy distribution of molecules

Eqn. (xiv) can be used to determine the number  $dn_E$  of molecules per unit volume whose kinetic energy lies between  $E$  and  $E+dE$ , by putting  $E = \frac{1}{2}mc^2$ .

$$dn_E = nf(E) dE = 4\pi n \left( \frac{m}{2\pi kT} \right)^{3/2} c^2 e^{-\frac{mc^2}{2kT}} dc$$

From  $E = \frac{1}{2}mc^2$  we have  $c^2 = \frac{2E}{m}$  and  $dc = \frac{dE}{\sqrt{2Em}}$

$$\therefore dn_E = 4\pi n \left( \frac{m}{2\pi kT} \right)^{3/2} e^{-\frac{E}{kT}} \frac{2E}{m} \frac{dE}{\sqrt{2Em}}$$

$$\text{or, } dn_E = 2n \frac{\sqrt{E}}{\sqrt{\pi} (kT)^{3/2}} e^{-E/kT} dE$$

Therefore, the fraction of the molecules having kinetic energy between  $E$  and  $E+dE$  is

$$\frac{dn_E}{n} = f(E)dE = \frac{2\sqrt{E}}{\sqrt{\pi} (kT)^{3/2}} e^{-E/kT} dE$$

The above expression is usually written in the form

$$f(E) = A(T) e^{-E/kT}$$

$$\text{where } A = \frac{2\sqrt{E}}{\sqrt{\pi} (kT)^{3/2}}$$

and is a function of temperature  $T$ .

### Temperature dependence of velocity distribution

If the equation

$$\frac{dn_c}{n} = 4\pi \left( \frac{m}{2\pi kT} \right)^{3/2} e^{-mc^2/2kT} c^2 dc$$

is integrated with respect to  $c$  for all possible values, we should get 1.

$$\therefore \int_0^{\infty} 4\pi \left( \frac{m}{2\pi kT} \right)^{3/2} e^{-mc^2/2kT} c^2 dc = 1. \quad (3)$$

$$\text{or, } \int_0^{\infty} F dc = 1 \quad \text{where } F = 4\pi \left( \frac{m}{2\pi kT} \right)^{3/2} e^{-mc^2/2kT} c^2.$$

The above expression means that if  $F$  is plotted against  $C$ , the area under the distribution curve will be unity.

But it has been observed earlier that the maximum of the  $F$ - $c$  curve

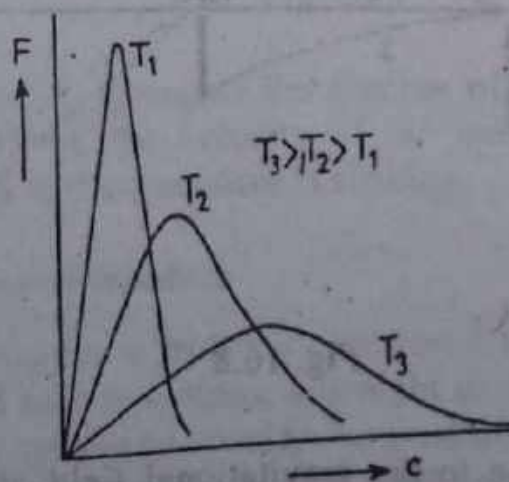


Fig. 16.7

occurs at  $c = c_m = \sqrt{\frac{2kT}{m}}$ . So when the temperature increases, the peak of the curve should get itself shifted to a higher  $c$ -value. But since, by relation (3), the area is fixed (unity), the curve would get broadened with increase in temperature. The broadening will naturally depend on the mass of the molecules. Fig. 16.7 shows this broadening of the distribution curve with increase in temperature.

### Experimental verification of velocity distribution

Zartman and Ko performed an experiment to study the distribution of velocity in 1934. In 1947, Estermann, Simpson and Stern designed a more precise apparatus to study the velocity distribution. This is described below.

The experimental arrangement is shown below diagrammatically (Fig. 16.8). O is an electrical oven from which a beam of caesium atoms emerges. The beam next passes through the collimating slit S and impinges upon the hot tungsten wire D. The whole apparatus is enclosed in an evacuated chamber. The pressure of the residual gas is very low :  $10^{-8}$  mm of Hg. The opening O and the slit S are horizontal. In the absence of a gravitational field, only the atoms emerging horizontally would pass through S and strike D irrespective

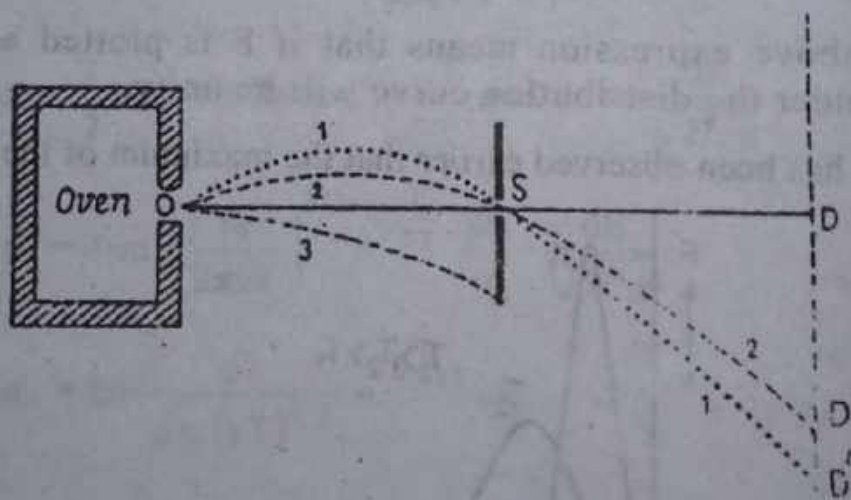


Fig. 16.8

of velocities. But due to the gravitational field, the path is in fact parabolic. The atoms going along the path 3 do not reach the wire



The atoms going along the paths 2 and 1 would pass through S and reach the wire at D' and D'' respectively. The molecules of path 2 move slower than those of path 1.

The caesium atoms strike the wire and get ionised and re-evaporate. They are collected by a negatively charged detecting cylinder (not shown) surrounding the tungsten wire. The magnitude of the current indicates the intensity of the atoms at various positions. The vertical height of the detector from the point D measured downwards represents the magnitude of the velocity and the ionisation current indicates the number of atoms striking the wire at a particular point.

Fig. 16.9 shows both the theoretical and experimental distribution of velocities – the dotted curve experimental and the continuous curve theoretical.

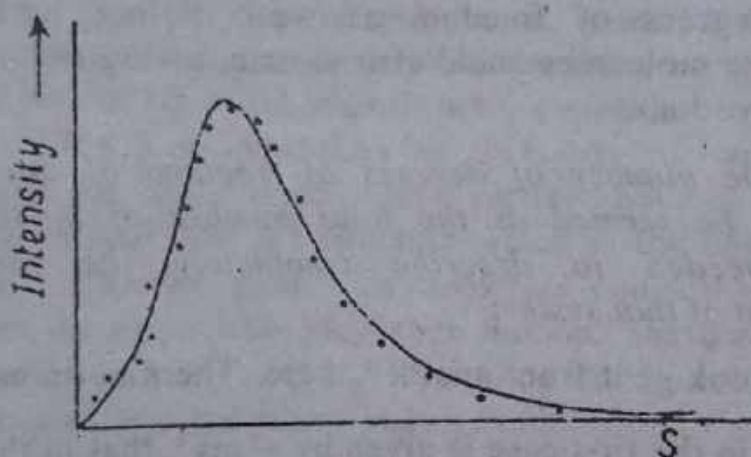


Fig. 16.9

The ordinate represents the number of molecules whereas the abscissa represents the velocity of the molecules. The agreement between theory and experiment is striking.

### 16.10 Degrees of freedom

Let us consider a molecule, supposed to be a rigid particle, to be constrained to move along a straight or a curved line. Only *one co-ordinate* is needed to specify the position and state of motion of the particle. The particle is then said to possess *one degree of freedom*. If the same molecule is now made to move in a plane, then

its velocity  $v$  can be resolved into two mutually perpendicular components  $v_x$  and  $v_y$  such that  $v = \sqrt{v_x^2 + v_y^2}$ . Thus two co-ordinates are now needed to specify the position and state of motion of the particle restricted to move in a plane. The particle is now said to possess *two degrees of freedom*. If the molecule can move freely in a three dimensional space, it will have *three degrees of freedom*, since its velocity can be resolved in three mutually perpendicular components ( $v^2 = v_x^2 + v_y^2 + v_z^2$ ) and three co-ordinates are now required to describe its position and state of motion.

Molecules, however, are not geometric points but are of finite size. If the molecule in the foregoing discussion is pictured as an object with internal structure, it will possess not only mass but also moment of inertia and therefore have, in addition to translational motion, rotational motion also. So molecules are expected to have rotational degrees of freedom as well. If not perfectly rigid structures, the molecules could also vibrate, giving rise to still more degrees of freedom.

Thus *the number of degrees of freedom of any dynamical system may be termed as the total number of independent co-ordinates needed to describe completely the position and configuration of that system.*

Let us look at it from another angle. The kinetic energy of the free particle in the first case is given by  $\frac{1}{2}mv^2$ , that in the second by  $\frac{1}{2}m(v_x^2 + v_y^2)$  and in the third by  $\frac{1}{2}m(v_x^2 + v_y^2 + v_z^2)$ . Similarly for a molecule with internal structure having rotational degrees of freedom, the Kinetic energy is given by  $\frac{1}{2}I\omega^2$  where  $I$  is the moment of inertia and  $\omega$  the angular velocity. The angular velocity vector of a rotating molecule can have a component along all the three coordinate axes. In addition the molecule may possess kinetic energy associated with the vibration of the atoms in a molecule (vibrational energy =  $\frac{1}{2}\mu v^2$ ). Thus *the number of degrees of freedom may be defined in a more general way as the number of independent*



*squared terms which enter into the expression for the energy of the system i.e., the total number of independent modes by which energy can be absorbed.*

### 16.11 Equipartition of energy

In the simplifying assumption of kinetic theory, a molecule was supposed to behave like a hard elastic sphere and its kinetic energy was supposed to be purely translational. The success of this simple model in correctly predicting the behaviour of gases of all kinds over a wide range of temperature led one to believe that it is the average kinetic energy of translation which determines what is measured as the temperature of the gas.

Although, the kinetic theory, in its simple form, was able to explain the specific heat of monatomic gas successfully, difficulty arose while explaining the specific heats of polyatomic gases. It then became necessary to look into all possible ways of absorbing energy and also to look into the possibility of whether or not a molecule can store energy internally, i.e., in a form other than kinetic energy of translation. This will certainly be the case if a molecule is pictured not as a rigid particle but as an object with internal structure which is the case in case of molecules of polyatomic gases. The molecule could then rotate and vibrate as well as move with translation motion. These rotational and vibrational modes of motion could be excited during collisions which would then contribute to the internal energy of the gas.

With this modification of the kinetic theory, let us now find the total energy of a large number of molecules, each thought of as an object having internal structure. Neglecting contributions of other kinds of energy which might exist, such as magnetic, the total energy will consist of kinetic energy of translation, rotational kinetic energy and kinetic energy associated with the vibration of the atoms in a molecule.

It can be shown from statistical mechanics that when the number of molecules is large and Newtonian mechanics holds, *the mean kinetic energy of a system of molecules in thermal equilibrium at the temperature  $T$ , is uniformly distributed among all the degrees of freedom, and for each degree of freedom of a molecule it equals*

*$\frac{1}{2}kT$ , where  $k$  is the Boltzmann constant.*



### Derivation

We shall now derive equipartition theorem. In deriving the theorem it will be assumed that (i) the system consists of free particles (no constraints) only, (ii) the energy of the system is expressible as the sum of the squares of positional or momentum coordinates whose number is equal to the number of degrees of freedom, and (iii) the energy, unlike quantum concept, is not discrete but continuous.

If a particle possesses  $F$  degrees of freedom, its energy  $E$  is given by

$$E = a_1 \alpha_1^2 + a_2 \alpha_2^2 + a_3 \alpha_3^2 + \dots + a_F \alpha_F^2$$

where  $a$ 's are constants and  $\alpha$ 's are either positional or momentum coordinates.

Now, from Boltzmann's law, the number of particles between  $\alpha_1$  and  $\alpha_1 + d\alpha_1$ ,  $\alpha_2$  and  $\alpha_2 + d\alpha_2$ , etc. having  $E$  as energy is

$$dN = A n e^{-E/kT} d\alpha_1 d\alpha_2 \dots d\alpha_F$$

$$\therefore n = \int dn = A n \int \dots \int e^{-E/kT} d\alpha_1 d\alpha_2 \dots d\alpha_F$$

$$\therefore A = \left[ \int \dots \int e^{-E/kT} d\alpha_1 d\alpha_2 \dots d\alpha_F \right]^{-1} \quad (i)$$

Since  $E$  is a homogeneous function of  $\alpha$ 's of degree 2, we have by Euler's theorem

$$\alpha_1 \left( \frac{\partial E}{\partial \alpha_1} \right) + \alpha_2 \left( \frac{\partial E}{\partial \alpha_2} \right) + \dots + \alpha_F \left( \frac{\partial E}{\partial \alpha_F} \right) = 2E$$

$$\therefore E = \frac{1}{2} \sum_i^F \alpha_i \left( \frac{\partial E}{\partial \alpha_i} \right) \quad (ii)$$

The total energy of the system of  $n$  - particles is

$$U = \int E dn$$

$$= \int \dots \int \left[ \frac{1}{2} \sum_i \alpha_i \left( \frac{\partial E}{\partial \alpha_i} \right) \right] \{ A n e^{-E/kT} d\alpha_1 \dots d\alpha_F \}$$

$$= \frac{1}{2} A n \int \dots \int \left[ \alpha_1 \frac{\partial E}{\partial \alpha_1} + \alpha_2 \frac{\partial E}{\partial \alpha_2} + \dots + \alpha_F \frac{\partial E}{\partial \alpha_F} \right] e^{-E/kT} d\alpha_1 \dots d\alpha_F \quad (\text{iii})$$

Let us consider the first integral, given by

$$I_1 = \int \dots \int \alpha_1 \frac{\partial E}{\partial \alpha_1} e^{-E/kT} d\alpha_1 \dots d\alpha_F$$

$$= \int \dots \int \alpha_1 e^{-E/kT} dE d\alpha_2 d\alpha_3 \dots d\alpha_F$$

since corresponding to  $\alpha_1$ ,  $dE = (\partial E / \partial \alpha_1) d\alpha_1$

$$\text{But } d \left[ e^{-E/kT} \right] = -e^{-E/kT} \frac{1}{kT} dE$$

$$\therefore e^{-E/kT} = kT d \left[ e^{-E/kT} \right]$$

$$\therefore I_1 = -kT \int \dots \int \alpha_1 d \left[ e^{-E/kT} \right] d\alpha_2 d\alpha_3 \dots d\alpha_F$$

$$= -kT \int \dots \int \left[ \int \alpha_1 d \left[ e^{-E/kT} \right] d\alpha_3 \right] d\alpha_2 d\alpha_3 \dots d\alpha_F$$

$$= -kT \int \dots \int \left[ \alpha_1 e^{-E/kT} - \int e^{-E/kT} d\alpha_1 \right] d\alpha_2 d\alpha_3 \dots d\alpha_F$$

Considering all possible values for  $\alpha_1$

$$\alpha_1 e^{-E/kT} \Big|_{-\infty}^{\infty} = 0$$

$$\therefore I_1 = kT \int \dots \int e^{-E/kT} d\alpha_1 d\alpha_2 \dots d\alpha_F = \frac{kT}{A}, \text{ using (i)}$$

Similarly, the other integrals of (iii) may be evaluated, each giving a value  $kT/A$ .

$$\therefore U = \frac{1}{2} A n \times F \times \frac{kT}{A} = \frac{1}{2} n F kT \quad (\because F \text{ integral exist})$$

The total number of degrees of freedom is  $nF$ .

Hence, energy per degree of freedom

$$= \frac{U}{nF} = \frac{\frac{1}{2} nFkT}{nF} = \frac{1}{2} kT$$

This theorem is called the *law of equipartition of energy*. J. C. Maxwell deduced this law for translational energy in 1859. Boltzmann extended it further to the energies of rotation and vibration. Each independent mode by which energy can be absorbed is called a *degree of freedom*.

### 16.12 Ratio of specific heats of gases ( $\gamma$ )

(i) Monatomic gas : According to kinetic theory, molecules of monatomic gases such as argon, neon, helium, etc., have no internal structure. The molecules have only three degrees of freedom.

Energy per molecule having three degrees of freedom at  $T^\circ\text{K}$ , =  
 $3 \times \frac{1}{2} kT = \frac{3}{2} kT$ .

Total energy per gram-molecule at  $T^\circ\text{K}$

$$= \frac{3}{2} kT \times N$$

$$= \frac{3}{2} \frac{R}{N} T \times N$$

$$= \frac{3}{2} RT$$

where  $N$  is the Avogadro's number (total number of molecules in 1 gm-molecule of a gas) and  $k = \frac{R}{N}$  is the Boltzmann's constant.

Total energy per gm-molecule at  $(T + 1)^\circ\text{K}$

$$\frac{3}{2} R (T + 1)$$

Hence the increase in energy per gm-molecule per degree increase in temperature.

$$dE = \frac{3}{2} R (T + 1) - \frac{3}{2} RT$$



$$= \frac{3}{2} R$$

But the increase in energy (*i.e.*, internal energy) is also given by the relation  $dE = n.C_v.dT$  (Art. 13.6)

where

$n$  = number of gm-molecules present in the gas.

$dT$  = increase in temperature

and  $C_v$  = molar specific heat at constant volume.

Putting  $n = 1$  and  $dT = 1^\circ$ , we get

$$dE = 1.C_v.1 = C_v$$

Thus the increase in the total energy per gm-molecule per degree rise in temperature is the molar specific heat at constant volume, *i.e.*,  $C_v$ .

$$\therefore C_v = \frac{3}{2} R$$

From  $C_p - C_v = R$ , we get

$$C_p = C_v + R = \frac{3}{2} R + R = \frac{5}{2} R$$

$$\text{Hence, } \gamma = \frac{C_p}{C_v} = \frac{\frac{5}{2} R}{\frac{3}{2} R} = \frac{5}{3} = 1.67.$$

(ii) *Diatomic molecules* : The molecules of a diatomic gas (oxygen, hydrogen, nitrogen, air, etc.) can be thought of as having a dumbbell shape (two spheres joined by a rigid rod). Such a molecule (Fig. 16.10) can rotate about any one of the three mutually perpendicular axes. It is, therefore, possible for a diatomic molecule to have three degrees of freedom due to rotational motion in addition to three translational degrees of freedom. Thus the molecule should possess a total of six degrees of freedom. However, the moment of inertia about an axis along the rigid rod should be negligible compared to that about axes perpendicular to the rod. The rotational energy should, therefore, consist of only two terms, such

as  $\frac{1}{2}I\omega_y^2$  and  $\frac{1}{2}I\omega_z^2$ . Hence the energy of a diatomic molecule is shared equally between five and not six degrees of freedom (3 translational and 2 rotational) and each degree of freedom is required by equipartition to contribute an energy of  $\frac{1}{2}kT$ .

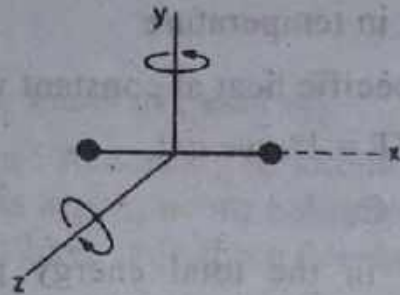


Fig. 16.10

Hence total energy per gm-molecule at TK

$$= 5 \times \frac{1}{2}kT \times N$$

$$= 5 \times \frac{1}{2} \frac{R}{N} T \times N = \frac{5}{2}RT$$

Total energy per gm-molecule at  $(T + 1)K$

$$\frac{5}{2}R(T+1)$$

Therefore, the increase in internal energy per gm-molecule for one degree increase in temperature

$$dE = C_v = \frac{5}{2}R(T+1) - \frac{5}{2}RT = \frac{5}{2}R$$

So, from  $C_p - C_v = R$ , we get

$$C_p = C_v + R = \frac{5}{2}R + R = \frac{7}{2}R$$

$$\therefore \gamma = \frac{C_p}{C_v} = \frac{\frac{7}{2}R}{\frac{5}{2}R} = \frac{7}{5} = 1.40.$$

(iii) *Triatomic molecules* : A triatomic molecule like that of carbon contains three spheres (atoms) joined together as shown in Fig. 16.10. The molecule is capable of rotating energetically about each of the three mutually perpendicular axes. Hence a triatomic gas molecule possesses six degree of freedom – three translational and three rotational.

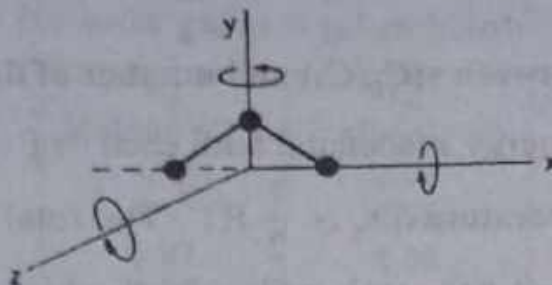


Fig. 16.11

Total energy per gm-molecule at TK

$$= 6 \times \frac{1}{2} kT \times N = 3 \cdot \frac{R}{N} T \cdot N = 3RT$$

Total energy per gm-molecule at  $(T + 1) K = 3R(T + 1)$

$$\therefore dE = C_v = 3R(T + 1) - 3RT = 3R$$

$$\text{and } C_p = C_v + R = 3R + R = 4R$$

$$\therefore \gamma = \frac{C_p}{C_v} = \frac{4R}{3R} = \frac{4}{3} = 1.33.$$

For a triatomic (polyatomic) gas possessing 7 degrees of freedom

Total energy per gm-molecule at TK

$$= 7 \times \frac{1}{2} \frac{R}{N} T \times N = \frac{7}{2} RT$$

$$\text{Energy at } (T + 1) K = \frac{7}{2} R(T + 1)$$



$$\therefore dE = C_v = \frac{7}{2} R (T+1) - \frac{7}{2} RT = \frac{7}{2} R$$

Hence the specific heat at constant pressure

$$C_p = \frac{7}{2} C_v + R = \frac{7}{2} R + R = \frac{9}{2} R$$

$$\therefore \gamma = \frac{C_p}{C_v} = \frac{\frac{9}{2} R}{\frac{7}{2} R} = \frac{9}{7} = 1.28$$

### 16.13 Relation between $\gamma(C_p/C_v)$ and number of degrees of freedom

The average energy associated with each degree of freedom of a molecule at a temperature  $TK$  is  $\frac{1}{2} RT$ . The total energy of 1 gm-molecule of a perfect gas, each molecule of which has  $n$  degrees of freedom, is  $N.n.\frac{1}{2} kT = N.n.\frac{1}{2} \frac{R}{N} T = \frac{1}{2} nRT$  where  $N$  is the Avogadro's number and  $R$  is the molar gas constant. The total energy at a temperature  $(T+1)K$  is then

$$\frac{1}{2} nR (T+1)$$

The increase in energy for one degree rise in temperature  $\frac{1}{2} nR (T+1) - \frac{1}{2} nRT = \frac{1}{2} nR$ , which by definition is the molar specific heat at constant volume of the gas.

$$\text{Also } C_p - C_v = R$$

$$\therefore C_p = C_v + R = \frac{1}{2} nR + R$$

$$= \frac{1}{2} R (n+2) = \frac{(n+2)}{2} R$$

$$\text{Then } \gamma = \frac{C_p}{C_v} = \frac{(n+2)R}{\frac{1}{2} nR} \times \frac{2}{nR}$$

$$= \frac{n+2}{2} = 1 + \frac{2}{n}$$

Thus, ratio of specific heats,

$$\gamma = 1 + \frac{2}{\text{number of degrees of freedom}}$$

In general, the more complex a molecule is the greater will be the number of its degrees of freedom and hence greater should be its specific heat. The value of  $\gamma$  will then approach unity.

The value of  $\gamma$  for some gases is given below :

Type of Gas	Gas	$C_p$ (cal/mole°C)	$C_v$ (cal/mole°C)	$C_p - C_v$	$\gamma = C_p/C_v$
Monatomic	He	4.97	2.98	1.99	1.67
	A	4.97	2.98	1.99	1.67
Diatomic	H <sub>2</sub>	6.87	4.88	1.99	1.41
	O <sub>2</sub>	7.03	5.03	2.00	1.40
	N <sub>2</sub>	6.95	4.96	1.99	1.40
	Cl <sub>2</sub>	8.29	6.15	2.14	1.35
Polyatomic	CO <sub>2</sub>	8.83	6.80	2.03	1.30
	SO <sub>2</sub>	9.65	7.50	2.15	1.29
	C <sub>2</sub> H <sub>6</sub>	12.35	10.30	2.05	1.20
	NH <sub>3</sub>	8.80	6.65	2.15	1.31

#### 16.14 Mean free path

According to kinetic theory, the molecules are continuously colliding with each other. The direction of the molecule is changed after every collision; but between any two consecutive collisions, the molecules move with constant speed along a straight line. This distance between any two consecutive collision is known as the *free path*. After a number of collisions, the total path appears to be zig-zag and the free path is not constant between any two consecutive collisions (Fig. 16.11). The average distance travelled by a molecule between two collisions is called the *mean free path* ( $\lambda$ ). If  $S$  is the total distance travelled during  $N$  collisions, then

$$\lambda = \frac{S}{N}$$



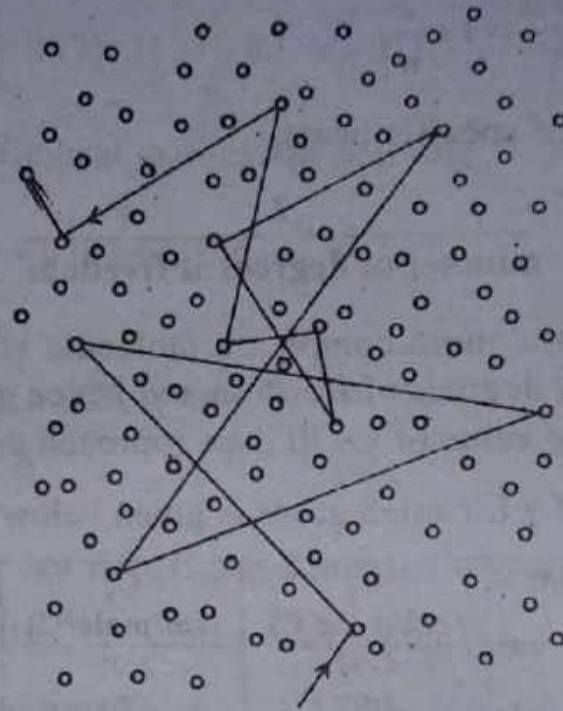


Fig. 16.12

### The distribution of free paths

We shall now investigate how the free paths are distributed among the molecules. This means that out of a large number of free paths, how many molecules have a specified free path length, say between  $x$  and  $x+dx$ .

Let us consider that a system has a large number of molecules, say  $N_0$ , at a given instant and follow this group as the molecules collide. Each collision knocks out the molecule from the group. After a distance  $x$ , as measured along free path of each molecule, has been travelled by the group, let the number of molecules remaining in the group be  $N$ . This means that  $N$  molecules have not yet made any collision. In the course of an additional short distance  $dx$ , some more molecules from this  $N$ , say  $dN$  will again be removed from the group by collision. Obviously,  $dN$  will be proportional to both the number  $N$  of the molecules as well as the distance  $dx$  travelled.

$$\therefore dN \propto N dx$$

$$\text{or, } dN = -P_c N dx \quad (i)$$

where  $P_c$  is a constant of proportionality known as the *collision probability*.

It depends on the physical conditions in the gas but neither on  $x$  nor on  $N$ . The negative sign implies that each collision removes a molecule



from the group thus decreasing  $N$ .

$$\text{From (i), } \frac{dN}{N} = -P_c dx$$

$$\text{Integrating, } \ln N = -P_c x + \text{const.} \quad (\text{ii})$$

$$\text{At } x = 0, N = N_0.$$

$$\text{So the integration constant} = \ln N_0$$

$$\therefore \ln N = -P_c x + \ln N_0$$

$$\text{or, } \ln \frac{N}{N_0} = -P_c x$$

$$\text{or, } \frac{N}{N_0} = e^{-P_c x}$$

$$\text{or, } N = N_0 e^{-P_c x} \quad (\text{iii})$$

Eqn. (iii) is an exponential equation. Thus the number of molecules remaining in the group falls off exponentially with  $x$ .

Substituting the value of  $N$  in eqn. (i), we get

$$dN = -P_c N_0 e^{-P_c x} dx \quad (\text{iv})$$

$|dN|$  thus gives the number of molecules with free path length between  $x$  and  $x+dx$  i.e., each of these  $dN$  molecules has survived at least a distance  $x$ . The mean free path  $\lambda$  can thus be obtained as

$$\lambda = \frac{x_1 dN_1 + x_2 dN_2 + \dots}{dN_1 + dN_2 + \dots} = \frac{\int x dN}{\int dN}$$

$$= \frac{1}{N_0} \int x dN$$

$$= \frac{1}{N_0} \int_0^{\infty} x P_c N_0 e^{-P_c x} dx, \quad \text{using (iv)}$$

$$= P_c \int_0^{\infty} x e^{-P_c x} dx$$

$$= P_C \frac{1}{P_C^2} \tau(2) = \frac{1}{P_C} \quad [\because \tau(2) = 1]$$

$$\therefore P_C = \lambda^{-1}$$

The collision probability is thus the reciprocal of the mean free path.

Substituting this value of  $P_C$  in eqns. (iii) and (iv), we get

$$N = N_0 e^{-x/\lambda} \quad (v)$$

$$\text{and } |dN| = \frac{N_0}{\lambda} e^{-x/\lambda} dx \quad (vi)$$

Eqn. (v) is also known as the *survival equation*. It gives the number of molecules  $N$  out of a group of  $N_0$  having free paths greater than zero, i.e., the molecules that survived over a path length  $x$ . Eqn. (vi), as already indicated, gives the number of molecules having free paths between  $x$  and  $x + dx$ . A plot of  $N/N_0$  against  $x/\lambda$  is shown in Fig. 16.13. The ordinate of the curve gives the fractional number of molecules with free paths greater than any fraction of the mean free path  $\lambda$ . It may be seen that the number with free paths longer than  $\lambda$  is  $e^{-1}$  or 37%, while the number with free paths shorter than  $\lambda$  is 63%.

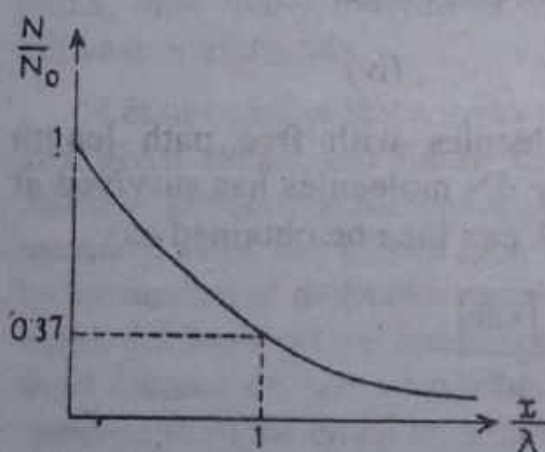


Fig. 16.13

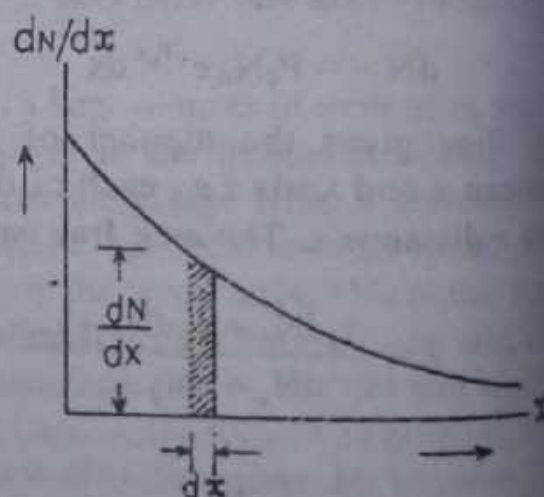


Fig. 16.14

A plot of  $|dN|$  as a function of  $x$  is shown in Fig. 16.14. The ordinate of the curve gives the number of molecules *per unit path length* with free paths between  $x$  and  $x+dx$ . The actual number  $dN$  having free paths between  $x$  and  $x+dx$  however is represented by the area of a narrow vertical strip.

### Expression for mean free path

It should be pointed out that if molecules were points, they would not collide at all and the mean free path would be infinite. Molecules, however, are not points and hence collisions occur. If the molecules are so numerous that they completely fill the space at their disposal, leaving no room for translational motion, the mean free path would be zero. It, therefore, becomes obvious that the mean free path of the molecules should be related to the density of the molecules and to their number per unit volume.

Before calculating the value of mean free path, the following simplifying assumptions will be made :

(i) Only the molecule under consideration is in motion and all other gas molecules are at rest.

(ii) The gas molecules are considered to be spheres of diameter  $c$ . Every gas molecule has a sphere of influence around it, the radius of which is equal to the diameter  $d$  of the molecules. A collision between two molecules will then take place when the centre of one molecule lies within the sphere of influence of the other, i.e., the centres of the two molecules approach within a distance  $d$  of one another.

An equivalent description of collision between two molecules can be made by regarding that one of the colliding molecules has a diameter twice the diameter of a single molecule ( $2d$ ) and that the other molecule is a single point. A collision is depicted in Fig. 16.15.

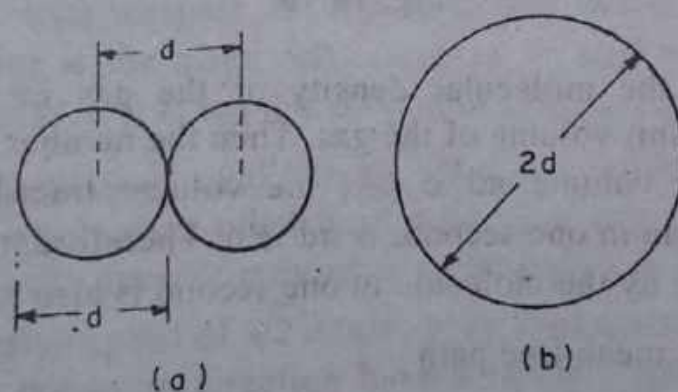


Fig. 16.15



Let us imagine a typical molecule of equivalent diameter  $2d$  to be moving within the gas with a velocity  $c$  while all the other molecules are at rest. It will be further assumed, for the time being, that the molecule and the point particles do not exert any force on each other. As the molecule moves, its sphere of influence sweeps out in one second a cylinder whose cross-section is  $\pi d^2$  ( $d$  being the radius of the sphere of influence or the diameter of a molecule) and whose length is the distance travelled in one second *i.e.*,  $c$  (it should, however, be remembered that the cylinder is not straight but is like a long jointed stove pipe, with a joint at every collision). In one second the molecule under consideration makes a collision with every other molecule whose centre lies within this volume (Fig. 16.16).

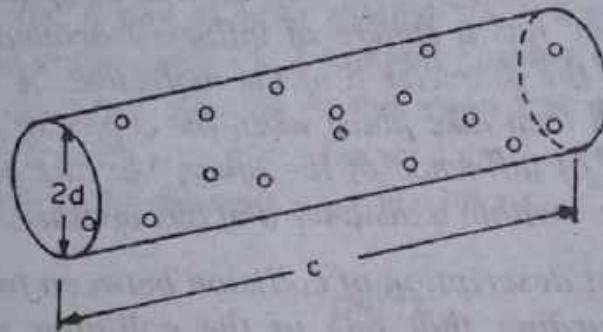


Fig. 16.16

Let  $n$  be the molecular density of the gas *i.e.*, number of molecules per unit volume of the gas. Then the number of molecules lying within a volume  $\pi d^2 \cdot c$  *i.e.*, the volume traced out by the moving molecule in one second, is  $\pi d^2 \cdot c \cdot n$ . Therefore, the number of collisions made by the molecule in one second is also  $\pi d^2 cn$ .

Therefore, mean free path

$$\begin{aligned}\lambda &= \frac{\text{total distance travelled in one second}}{\text{number of collisions made in one second}} \\ &= \frac{c}{\pi d^2 cn}\end{aligned}$$

$$= \frac{1}{\pi d^2 n} \quad (16.15)$$

$$= \frac{m}{\pi d^2 \cdot mn}$$

$$= \frac{m}{\pi d^2 \cdot \rho} \quad (16.16)$$

where  $m$  is the mass of a single molecule and  $mn$  is the total mass per unit volume (*i.e.*, the density  $\rho$  of the gas).

Thus the mean free path is inversely proportional to the density of the gas. Since density itself varies directly with the pressure and inversely as the temperature in K,  $\lambda$  is inversely proportional to the pressure and directly proportional to the temperature in K.

The above relation was deduced by Clausius. More rigorous treatment by Boltzmann puts the value of  $\lambda$  as

$$\lambda = \frac{3}{4} \cdot \frac{1}{\pi d^2 n}$$

Both Clausius' and Boltzmann's relations are based on the picture of a molecule hitting stationary targets. Actually the molecule hits moving targets. When the target molecules are moving, the two  $c$ 's in the expression for  $\lambda$  above are not the same. The one in the numerator is the mean molecular speed ( $= c$ ) measured with respect to the container while the one in the denominator is the mean relative speed ( $= c_{rel}$ ) with respect to the other molecules. It is  $c_{rel}$  that determines the collision rate.

It can be seen qualitatively, that two molecules of speed  $c$  moving towards each other have a relative speed of  $2c$  ( $> c$ ); two molecules with speed  $c$  moving at right angles on a collision course have a relative speed of  $\sqrt{2}c$  (also  $> c$ ); two molecules with speed  $c$  moving in the same direction have a relative speed of zero ( $< c$ ). Thus the molecules arriving from all of the forward hemisphere and part of the backward hemisphere have  $c_{rel} > c$ . Since the number of molecules arriving from all of the forward hemisphere and part of the backward hemisphere are larger, the relative speed  $c_{rel}$  when averaged over both the hemisphere is greater than  $c$  ( $c_{rel} > c$ ). A



quantitative calculation, taking into account the actual speed distribution of the molecules gives  $c_{rel} = \sqrt{2} \cdot c$ . When this value is substituted in the equation (16.15), the collision frequency is increased. The mean free path  $\lambda$  then reduces to

$$\lambda = \frac{1}{\sqrt{2} \cdot \pi d^2 n}$$

### 16.15 Viscosity of gases from the stand-point of kinetic theory

When two adjacent layers of a fluid (liquid or gas) move with different velocities, the fast moving layer tend to accelerate the slower moving layer while the slower moving layer tend to retard the motion of the fast moving layer. The two layers taken together tend to destroy their relative motion as though a tangential backward dragging force, frictional in nature, comes into play between the layers tending to oppose their relative motion. The property of the fluid which gives rise to this force, called the *viscous force*, is known as *viscosity or internal friction* of the fluid.

For a qualitative explanation of viscosity let us consider three layers A, B and C of a gas such that the distance between A and B, and B and C are each equal to the *mean free path* of the molecules of the gas (Fig. 16.17).

Let us suppose that the gas in the immediate neighbourhood of the plane A is moving bodily from left to right and that the rate of flow of gas decreases as we move from A through B to C.

Some of the molecules of the gas in the layer A moves towards B and reaches it without suffering an encounter with other molecules. These molecules become part of the gas below B retaining the same velocity as those of the molecules in the layer A. Likewise molecules from C move up and pass through B and become part of the gas above B retaining the velocity of the molecules in layer C. Thus as a result of this interchange of molecules having different velocities between the different layers of the gas, the motion of the gas above the layer B is decreased while that below it is increased. These tend to oppose the relative motion of the adjacent layers and endow the medium with the property of viscosity.



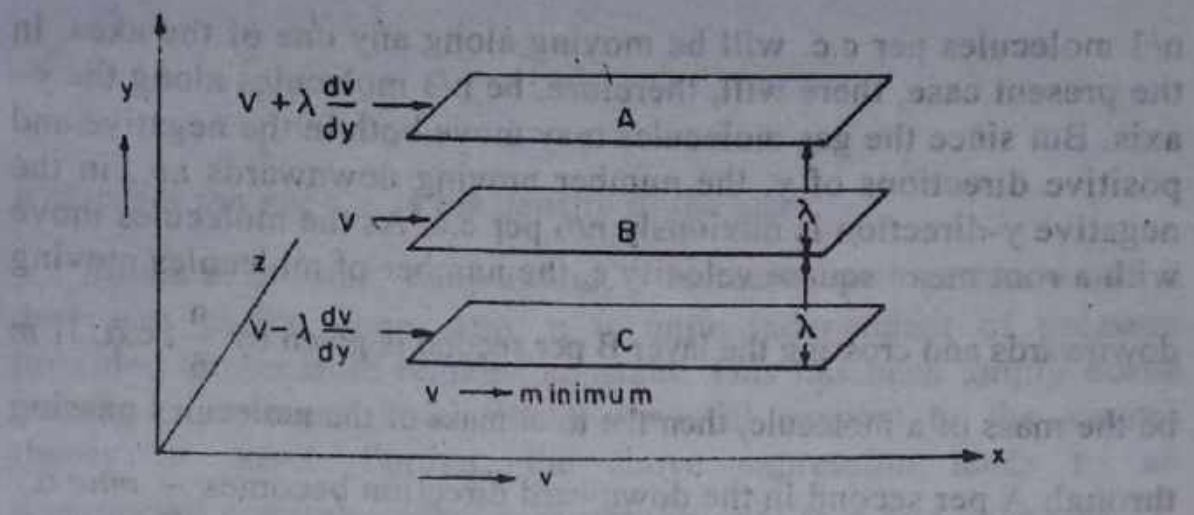


Fig. 16.17

An expression for the coefficient of viscosity may be obtained as follows :

If the velocity gradient between the two adjacent layers is  $\frac{dv}{dy}$  and  $\alpha$  is the surface area in contact, then the tangential force tending to accelerate or retard one of the layers is given by

$$F = \eta \alpha \cdot \frac{dv}{dy}$$

where  $\eta$  is called the coefficient of viscosity.

According to kinetic theory, the tangential force comes into play due to transfer of momentum per second across the surface of contact between two layers, having a velocity gradient between them. To deduce an expression for  $\eta$  let us consider the layers to be parallel to  $x$ - $z$  plane and let the middle layer B has a velocity  $v$ . Let the velocity gradient along the  $y$  - axis be  $\frac{dv}{dy}$ . Then the velocities

of the upper and lower layers are  $v + \lambda \frac{dv}{dy}$  and  $v - \lambda \frac{dv}{dy}$  respectively.

Due to thermal agitation, the molecules are moving in all possible directions. If  $n$  be the number of molecules per c.c., i.e., the molecular density of the gas, then let us suppose for simplicity that

$n/3$  molecules per c.c. will be moving along any one of the axes. In the present case, there will, therefore, be  $n/3$  molecules along the  $y$ -axis. But since the gas molecules may move both in the negative and positive directions of  $y$ , the number moving downwards *i.e.*, in the negative  $y$ -direction is obviously  $n/6$  per c.c. As the molecules move with a root mean square velocity  $c$ , the number of molecules moving downwards and crossing the layer B per second is given by  $\frac{n}{6} \cdot c \cdot \alpha$ . If  $m$  be the mass of a molecule, then the total mass of the molecules passing through A per second in the downward direction becomes  $\frac{1}{6} mnc \alpha$ .

Hence the transfer of momentum from the upper layer on to B is given by

$$\frac{1}{6} mnc \alpha \left( v + \lambda \frac{dv}{dy} - v \right) = \frac{1}{6} mnc \alpha \lambda \frac{dv}{dy}$$

Similarly, the transfer of momentum due to the upward stream of molecules from the lower layer on to B is given by

$$\frac{1}{6} mnc \alpha \left( v - \lambda \frac{dv}{dy} - v \right) = - \frac{1}{6} mnc \alpha \lambda \frac{dv}{dy}$$

But as this negative momentum transfer is due to the upwards transport of the molecules, it may be supposed to be equivalent to an identical positive amount of momentum transfer due to the downward transport of the molecules. So the net transfer of momentum per second through B is equal to

$$\begin{aligned} & \frac{1}{6} mnc \alpha \lambda \frac{dv}{dy} + \frac{1}{6} mnc \alpha \lambda \frac{dv}{dy} \\ &= \frac{1}{3} mnc \alpha \lambda \frac{dv}{dy} \end{aligned}$$

But this net transfer of momentum is equal to the tangential force  $F$ .

$$\therefore F = \eta \alpha \cdot \frac{dv}{dy} = \frac{1}{3} mnc \alpha \lambda \cdot \frac{dv}{dy}$$



$$\begin{aligned}\text{or, } \eta &= \frac{1}{3} \text{ mnc } \lambda \\ &= \frac{1}{3} \rho c \lambda\end{aligned}$$

where  $\rho = \text{mn per c.c.}$  is the density of the gas.

Since at constant temperature,  $\rho$  increases with pressure and  $\lambda$  decreases in the same ratio,  $\eta$  is quite independent of pressure provided temperature remains constant. This has been amply borne out by experiment which lends powerful support to the kinetic theory of gases. Further, the above expression leads to an unexpected conclusion that  $\eta$  is proportional to  $c$  and hence to  $\sqrt{T}$  ( $\because c \propto \sqrt{T}$ ). The prediction has been borne out quite well over a wide range of density (or pressure). The prediction breaks down at very low pressure where  $\lambda$  becomes so long that it can travel across a container without making numerous collisions on the way, and at very high pressures, where the molecules are so close to each other that intermolecular forces becomes significant.

**Example 16.16.** *The number of molecules per c.c. of a gas at N.T.P. is  $2.7 \times 10^{19}$ . Calculate the number of molecules per c.c. of the gas at  $39^\circ\text{C}$  and  $10^{-6}$  mm pressure of mercury.*

**Soln.**

Let the number of molecules per c.c. at N.T.P. be  $n_1$  and that at  $39^\circ\text{C}$  and  $10^{-6}$  mm of pressure of mercury be  $n_2$ .

Then

$$n_1 = \frac{P_1 N}{RT_1}$$

$$P_1 = 760 \text{ mm. pressure of mercury}$$

$$T_1 = 273 \text{ K}$$

$$\text{and } n_2 = \frac{P_2 N}{RT_2}$$

$$P_2 = 10^{-6} \text{ mm pressure of mercury}$$



$$T_2 = (273 + 39) = 312 \text{ K.}$$

$$\therefore \frac{n_1}{n_2} = \frac{P_1 N}{RT_1} \times \frac{RT_2}{P_2 N} = \frac{P_1}{P_2} \times \frac{T_2}{T_1}$$

$$= \frac{760}{10^{-6}} \times \frac{312}{273} = 868.6 \times 10^6$$

$$\text{or, } n_2 = \frac{n_1}{868.6 \times 10^6} = \frac{2.7 \times 10^{19}}{868.6 \times 10^6}$$

$$= 3.1 \times 10^{10}.$$

**Example 16.17.** Calculate the mean free path and the collision frequency for air molecules at  $0^\circ\text{C}$  and 1 atm. pressure. Given the effective diameter of air molecule  $= 2 \text{ \AA}$ , the root mean square speed of air molecules at N.T.P. is about  $1 \times 10^5 \text{ cm/sec}$  and the number of air molecules per c.c.  $= 3 \times 10^{19}$ .

**Soln.**

Using Maxwell's relation, the mean free path.

$$\lambda = \frac{1}{\sqrt{2} \cdot \pi n d^2}$$

$$d = 2 \text{ \AA} = 2 \times 10^{-8} \text{ cm}$$

$$n = 3 \times 10^{19} / \text{cm}^3$$

$$\therefore \lambda = \frac{1}{\sqrt{2} \times 3.14 \times (3 \times 10^{19}) \times (2 \times 10^{-8})^2}$$

$$= 1.9 \times 10^{-5} \text{ cm.}$$

The corresponding collision frequency is

$$\frac{c}{\lambda} = \frac{1 \times 10^5}{1.9 \times 10^{-5}} = 5.2 \times 10^9 / \text{sec.}$$

Thus, on the average, each molecule makes five billion collisions per seconds.

**Example 16.18.** Calculate the number of molecules per c.c. of a gas if the mean free path of the molecules is  $2.4 \times 10^{-6}$  cm. and the molecular diameter is equal to  $2.0 \times 10^{-8}$  cm. What will be the collision frequency if the r.m.s. speed of the molecules under the given condition is of the order of  $1 \times 10^5$  cm?

**Soln.**

Using the more accurate Maxwell's relation, the mean free path,

$$\lambda = \frac{1}{\sqrt{2} \cdot n \pi d^2}$$

$$\text{or, } n = \frac{1}{\sqrt{2} \cdot \lambda \pi d^2}$$

$$\lambda = 2.4 \times 10^{-6} \text{ cm}$$

$$d = 2 \times 10^{-8} \text{ cm}$$

$$\therefore n = \frac{1}{\sqrt{2} \times 3.14 \times (2.4 \times 10^{-6}) \times (2 \times 10^{-8})^2}$$

$$= 2.345 \times 10^{20} \text{ molecules/c.c.}$$

$$\text{collision frequency} = \frac{\text{r.m.s. velocity}}{\text{mean free path}}$$

$$\frac{c}{\lambda} = \frac{1 \times 10^5}{2.4 \times 10^{-6}} = 41.66 \times 10^9.$$

**Example 16.19.** Calculate the mean free path of gas molecules in a chamber at  $0^\circ\text{C}$  and  $10^{-6}$  mm pressure of mercury. Assume the molecular diameter to be  $2 \text{ \AA}$ . One gm-molecule of the gas occupies 22.4 litres at N.T.P.

**Soln.**

At N.T.P. the number of molecules in 22.4 litres of gas

$$= 6.023 \times 10^{23}$$

Therefore, the number of molecules per c.c. in the chamber at N.T.P.

$$= \frac{6.023 \times 10^{23}}{22.4 \times 10^3}$$

The number of molecules per c.c. in the chamber at 273 K (0°C) and  $10^{-6}$  mm pressure of mercury

$$n = \frac{10^{-6}}{760} \times \frac{6.023 \times 10^{23}}{22.4 \times 10^3} \text{ (example 16.15)}$$

$$= 3.538 \times 10^{10} \text{ molecules/c.c.}$$

Mean free path,

$$\lambda = \frac{1}{\sqrt{2} \cdot \pi n d^2} \text{ using Maxwell's relation.}$$

$$n = 3.538 \times 10^{10} / \text{c.c.}$$

$$\therefore \lambda = \frac{1}{\sqrt{2} \times 3.14 \times (3.538 \times 10^{10}) \times (2 \times 10^{-8})^2}$$

$$= 1.59 \times 10^4 \text{ cm.}$$

**Example 16.20.** Estimate the size of a helium atom, assuming its mean free path at N.T.P. to be  $28.5 \times 10^{-6}$  cm. Given that the density of helium at N.T.P. is 0.178 g/litre and the mass of the helium atom is  $6 \times 10^{-24}$  g.

**Soln.** From Maxwell's expression, the mean free path  $\lambda$  is given by  $\lambda = \frac{1}{\sqrt{2} \cdot \pi n d^2}$

Here density  $\rho = mn = 0.178 \times 10^{-3}$  g/c.c.

and  $m = 6 \times 10^{-24}$  g.

$$\therefore n = \frac{\rho}{m} = \frac{0.178 \times 10^{-3}}{6 \times 10^{-24}} = 29.7 \times 10^{18} \text{ per c.c.}$$

Also  $\lambda = 28.5 \times 10^{-6}$  cm.

From  $\lambda = \frac{1}{\sqrt{2} \cdot \pi n d^2}$ , we have



$$28.5 \times 10^{-6} = \frac{1}{(\sqrt{2})(3.14)(d^2)(29.7 \times 10^{18})}$$

$$\therefore d^2 = \frac{1}{(\sqrt{2})(3.14)(28.5 \times 10^{-6})(29.7 \times 10^{18})}$$

$$\therefore d \simeq 1.63 \times 10^{-8} \text{ cm.}$$

**Example 16.21.** If the molecular diameter of hydrogen is  $1.9 \times 10^{-8} \text{ cm}$ , find the number of collisions by a hydrogen molecule in 1s, if  $T = 300 \text{ K}$ ,  $P = 1 \text{ atmos}$ .

**Soln.** From the relation  $P = nkT$  (ex 16.13), we have

$$76 \times 13.6 \times 980 = n \times 1.38 \times 10^{-16} \times 300$$

$$\therefore n = \frac{(76)(13.6)(980)}{(1.38 \times 10^{-16})(300)} = 2.44 \times 10^{19}$$

$$\text{Mean free path } \lambda = \frac{1}{\sqrt{2} \cdot \pi n d^2}$$

$$= \frac{1}{(\sqrt{2})(3.14)(1.9 \times 10^{-8})^2 (2.44 \times 10^{19})}$$

$$= 25.5 \times 10^{-6} \text{ cm.}$$

$$\text{Root mean square speed, } c = \sqrt{\frac{3kT}{m}}$$

$$= \sqrt{\frac{(3)(1.38 \times 10^{-16})(300)}{(2/6.01 \times 10^{23})}}$$

$$= \sqrt{3732.21 \times 10^7}$$

$$= \sqrt{373.2 \times 10^8}$$

$$= 19.3 \times 10^4 = 1.93 \times 10^5 \text{ cm/s.}$$

$$\therefore \text{collision frequency, } \frac{c}{\lambda} = \frac{1.93 \times 10^5}{25.5 \times 10^{-6}} = 7.5 \times 10^{-2} \times 10^5 \times 10^6$$

$$= 7.5 \times 10^9$$

**Example 16.22.** A shower of 5000 molecules, each moving with the same velocity originally, traverses a gas. Compute the number that would travel undeflected even after traversing a distance equal to 0.5 and 1 time the mean free path.

**Soln.** We have, from the survival equation

$$N = N_0 e^{-x/\lambda}$$

In the first case  $x = 0.5 \lambda$  and in the second case  $x = \lambda$ .

$$\therefore N = N_0 e^{-0.5\lambda/\lambda} = 5000 \times e^{-0.5} = 3032$$

$$\text{and } N = N_0 e^{-1/\lambda} = 5000 \times e^{-1} = 1839.$$

**Example 16.23.** Calculate what fraction of gas molecules (a) traverses, without collision, distances exceeding the mean free path  $\lambda$ , (b) has free paths lying within  $\lambda$  to  $2\lambda$ .

**Soln.** From eqn.  $N = N_0 e^{-x/\lambda}$ , we have

$$(a) \quad \frac{N}{N_0} = e^{-\lambda/\lambda} = e^{-1} = 0.368$$

$$(b) \quad \frac{N}{N_0} = e^{-2\lambda/\lambda} = e^{-2} = 0.135$$

$$\therefore \text{the fraction having free paths between } \lambda \text{ and } 2\lambda \\ = 0.368 - 0.135 = 0.233.$$

**Example 16.24.** Show that the mean energy of translation is  $3/2$  of the total energy where  $n$  is the number of degrees of freedom.

**Soln.** Total energy = no. of degrees of freedom  $\times$  energy associated with each degree of freedom.

$$= n \times \frac{1}{2} kT = \frac{nkT}{2}$$

$$\text{Mean energy of translation of a molecule} = \frac{3}{2} kT$$

$$\therefore \frac{\text{Mean energy of translation}}{\text{Total energy}} = \frac{\frac{3}{2} kT}{\frac{nkT}{2}} = \frac{3}{n}$$



**EXERCISES**

- [1] State the fundamental assumptions of the kinetic theory of gases.  
Obtain an expression for the pressure exerted by an ideal gas on the basis of kinetic theory of gases.
- [2] Applying the principle of kinetic theory of gases show that the pressure exerted by a perfect gas is equal to two-thirds of the kinetic energy of the molecules per unit volume.  
Give the interpretation of temperature from the stand point of kinetic theory.
- [3] Explain what is meant by root mean square velocity of the molecules of a gas.  
Obtain an expression for the pressure of a gas in terms of its density and the root mean square velocity of the molecules from the standpoint of kinetic theory.
- [4] How do you interpret (i) pressure and (ii) temperature on the basis of the kinetic theory of gases.  
Show that the kinetic energy of translation per gram-molecule of any gas at a given temperature is constant and is equal to  $\frac{3}{2} RT$ .
- [5] What is the kinetic theory view-point of gaseous systems? How would you relate the pressure, volume and temperature of an ideal gas from this view-point? Hence, explain the physical significance of the gas constant  $R$ .
- [6] Deduce Boyle's law, Charles' law and Avogadro's law for a perfect gas from the principle of kinetic theory.
- [7] Deduce from the kinetic theory of gases, an expression for the pressure exerted by a gas. Hence prove that  $PV = RT$ .
- [8] Deduce Avogadro's law, Dalton's law of partial pressure and Graham's law of diffusion from the standpoint of kinetic theory of gases. What interpretations of the gas constant and temperature are given according to this theory?
- [9] Show from the kinetic theory that the mean kinetic energy of translation of one molecule of a perfect gas is  $\frac{3}{2} kT$ , where  $k$  is Boltzmann's constant, and  $T$  the absolute temperature of the gas.



- [10] Show that  $n$ , the number of molecules per unit volume of an ideal gas is given by

$$n = \frac{PN}{RT}$$

where  $N$  is the Avogadro's number.

- [11] Explain clearly what you mean by the principle of equipartition of energy. Explain the term degree of freedom in this connection
- [12] State the principle of equipartition of energy and explain the term degree of freedom in this connection. Calculate theoretically the ratio of two specific heats for monatomic and diatomic gases.
- [13] State the law of equipartition of energy. Prove that for a monatomic gas, the value of  $\gamma$ , the ratio of the two specific heats, is  $\frac{5}{3}$ , for a diatomic gas it is  $\frac{7}{5}$  and for a triatomic gas it is  $\frac{4}{3}$ .
- [14] What is meant by mean free path of a molecule? Obtain an expression for the mean free path of a molecule in term of its mass and diameter and density of the gas.
- [15] Explain the term mean free path of a molecule of a gas. Show that it is equal to  $\frac{1}{\pi n d^2}$ , where  $n$  is the number of molecules per c.c. and  $d$  the diameter of each molecule. Show that the mean free path is proportional to pressure.
- [16] Write short notes on :
- (i) Fundamental assumptions of kinetic theory of gases.
  - (ii) Root mean square velocity of a gas.
  - (iii) Degrees of freedom of a gas.
  - (iv) Equipartition of energy.
  - (v) Mean free path of a gas molecule.
- [17] Calculate the root mean square velocity of the molecules of nitrogen at N.T.P. The density of nitrogen at N.T.P. = 0.00125 gm per c.c. ( $4.93 \times 10^4$  cm/sec).
- [18] Deduce Maxwell's velocity distribution law. Hence obtain an expression for the energy distribution of the gas molecules.

- [19] Deduce an expression for the most probable velocity of the molecules of a gas. Also deduce the mean and r.m.s. velocities in terms of the most probable velocity. Indicate the three velocities in a diagram showing the velocity distribution of the gas molecules.
- [20] Derive Maxwell's law of distribution of velocities of the molecules of a gas. Find the ratio of the average velocity to r.m.s. velocity of the molecules. Describe an experiment to verify Maxwell's distribution law.
- [21] Write down Maxwell's law of distribution of molecular velocities and explain the symbols you use. Indicate graphically how the distribution changes with rise of temperature and pressure.
- [22] What do you mean by "degrees of freedom" of a dynamical system? State and establish the equipartition theorem. How is the atomicity of gas molecules related with the ratio of specific heats? What specific heats are predicted for dia-and triatomic molecules by the theory.
- [23] Deduce the law of equipartition of energy. What assumptions are made in the deduction of the law?
- [24] Define 'free path' and 'collision frequency'. Show that the probability of a gas molecule traversing a distance  $x$  without collision is  $e^{-x/\lambda}$  where  $\lambda$  is the mean free path of the gas molecule.
- [25] Explain (i) degrees of freedom, (ii) equipartition of energy. Show that the mean energy of translation is  $3/n$  of their total energy,  $n$  = number of degrees of freedom. Deduce the formula you would use for the purpose.
- [26] If the density of hydrogen at N.T.P. is  $0.09 \text{ kg/m}^3$ , find the root mean square velocity of hydrogen molecules. ( $1.84 \times 10^3 \text{ m/sec}$ ).
- [27] At what temperature, pressure remaining constant, will the root mean square velocity of hydrogen molecules be double its value at N.T.P.? ( $819^\circ\text{C}$ ).
- [28] Find the root mean square velocity of the molecules of nitrogen at  $20^\circ\text{C}$ . Molecular weight of nitrogen is 28, the gas constant  $R$  for a gm-molecule is  $8.315 \times 10^7$  ergs per degree centigrade. ( $5.11 \times 10^4 \text{ cm/sec}$ ).
- [29] Calculate the root mean square velocity of the molecules of a gas whose density is 1.4 gm per litre at a pressure of 76 cm of mercury. Density of mercury = 13.6 gm per c.c.,  $g = 981 \text{ cm/sec}^2$ . ( $4.66 \times 10^4 \text{ cm/sec}$ ).



- [30] The density of helium is 0.178 gm/per litre at N.T.P. What is the molecular velocity of helium under these conditions? Compare it with the velocity at 200°C. ( $1.306 \times 10^5$  cm/sec.; 1 : 1.316).
- [31] Calculate the total random kinetic energy of 2 gm of helium at 200K. Molecular weight of helium is 4 and the molar gas constant R is 8.315 joules/kg -K. (1245 joules).
- [32] Calculate the mean translational kinetic energy per molecule of a gas at 727°C, given  $R = 8.32$  joules/mole-K and Avogadro's number  $= 6.023 \times 10^{23}$ . ( $2.059 \times 10^{-20}$  joules).
- [33] What is the translational kinetic energy of an oxygen molecule at 27°C? If an oxygen molecule has five degrees of freedom, what is its total kinetic energy at this temperature? ( $6.2 \times 10^{-14}$  ergs;  $1.03 \times 10^{-13}$  ergs).
- [34] Calculate the number of molecules in one litre of an ideal gas at 136.5°C temperature and 3 atmospheres pressure. ( $5.376 \times 10^{22}$ ).
- [35] The number of molecules per c.c. of a gas is  $2.7 \times 10^{19}$  at N.T.P. Calculate the number of molecules per c.c. of the gas at 0°C and  $10^{-6}$  mm pressure of mercury. ( $3.553 \times 10^{10}$ ).
- [36] Calculate the volume occupied by 3.2 grammes of oxygen at 76 of mercury and 27°C. Molecular weight of oxygen = 32.  $R = 8.31 \times 10^7$  ergs/gm mole-K. (2461 c.c.)
- [37] The mean free path of the molecules of a gas is  $2 \times 10^{-5}$  cm, when there are  $1.225 \times 10^{19}$  molecules per c.c. Compute the diameter of the molecules. ( $3 \times 10^{-8}$  cm).
- [38] Calculate the number of molecules per c.c. of a gas, taking the mean free path as  $1.83 \times 10^{-5}$  cm and the molecular diameter equal to  $2.3 \times 10^{-8}$  cm ( $2.3 \times 10^{19}$ ).
- [39] Calculate the mean free path and the collision frequency of air molecules at 0°C and 1 atmos pressure. Given :  $d = 2 \times 10^{-8}$  cm,  $c = 10^5$  cm/s and  $n = 3 \times 10^9$  molecules/cc. [ $2.65 \times 10^{-5}$  cm,  $4 \times 10^9$ /s]
- [40] A shower of  $10^4$  molecules, each molecule originally moving with the same velocity, traverses the gas. Find the number of molecules which will remain undeflected even after traversing a length twice the mean free path.
- [41] Find the ratio of number  $n_1$  of molecules of hydrogen whose speeds lie between 3000 and 3010 m/s to the number  $n_2$  of molecules having speeds between 1500 and 1510 m/s, if the temperature of hydrogen is 300 K.



## CHAPTER XVII

## EQUATIONS OF STATE

*Equation of state-Deviation from Boyle's law-Andrews' experiment on carbon dioxide-Vapours and gases-Critical point and critical constants-Van der Waals' equation -Van der Waals' constants-Boyle temperature from Van der Waals' equation-Reduced equation of state-Defects of Van der Waals' equation-Experimental determination of the critical constants-Solved problems-Exercises.*

## 17.1 Equation of state

An equation which shows the relation between pressure, volume and temperature of a given mass of a homogeneous substance is called the *equation of state* of the substance.

The equation of state of a perfect gas, obtained by combining the laws of Boyle and Charles, is

$$PV = RT \quad (17.1)$$

where  $P$  is the pressure,  $V$  is the volume,  $T$  is the absolute temperature of the gas and  $R$  is a constant known as the gas constant.

The relation given by eqn. (17.1) applies to real gases under ideal conditions only, viz., *high temperature and low pressure*. A more general equation of state suitable for all gases, was proposed in 1901 by K. Onnes and is as follows:

$$PV = A + BP + CP^2 + DP^3 + \dots \quad (17.2)$$

where  $V$  is the volume of a gram-molecule of the gas and  $P$  is the pressure at a given temperature.  $A, B, C, \dots$  are constants whose values depend upon the nature of the gas and the temperature and  $A > B > C, \dots$ . These are called *virial coefficients*. The first virial coefficient  $A$  is equal to  $RT$  in eqn. (17.1). The other coefficients  $B, C, D$ , etc., diminish very rapidly and hence for all practical purposes, terms involving these coefficients, except  $BP$ , may be neglected.  $B$  varies in the same manner for all gases. Starting with a negative value at low temperatures,  $B$  gradually increases to zero as the temperature is raised and finally becomes positive with farther rise in temperature. The temperature at which  $B = 0$  is called the *Boyle*

temperature of the gas. At this temperature, the gas obeys Boyle's law for a wide range of pressure.

## 17.2 Deviation from Boyle's law

Boyle's law states that temperature remaining constant, the volume of a given mass of any gas varies inversely as its pressure. This law strictly holds for a perfect gas only. But a perfect gas is an ideal

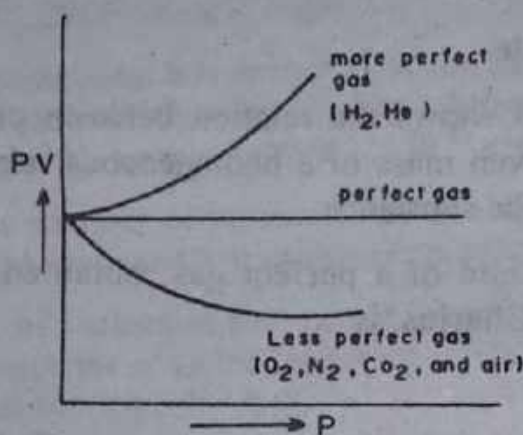


Fig. 17.1

substance and cannot be obtained in nature. Real gases obey Boyle's law only at low pressure and high temperature. If gases obeyed Boyle's law,  $V = \frac{1}{2} V_0$  would give rise to a pressure  $P = 2P_0$ . This means that if  $P_0 V_0 / PV$  is plotted as ordinate against  $P$  as abscissa, the resulting graph would be a straight line parallel to the x-axis. In case of real gases, instead of straight lines, curves inclined to x-axis were obtained;  $PV$  decreasing for air, nitrogen, carbon dioxide and increasing in the case of hydrogen and helium (Fig 17.1). Thus the permanent gases were shown to be imperfect. Regnault termed the increasing  $PV$ - gases as the more perfect gases and the decreasing  $PV$ -gases as the less perfect gases. However, for each gas there is a certain temperature, the so called Boyle temperature, at which  $PV$  is approximately independent of  $P$  and Boyle's law is obeyed. For higher temperatures  $PV$  increases as  $P$  increases, while for temperatures below the Boyle temperature  $PV$  at first decreases and then increases (Fig, 17.2). The Boyle temperatures for hydrogen and nitrogen are  $-177^\circ\text{C}$  and  $50^\circ\text{C}$  respectively. Hence at room temperature



nitrogen behaves like a less perfect gas while hydrogen behaves like a more perfect gas.

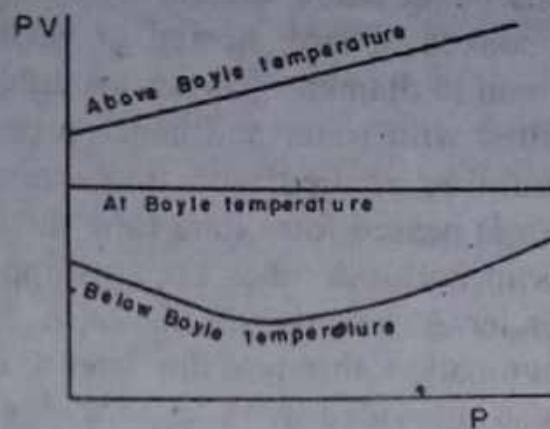


Fig. 17.2

Boyles's experiment is not sufficiently accurate to detect the deviation of the real gases from the Boyle's law. Such deviations were studied by Regnault, Amagat and Andrews at high pressures. We shall here discuss Andrew's experiment on carbon dioxide.

### 17.3 Andrews' experiment on carbon dioxide

In course of an attempt to liquefy some of the so called permanent gases, Andrews studied the isothermals of carbon dioxide at different temperatures. The principle of the experiment was to subject a fixed mass of a gas to varying pressure and note down the corresponding volumes occupied by it, keeping the temperature constant. The observations were repeated at different temperatures;  $P$  and  $V$  were then plotted for each temperature to obtain the different isothermals. The variation of the curves from Boyle's law was then studied.

The apparatus used by Andrews is shown in Fig. 17.3.  $ab$  and  $a'b'$  are two similar glass tubes having thick capillary tubes at the top ( $a$  to  $e$  and  $a'$  to  $e'$ ) and bulb in the middle ( $b$ ). Carefully dried carbon dioxide was passed for several hours through the tube  $ab$ , open at both ends, to expel all air. The capillary end was then sealed. The other end was also temporarily sealed. This end was then immersed under mercury and opened. The tube was then slightly heated to expel some of the gas. On cooling it, a pellet of mercury was drawn in. The tube was then firmly fixed in a strong



copper cylinder A with its capillary portion projecting outside. The copper cylinder was fixed with two massive brass flanges at either ends, the connections being made airtight by using leather washers which were earlier soaked in lard, heated in *vacuo*. A steel screw 180 mm long and 4 mm in diameter passes through the lower flange. The cylinder was filled with water and hence a pressure as high as 400 atmospheres could be applied with this screw. To record this pressure, pure dry air is passed for a long time through the tube a'b'. The mercury pellets in both the tubes act as stoppers. The air tube was fixed in a cylinder A' exactly similar to A and both of them were put in communication through the lateral opening cd. The cylinder A' was also provided with a screwing system S' and pressure, which is identical in both the tubes, could be varied by using either S or S'.

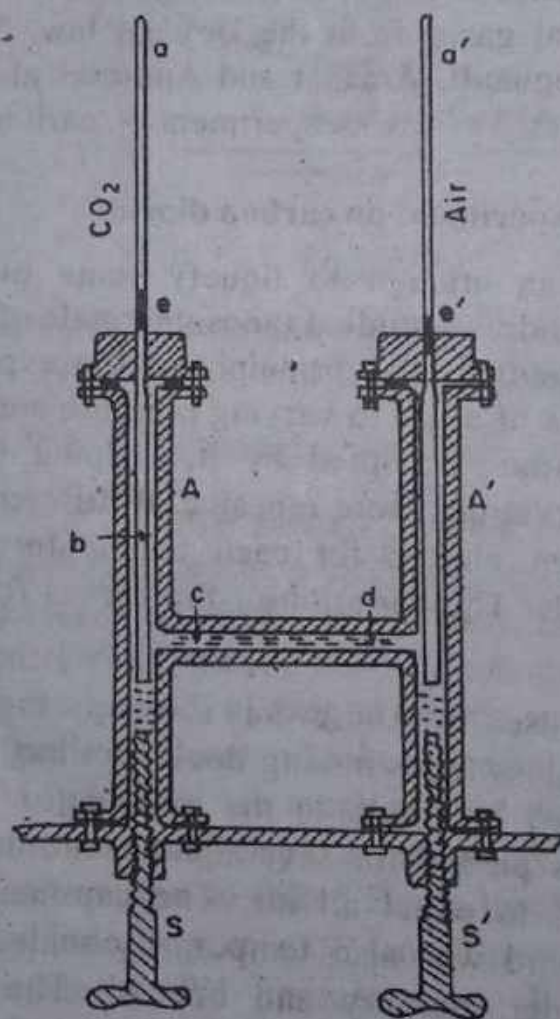


Fig. 17.3

The screws are turned so that mercury pellets in the two tubes appear above the cylinders A and A'. The portions of the capillary tubes projecting above the cylinders are surrounded by jackets having a bath of desired temperature or a freezing mixture as the case may be so as to maintain them at any constant temperature. The capillary tubes of A and A' are graduated to read volume directly.

The pressure on carbon dioxide was gradually increased by screwing in either S or S'. Since the pressures of air in A' and carbon dioxide in A are the same, the pressure on carbon dioxide at each step was equal to that on air in A'. The pressure on air was calculated by considering air to be a perfect gas and applying the formula  $PV = RT$ . The volume was read from the graduations. The temperatures of upper jacket tubes were kept constant for each set of observation. During these observations, the volume of carbon dioxide corresponding to a particular value of pressure was read off directly from the graduation. Several such set of observations at different constant temperatures were recorded. For each set, a curve was plotted with P against V. Each of these curves at a definite temperature is known as an *isothermal*.

### Discussion of Andrews' results

The results of Andrews' experiments are shown in Fig. 17.4. The ordinate represents pressures in atmospheres while *specific volume in c.c. i.e.*, the volume of unit mass of the gas at some standard temperature and pressure is represented along the abscissa. The figures on the right hand side denote the temperatures in degree celsius of the various isothermals.

Considering the isothermal at  $13.1^{\circ}\text{C}$ , it can be seen that in the portion AB of the curve, carbon dioxide roughly obeys Boyle's law. Here the volume diminishes with pressure. At B the gas is like saturated vapour and liquefaction begins at a pressure of about 49 atmospheres. The pressure remains constant as long as liquefaction continues and over the region BC the volume continually decreases as more and more of the gas becomes liquid. The liquid and vapour states coexist in this horizontal portion BC. At C, the gas is completely liquefied. On further increase of pressure the volume remains practically constant which shows that liquids are practically incompressible as indicated by the almost vertical rise of the portion



CD of the curve. The ratio of volumes of liquid and vapour states is equal to the ratio of the two values of the volumes at the two extreme points C and B of the horizontal portion of the curve.

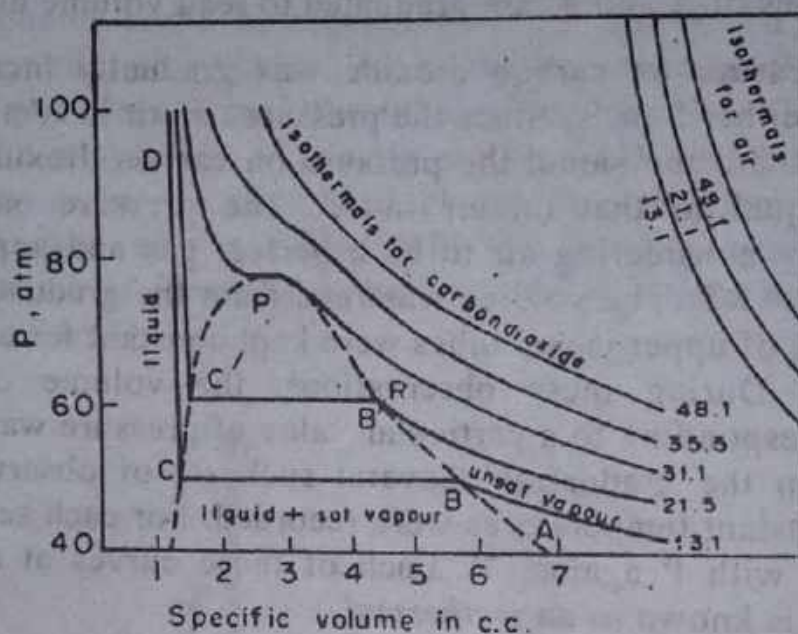


Fig. 17.4

The isothermal corresponding to  $21.5^{\circ}\text{C}$  is of the same general form, liquefaction beginning at a pressure of about 60 atmospheres. But the horizontal portion  $B'C'$  of the curve is shorter. This means that the specific volume of the vapour when condensation begins is smaller while that of the liquid when condensation is complete is greater than the corresponding quantities for the isothermal at  $13.1^{\circ}\text{C}$ . With further increase of temperature the curves show the same general feature. The horizontal portion of the curves gets shorter and shorter as the temperature increases meaning thereby that the difference between the volumes of saturated vapour and that of the liquid is lessened *i.e.*, the density of the vapour is approaching that of the liquid. It is further noticed that at higher temperature a higher pressure is needed to liquefy the gas.

The isothermal at  $31.1^{\circ}\text{C}$  is very important. The horizontal portion for this isothermal just vanishes and the two volumes have become identical. However, although the isothermal has no



horizontal portion as in lower temperatures, the gas could still be liquefied at  $31.1^{\circ}\text{C}$  by the application of pressure only. Above this temperature the horizontal portion is absent from all the isothermals. There is no visible formation of liquid as pressure is increased, but the volume diminishes rapidly till it becomes equal to that of the liquid at the temperature *i.e.*, the isothermals show a tendency towards the horizontal portion. This continues up to a certain range of temperature. But above this range, say from  $48.1^{\circ}\text{C}$ , the curves show no such tendency. The curves are approximately hyperbolic, thereby indicating that at these temperatures the nature of the gas is similar to that of the perfect gas and it obeys Boyle's law. Some isothermals for air are shown on the right hand top of the figure.

The temperature at or below which a gas can be liquefied by pressure alone is called the *critical temperature* and the isothermal at that temperature is called the *critical isothermal*. The critical isothermal divides the isothermals into two essentially different regions, one above the critical isothermal and the other below it. In the region above the critical isothermal no liquid state is at all possible even under the greatest pressure, while below it there are three separate regions. In the region enclosed by the dotted curve both liquid and gaseous states coexist. Below the critical isothermal, there is the liquid region to the left of the line  $PC$  while to the right of  $PB$  there is the gaseous region. The dotted curve  $CC'PB'B$ , drawn along the bends of the isothermals below  $31.1^{\circ}\text{C}$  is known as the border curve. It is the locus of all points where change of state between liquid and vapour starts. The highest point  $P$  of this curve, where the tangent to the curve is horizontal, is known as the *critical point*. At point  $P$ , the liquid, vapour and gaseous states meet.

### Continuity of state

The liquid and the vapour states of substances appear to be widely different. According to Andrews, a substance may be made to pass from one state to another by a series of graded operations without any discontinuity occurring *i.e.*, without *coexistence of liquid and vapour states*. This view is supported by the following experiment.

Let gaseous carbon dioxide at  $25^{\circ}\text{C}$  and 60 atmospheres pressure (represented by the point  $R$ ) is to be transformed into liquid



carbon dioxide at the same temperature (represented by the point S) without reaching the inside of the dotted curve *i.e.*, without any discontinuity appearing. For this, the gas is first heated to a temperature above  $31.1^{\circ}\text{C}$ ; then compressed at this temperature till its volume becomes equal to the volume of the liquid at the point S at  $25^{\circ}\text{C}$ . The gas is then cooled to  $25^{\circ}\text{C}$  and the pressure is reduced. The substance stands at S. Thus starting from the point R which undoubtedly represents a gaseous state, one is able to reach the point S, which undoubtedly represents a liquid state while at no stage of the operation did the liquid and the vapour states exist together *i.e.*, any discontinuity appeared. The result was summed up by Andrews as follows:

(i) There is no physical distinction between the liquid and gaseous states of a substance as shown by the horizontal portions of the isothermals. The liquid and gaseous states are two distant points of the same continuous physical process *i.e.*, these are the two forms of the same condition of matter. It is possible to go from one form to the other form in a continuous process as mentioned above. In other words *there exists a state of continuity of the liquid and gaseous states.*

(ii) At the critical temperature the boundary line between the vapour and the liquid vanishes. The density of the vapour and the liquid become equal at the critical point. Moreover, the compressibility of the vapour becomes infinite at the critical point.

(iii) There is no essential difference between *liquefiable and permanent gases*. Below the critical temperature a permanent gas can be liquefied by subjecting it to pressure. Above and remote from the critical temperature, the liquefiable gases behave the same way as the permanent gases *viz.*, oxygen, nitrogen and hydrogen.

### 17.5 Vapours and gases

The scientific difference between the terms gas and vapour may now be understood. When a substance is below its critical temperature it is referred to as vapour, thereby implying that the liquid form is easily obtainable. However, when the substance is above its critical temperature it is referred to as gas since it cannot be liquefied. The critical temperatures of hydrogen and nitrogen are  $-241^{\circ}\text{C}$  and  $-147^{\circ}\text{C}$  respectively, and hence these substances are normally well above their critical temperatures. Hydrogen and nitrogen

are consequently known as permanent gases. Thus the terms vapour and gas are artificial distinction; above the critical temperature the substance is gas and below the critical temperature it is termed vapour. The name permanent gas has no real significance.

### 17.5 Critical point and critical constants

The point  $P$  on the critical isothermal (Fig 17.4) where the three states - the liquid, the vapor and the gaseous states of a substance meet is described as the critical point. And, because all the three states of matter meet at the critical point, a substance at the critical point possesses some remarkable properties.

For every substance there are three constants known as its critical constants. They are *critical temperature*, *critical pressure* and *critical volume*.

**Critical temperature ( $T_c$ ):** The critical temperature of a substance is the temperature at or below which the substance in the gaseous state can be liquefied by the application of pressure alone. Above this temperature, no liquid can make its appearance, however great the pressure may be.

**Critical pressure ( $P_c$ ):** The minimum pressure which will be sufficient to liquefy the substance at its critical temperature is called its critical pressure. This is also the saturation vapour pressure of the substance at the highest temperature at which its liquid form can exist.

**Critical volume ( $V_c$ ):** The critical volume of a substance is the volume of one gramme of the substance at critical temperature and under critical pressure.

The isothermal at  $31.1^\circ\text{C}$  in Andrews' isothermals for carbon dioxide is generally referred to as the critical isothermal for carbon dioxide. The point  $P$  where the tangent to this isothermal becomes horizontal is known as the critical point. For carbon dioxide, the values for the critical constants are given below:

$T_c = 31.1^\circ\text{C}$  (subsequently given as  $30.92^\circ\text{C}$  by Andrews and  $30.98^\circ\text{C}$  by Keesom)

$P_c = 73$  atmospheres



$$V_C = 2.17 \text{ c.c.}$$

The values for the critical constants of some gases are given in Table 1.

### 17.6 Van der Waals' equation

The equation of state of an idea gas  $PV = RT$  was derived on the basis of kinetic theory assuming (i) *that the molecules of the gas are mere point-masses occupying no space and (ii) that there are no inter-molecular forces what so ever between them.* The equation is the limit to which all real gases approach but is not accurately obeyed by any one of them. The deviations from the equation are all the more prominent when the gas is at low temperatures and under high pressures. The deviation may be ascribed to the assumptions of the kinetic theory which were rather too simplifying. Subsequently scientists tried to modify the gas equation or deduce new equations discarding some of these assumptions. Clausius recognized the fact that *physical molecules cannot be point-masses occupying no space but must have a finite volume of their own no matter how small they may be.* And Hirn pointed out that *in real gases there must exist some sort of cohesive force of attraction between the molecules.* But credit goes to the Dutch physicist J.D. Van der Waals who first corrected the equation  $PV = RT$  by making due allowance for the above two factors. The corrections regarding molecular volume and inter-molecular forces have come to be known as *volume correction* and *pressure correction* respectively. The gas equation, as corrected by Van der Waals, fits the behavior of actual gases more closely than the perfect gas equation  $PV = nRT$ .

#### Deduction

(i) *Volume correction:* Let us suppose that the molecules are of finite size, being spheres of radius  $r = d/2$  where  $d$  is the diameter of the molecules. Since the centres of two molecules cannot be within a distance less than  $d = 2r$  from each other, it is impossible for the centre of a molecule to lie within any of the spheres of radius  $d$  surrounding the other molecules. In other words, a spherical space of radius  $d$  and volume  $\frac{4}{3}\pi d^3$  around each one of the molecules is not available to the rest (Fig. 17.5).

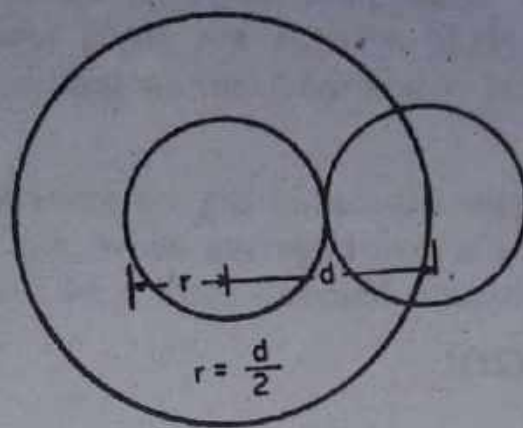


Fig. 17.5

Let us now consider a perfectly evacuated enclosure of volume  $V$  and imagine that a number of  $N$  molecules are being introduced inside this enclosure one by one. The first molecule has an available volume  $V$ . As a volume  $\frac{4}{3}\pi d^3$  is not available around this molecule; the second molecule will have an available volume

$$V - \frac{4}{3}\pi d^3 = V - v \text{ (say)}$$

As a volume  $\frac{4}{3}\pi d^3$  is not available around each of the first and second molecule, the third molecule will have an available volume  $V - 2v$ . The fourth molecule will then have an available volume  $V - 3v$ , the fifth  $V - 4v$ , the sixth  $V - 5v$  ... and the  $N$ th  $V - (N-1)v$ .

The average volume available to a molecule is no longer the volume  $V$  of the vessel but is given by

$$\frac{V + (V - v) + (V - 2v) + \dots + [V - (N-1)v]}{N}$$

$$= \frac{NV - [1 + 2 + 3 + \dots + (N-1)]v}{N}$$

$$= V - \frac{N(N-1)v}{2N}$$

$$= V - \frac{N-1}{2}v \simeq V - b \text{ (say)}$$



where

$$\begin{aligned}
 b &= \frac{N-1}{2} \cdot v \\
 &= \frac{N-1}{2} \cdot \frac{4}{3} \pi d^3 \\
 &= \frac{2}{3} (N-1) \pi (2r)^3 \\
 &= \frac{2}{3} (N-1) 8 \pi r^3 \\
 &\simeq 4N \cdot \frac{4}{3} \pi r^3 \quad (\text{since } N \text{ is very very large}).
 \end{aligned}$$

But  $\frac{4}{3} \pi r^3$  is the volume of a single molecule. Hence  $b$  is four times the total volume of all the molecules. Without loss of generality, we can assume that the  $N$  molecules together constitute one gm-molecule of the gas. Thus, with this correction Van der Waals' equation becomes

$$P(V-b) = RT \quad (17.3)$$

(ii) *Pressure correction*: If the forces between the molecules of the gas are no longer negligible, a second correction is necessiated by the presence of this force. We can suppose that the molecules being material particles, a certain force of cohesion acts between any molecule and its neighbor. The magnitude of this attractive force varies inversely as some power of the distance between the two molecules. So long as a molecule is sufficiently remote from the surface, it is equally surrounded by gas molecules on all sides. Forces act on it which will vary continually both in direction and magnitude, but which, when averaged over a sufficient interval of time, are likely to cancel out and the resultant force acting on the molecule is zero. But this is not so for a molecule on the surface or close to the surface.

Let us fix our attention on a molecule which is as near to the boundary as it can go. If the force from each adjacent molecule be



resolved into normal and tangential components, all directions in the boundary (tangential) plane are equally likely and so they will balance. But the resultant normal component is invariably directed inwards.

This is because there are gas molecules only on one side of the wall. (Fig. 17.6). Thus, when averaged over a sufficient interval of time, the surface will be a force directed inwards at right angles to the boundary.

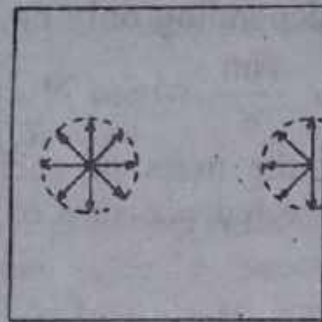


Fig. 17.6

Thus the effect of all the inter-molecular forces can be regarded as a steady inward pull which is exerted on a molecule in the layer closest to the boundary. Then the molecules, when they reach the boundary, are no longer deflected by impact alone, (both the molecules and the wall are supposed to behave like perfectly elastic bodies), but by the joint action of their impact with the boundary and of this inward pull. The change of momentum suffered by the molecules may be supposed to be produced by a total pressure ( $P + p_1$ ) per unit area, instead of the simple pressure,  $P$  which is observed.

[Alternately, when a molecule experiences a pull directed normally towards the interior, it will impinge on the wall with a velocity less than that corresponding to no inter-molecular attraction. The momentum with which the molecules strike the walls will then be reduced. And, when the momentum is less, the pressure exerted by the gas will also be less. Therefore, if the effect of the inter-molecular attraction is taken into account, the corrected or the actual pressure must be equal to the sum of the observed pressure  $P$  and the decrease  $p_1$  in the pressure as a result of this attraction. Thus the observed pressure  $P$  will be less than the actual pressure,  $P + p_1$ ].

It is obvious that the magnitude of the pressure  $p_1$  due to the inward pull will be proportional jointly to (i) the number of molecules striking unit area of the wall per second and (ii) the resultant normal component of the cohesive forces. As each of these two factors is directly proportional to the number of molecules present in a given volume i.e., the density of the gas,  $p_1$  will be proportional to the square of the density. Thus

$$p_1 \propto \rho^2; \text{ or, } p_1 = C\rho^2$$

where  $C$  is a constant depending only on the nature of the gas.

Now  $\rho$  can be replaced by  $\frac{Nm}{V}$  where  $N$  is the total number of molecules in a volume  $V$ , the mass of each molecule being  $m$ . Putting  $CN^2m^2 = a$ , the corrected pressure of a real gas should be given by

$$P + p_1 = P + \frac{a}{V^2}$$

Eqn. 17.3 then takes the form

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT \quad (17.4)$$

Eqn. (17.4) is known as the *Van der Waals' equation of state of a gas*.

So long as we confine our attention to a single mass of gas,  $a$  and  $b$  are constants. These are referred to as *Van der Waals' constants*. But, being functions of the total number of molecules, they are not absolute constants; they depend on the mass of the gas. Eqn. (17.4) refers to one gram-molecule of a gas. Since  $a \propto N^2$  and  $b \propto N$ ,  $N$  being the number of molecules in one gm-molecule of the gas, it can be easily inferred that for  $n$  gm-molecules, these values should change by factors  $n^2$  and  $n$  respectively while the right hand side changes by a factor  $n$ . Thus for  $n$  gm-molecules, Van der Waals' equation becomes

$$\left(P + \frac{an^2}{V}\right)(V - nb) = nRT \quad (17.5)$$



Again, since one gm-molecule occupies 22.4 liters at N.T.P., the values for 1 c.c of the gas at N.T.P are obtained by dividing the gm-molecular values of  $a$  and  $b$  by  $(22.4 \times 10^3)^2$  and  $(22.4 \times 10^3)$  respectively

The gm-molecular values of  $a$  and  $b$  for a few gases are listed in Table 2.

### 17.7 Van der Waals' equation and Andrews' isothermals for carbon dioxide

Van der Waals' equation may be written in the form

$$P = \frac{RT}{V-b} - \frac{a}{V^2} \quad (17.6)$$

Eqn. (17.6) is more convenient for plotting isothermal curves of pressure against volume. It can be seen that when  $V$  is very small approaching  $b$ ,  $P$  tends to be infinity. Again for large values of  $V$ ,  $P$  is small and in the limit  $P = 0$  when  $V \rightarrow \infty$ . Thus the  $P$ - $V$  curves must have a concavity upwards, which is found to be the case with all isothermals at higher temperatures. But at temperatures below the critical temperature, the Van der Waals' isothermals are wavy in nature having maxima and minima. Further  $V$  cannot be less than  $b$  as  $P$  would then be negative which is physically absurd.

Substituting the known values of  $a$  and  $b$  viz,  $a = 0.00717$  atmos and  $b = 0.00191$  c.c for carbon dioxide and of  $R$ , the curves of Fig. 17.7 are obtained. Comparing these curves with those obtained by Andrews' experimentally, it is readily seen that the former gives maxima and minima in a region represented by a straight line in the latter. Experimentally this is the region where condensation or vaporization begins and the pressure remains constant as long as the process continues. The difference can be easily explained by a proper interpretation of the theoretical curves. The part BDE of the curve indicates that the volume should decrease with decrease of pressure and is supposed to represent bumping. This is a *collapsible state*, for any decrease of volume is accompanied by a decrease of pressure which tends to decrease the volume further. Thus the state BDE represents a state of unstable equilibrium and consequently can never be realized in practice. Consequently this portion BDE of the curve was not obtained in Andrews' experiment.



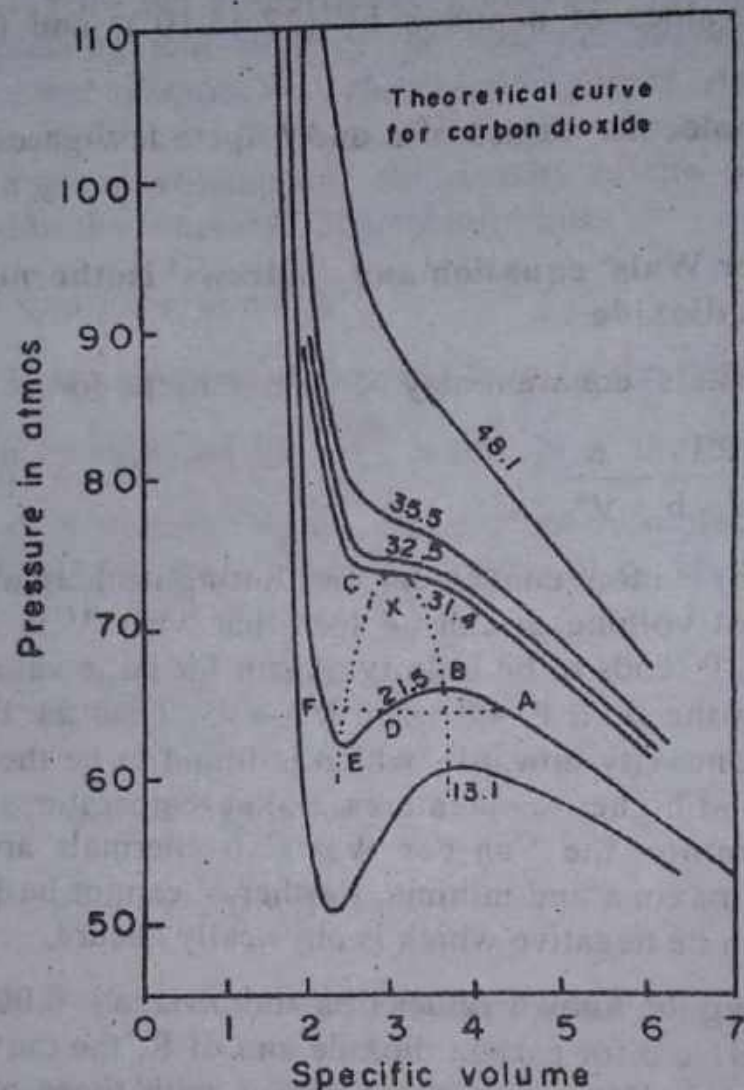


Fig. 17.7

The region AB of the Van der Waals' curve represents a liquid state at a higher temperature as is evident by a reference to Andrews' experimental curves. But here in Van der Waals' curve it is still gaseous at a lower temperature. Thus the portion AB obviously represents a *supercooled vapors state*. Now this state of super cooled equilibrium is highly unstable, as introduction of a little dust or smoke will disturb this supercooled or supersaturated vapour which suddenly condenses. Thus the portion AB does not appear in Andrews' experimental curves.

Similarly the portion EF represents a *super heated liquid state*.

By referring to Andrews' experimental curves this is a region which represents a gaseous state at a lower temperature; but here it is still liquid at a higher temperature. This is also a state of unstable equilibrium and hence does not appear in Andrews' curves. In case of pure liquids boiling is retarded and the liquid can be heated to a temperature higher than its boiling point before it begins to boil. Thus the apparent divergence between Van der Waals' curves and Andrews' experimental isothermals for carbon dioxide can be explained.

Van der Waals' theory does not, however, tell us when condensation or vaporization begins, *i.e.*, where the straight part commences. A simple thermo-dynamic argument shows that the straight line should be so drawn that the area above the line is equal to the area below the line, *i.e.*, area ABD = area DEF. The curve BXE represented by the dotted line is the locus of the maxima and minima points in the theoretical curves.

### 17.8 Van der Waals' equation and critical constants

Van der Waals' equation can also be written as

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT$$

$$\text{or, } V^3 - \left(b + \frac{RT}{P}\right)V^2 + \frac{aV}{P} - \frac{ab}{P} = 0 \quad (17.7)$$

This is a cubic equation in  $V$ ; hence, in general there are three values of  $V$  for given values of  $P$  and  $T$ . Further, from the theory of equation, either all the three values are real or one real and two imaginary (imaginary roots occur in pairs). However, at point  $X$  on the isothermal  $C$ , the three values of  $V$  merge into one common value. The point  $X$  is called the *point of inflexion* and the tangent to the curve drawn at this point is horizontal. At this point, the isothermal  $C$  changes its curvature. This point is known as the *critical point* and the pressure, volume and temperature corresponding to this point are referred to as *critical pressure* ( $P_c$ ), *critical volume* ( $V_c$ ) and *critical temperature* ( $T_c$ ). Isothermal  $C$  is known as the *critical isothermal*. The values of these critical constants can be found by keeping in mind that  $X$  is a maximum point. So at  $X$



$$\frac{dP}{dV} = 0 \quad \text{and} \quad \frac{d^2P}{dV^2} = 0$$

At the critical point X, Van der Waals' equation can be written as

$$P_c = \frac{RT}{V_c - b} - \frac{a}{V_c^2} \quad (\text{i})$$

Differentiating with respect to volume, we have

$$\frac{dP_c}{dV_c} = -\frac{RT}{(V_c - b)^2} + \frac{2a}{V_c^3} = 0 \quad (\text{ii})$$

$$\text{or, } \frac{RT_c}{(V_c - b)^2} = \frac{2a}{V_c^3} \quad (\text{iii})$$

Differentiating eqn. (i) twice with respect to volume, we get

$$\frac{d^2P_c}{dV_c^2} = \frac{2RT_c}{(V_c - b)^3} - \frac{6a}{V_c^4} = 0$$

$$\text{or, } \frac{2RT}{(V_c - b)^3} = \frac{6a}{V_c^4} \quad (\text{iv})$$

Dividing eqn. (iii) by eqn. (iv), we have

$$\frac{V_c - b}{2} = \frac{V_c}{3}; \quad \text{or, } V_c = 3b \quad (\text{v})$$

Substituting the value of  $V_c$  as given by eqn (v) in eqn (iii), we have

$$\frac{RT_c}{(3b - b)^2} = \frac{2a}{(3b)^3}$$

$$\text{or, } T_c = \frac{8a}{27bR} \quad (\text{vi})$$

Substituting the values of  $V_c$  and  $T_c$  in eqn. (i) above, we get

$$P_c = \frac{R \cdot \frac{8a}{27bR}}{(3b - b)} - \frac{a}{(3b)^2} = \frac{a}{27b^2}$$



The ratio  $\frac{RT_c}{P_c V_c}$  is known as the critical coefficient of a gas.

Substituting the values of  $T_c$ ,  $P_c$  and  $V_c$  we get

$$\frac{RT_c}{P_c V_c} = \frac{R \times \frac{8a}{27bR}}{\frac{a}{27b^2} \times 3b} = \frac{8}{3} = 2.67 \quad (\text{viii})$$

*According to Van der Waals' equation the critical coefficient of any gas is same and is equal to 2.67.*

### 17.9 Values of Van der Waals' constants

Measurements of  $P_c$  and  $T_c$  are comparatively easier than of  $V_c$ . Van der Waals' constants  $a$  and  $b$  are, therefore, generally determined from a knowledge of  $P_c$  and  $T_c$ . Let us obtain expressions connecting  $a$  and  $b$  with  $T_c$  and  $P_c$ .

Now

$$P_c = \frac{a}{27b^2}$$

$$\text{and } T_c = \frac{8a}{27Rb}$$

Squaring  $T_c$  and dividing by  $P_c$

$$\begin{aligned} \frac{T_c^2}{P_c} &= \frac{64a^2}{(27)^2 R^2 b^2} \times \frac{27b^2}{a} \\ &= \frac{64a}{27R^2} \end{aligned}$$

$$\therefore a = \frac{27}{64} \frac{R^2 T_c^2}{P_c} \quad (17.8)$$

Again dividing  $P_c$  by  $T_c$ , we get

$$\frac{P_c}{T_c} = \frac{a}{27b^2} \times \frac{27Rb}{8a} = \frac{R}{8b}$$

$$\therefore b = \frac{RT_c}{8P_c} \quad (17.9)$$

### 17.10 Boyle temperature from Van der Waals' equation

Van der Waals' equation may be written in the form

$$P = \frac{RT}{V-b} - \frac{a}{V^2}$$

$$\text{or, } PV = \frac{RTV}{V-b} - \frac{a}{V}$$

At Boyle temperature  $T = T_B$

$$PV = RT_B$$

Hence, from eqn. (i),

$$RT_B = \frac{RT_B V}{V-b} - \frac{a}{V}$$

$$\text{or, } RT_B \left( \frac{V}{V-b} - 1 \right) = \frac{a}{V}$$

$$\text{or, } RT_B \left( \frac{b}{V-b} \right) = \frac{a}{V}$$

$$\text{Hence } T_B = \frac{a(V-b)}{RVb} \simeq \frac{a}{Rb} \quad (17.11)$$

(Since  $a$  and  $b$  are both very small quantities)

It may be noted that the critical temperature  $T_c = \frac{8a}{27bR}$  while

$$\text{the Boyle temperature } T_B = \frac{a}{Rb} = \frac{27}{8} T_c$$

### 17.11 Reduced equation of state

Expressing the actual pressure, volume and temperature as fractions of the critical pressure, volume and temperature, we obtain what is called the *reduced equation of state*

Let

$$\frac{P}{P_c} = \alpha, \quad \frac{V}{V_c} = \beta, \quad \frac{T}{T_c} = \gamma$$

Then,  $P = \alpha \cdot P_c$ ,  $V = \beta \cdot V_c$ ,  $T = \gamma T_c$

The Van der Waals' equation

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT \text{ reduces to}$$

$$\left(\alpha \cdot P_c + \frac{a}{\beta^2 V_c^2}\right)(\beta V_c - b) = R \cdot \gamma \cdot T_c$$

Substituting the values  $P_c = \frac{a}{27b^2}$ ,  $V_c = 3b$  and  $T_c = \frac{8a}{27Rb}$  in the above equation, we get

$$\left(\frac{\alpha}{27} \cdot \frac{a}{b^2} + \frac{a}{\beta^2 9b^2}\right)(\beta \cdot 3b - b) = \frac{R \cdot \gamma}{27} \frac{8a}{Rb}$$

which simplifies to

$$\left(\alpha + \frac{3}{\beta^2}\right)\left(\beta - \frac{1}{3}\right) = \frac{8}{3}\gamma \quad (17.12)$$

Eqn (17.12) is called the reduced equation of state, which does not contain any constants like  $a$  or  $b$  but depends upon the nature of the substance. It should, therefore, be applicable to all gases. But in reality it is not so. The defect, however, is in-built in the parent equation and is not a defect of the reduced equation of state. Thus eqn. (17.12) will apply to all substances.

$\alpha$ ,  $\beta$  and  $\gamma$  in eqn. (17.12) are known as the *reduced pressure*, *reduced volume* and *reduced temperature* respectively.

### Law of corresponding states

If the values of  $\alpha$ ,  $\beta$  and  $\gamma$  be the same for any two gases, then the gases are said to be in the corresponding states.

From the reduced equation of state it can be seen that if any two of the quantities  $\alpha$ ,  $\beta$  and  $\gamma$  are the same for two gases, the third



quantity will also be same. In other words, if the temperature and pressure of any two gases bear the same ratio to their critical temperatures and pressures, then their volumes also will have the same ratio to their respective critical volumes. This is known as the *law of corresponding states*.

### 17.12 Defects of Van der Waals' equation

Van der Waals' equation, in spite of its success in explaining the behaviors of real gases over a wide range of temperature and pressure, has several drawbacks. These are discussed below.

(1) As can be seen in the case of carbon dioxide, the isothermals obtained theoretically from Van der Waals' equation do not exactly tally with those obtained experimentally by Andrews. A particular isothermal may be made to agree much more closely with suitably chosen values of  $a$  and  $b$ , but then for the same values of  $a$  and  $b$ , the agreement is not so good for other isothermals. This implies that  $a$  and  $b$  vary with temperature as has been directly found from experiments. The limit of this variation with temperature may be sometimes as high as 30 per cent. But Van der Waals' equation assumes them to be constant. This is a fundamental drawback in Van der Waals' equation which cannot be easily remedied by suitable corrections.

(2) Van der Waals' equation gives  $V_C = 3b$ , whereas experimentally it is found to be more nearly equal to  $2b$ , ranging from  $1.41b$  for argon to  $2.8b$  for hydrogen. Experimentally determined values of  $V_C$  for certain gases are listed in Table 4.

(3) A more definite discrepancy however lies in the values of the critical coefficients  $\frac{RT_c}{P_c V_c}$ . According to Van der Waals' equation

$\frac{RT_c}{P_c V_c}$  should be  $\frac{8}{3}$  for all gases independent of their nature.

However, it actually varies from gas to gas with an average value of 3.7 for most substances. This constant, therefore, appears to depend upon the molecular structure of the gas. The values of critical coefficients of certain gases are given in Table 3.

(4) Van der Waals' equation gives a value for the Boyle temperature ( $T_B$ ) which is equal to  $\frac{27}{8} T_c = 3.375 T_c$ . Actually it is  $3.65 T_c$  for helium and is below  $3.3 T_c$  for other gases.

In spite of the above limitations, the Van der Waals' equation is, however, frequently employed because of its inherent simplicity. It lends itself to mathematical treatment readily and even if the results obtained from it are not exact, they are at least qualitatively correct in so far as the general features of transition from liquid to gaseous state are concerned.

### 17.13 Experimental determination of the critical constants

(a) The critical constants can be determined graphically by plotting the isothermals between  $P$  and  $V$  at different temperatures of the gas. The point where the horizontal portion of the curve just disappears *i.e.*, the specific volumes of the gaseous and liquid states just coincide, is the critical point and the corresponding isothermal the critical isothermal. The pressure, volume and temperature corresponding to the critical point give the critical constants of the gas.

(b) *The following experiment for the determination of the critical constants was carried out by Cagniard de la Tour.*

The apparatus used consisted of a strong V-shaped glass tube AB, (Fig. 17.8) filled with mercury from A to B. The space above mercury in A contained the experimental liquid and its vapor, while that in B contained air. The volume of this air was used to calculate its pressure by applying Boyle's law which also determines the pressure of the vapor above A. The whole arrangement was heated gradually in an oil bath. With the rise of temperature evaporation of the liquid took place; the density of the vapor increased while that of the liquid decreased. Since the liquid was always under the pressure, more and more liquid evaporated with the rise of temperature to maintain the saturation vapor pressure until at a definite temperature a striking phenomenon was witnessed. The meniscus separating the vapor and liquid grew indistinct and finally disappeared altogether *i.e.*, the layers of liquid and gas could not be distinguished as two separate phases. The disappearance of the meniscus was accompanied by the formation of a peculiar mist. At a temperature slightly above this,



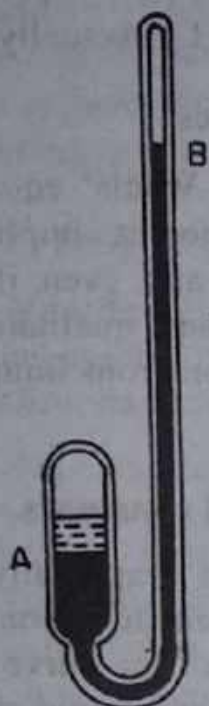


Fig. 17.8

the space above A was filled with a homogeneous substance undistinguishable as liquid and vapour. If the tube is again allowed to cool down to the same temperature as before, a mist appears at first and then a faint meniscus appears again. At a slightly lower temperature the mist disappears leaving once again the distinct meniscus separating the liquid and its vapour.

The temperature at which the meniscus just disappeared while heating and just reappeared while cooling were noted. The mean of these two temperatures gives the critical temperature. The critical pressure for the liquid under investigation was obtained by adding the pressure of air to the difference of the two mercury columns. The pressure of the air is calculated by applying the formula  $PV = RT$  and considering the air to behave like a perfect gas.

### Determination of critical volume

The direct measurement of critical volume is very difficult as a very small change in pressure leads to a large change in volume. Moreover, for correct recording of the volume, the substance must be exactly at the critical temperature. Even a small variation of temperature, say by  $0.1^\circ\text{C}$ , produces a large change in volume and so a small error in temperature leads to a considerable error in the volume.

Cailletet and Mathias have suggested a method for the determination of critical volume. Density of the liquid is determined at different temperature. The density of the saturated vapour pressure of the liquid is also determined accurately for the same temperatures. As the density of liquid decreases with temperature while that of the vapour increases, a curve roughly parabolic in shape is obtained when a graph is plotted between density and temperature (Fig. 17.9). The two halves of this curve meet at the critical point. If the mean of the liquid and vapour densities is plotted on the same graph, the result is also a straight line passing through the critical point. The density corresponding to the critical point is obtained from the graph. Let it be  $\rho$ .



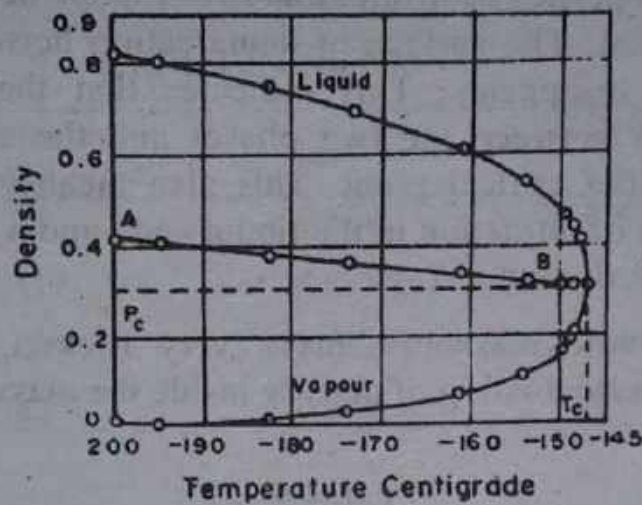


Fig. 17.9

The specific volume, *i.e.*, the volume per unit mass  $V_c = \frac{1}{\rho}$ .

This represents the critical volume.

### Behavior of a substance near its critical point.

Based on the experiments of Andrews, Amagat and others, the behaviour of a substance near its critical point may be summarized as follows:

Table 1

Gas	Critical temperature $T_c$ in $^{\circ}\text{C}$	Critical pressure $P_c$ in atm.	Specific volume $V_c$ in c.c	$\frac{RT_c}{P_c V_c}$
Helium	-267.9	2.25	15.4	3.13
Hydrogen	-239.9	12.80	32.2	3.28
Argon	-122.4	48.00	1.88	3.43
Oxygen	-118.8	49.70	2.32	3.42
Nitrogen	-147.1	33.50	3.21	3.42
Carbon dioxide	31.0	72.80	2.17	3.48
Ammonia	132.2	112.30	4.24	4.12
Ether	193.8	35.60	3.85	3.81
Sulphur dioxide	157.2	77.60	1.95	3.60
Chlorine	146.0	76.00		
Water	374.0	218.00	3.181	4.30
Methyl chloride	143.1	65.90	2.71	3.80

(a) The properties of a liquid and its vapour become identical at its critical point. The surface of demarcation between the liquid and its vapour disappears. This indicates that there must exist mutual diffusion between the two phases and the surface tension must vanish at the critical point. This also means that the inter-molecular forces of attraction in the liquid and vapour states must be equal at the critical point.

(b) The whole mass presents a very flickering appearance which suggests a fluctuation of density inside the mass.

**Table 2**  
Van der Waals' constants

Gas	a atm. $\times$ (c.c.) <sup>2</sup> mole <sup>2</sup>	b c.c./mole
Hydrogen	$0.10 \times 10^6$	23.0
Oxygen	$1.36 \times 10^6$	31.6
Nitrogen	$1.35 \times 10^6$	38.6
Ammonia	$4.17 \times 10^6$	37.1
Carbon dioxide	$3.66 \times 10^6$	42.9
Ethylene	$4.47 \times 10^6$	57.1
Chlorine	$5.35 \times 10^6$	46.1
Water	$5.46 \times 10^6$	30.5

(c) the compressibility of the vapour becomes very high near the critical point-becoming almost infinite at the critical point. Guoy pointed out that this explains the variation of density throughout the mass referred to in (b) above.

(d) The densities of the liquid and the vapour gradually approach each other till they become equal at the critical point. However, it was shown by experiments of Callendar, that for pure water freed from all traces of dissolved air, the densities of liquid and vapour did not become equal at  $374^{\circ}\text{C}$ , the critical temperature at which the surface between the liquid and vapour phases disappear. The differences in densities of the two phases continue upto  $380^{\circ}\text{C}$  where they become identical. This suggests the existence of a critical region rather than a critical point.

**Table 3**  
Critical coefficient

Gas	Critical coefficient	Gas	Critical coefficient
Helium	3.27	Xenon	3.605
Oxygen	3.419	Marsh gas	3.67
Nitrogen	3.421	Carbon dioxide	3.61
Argon	3.424	Acetic acid	4.99

**Table 4**  
Experimental value of  $V_c$  for some gases

Gas	$V_c$	Gas	$V_c$
Hydrogen	2.8 b	Oxygen	1.46 b
Nitrogen	1.5 b	Carbon dioxide	1.86 b
Air	1.89 b	Argon	1.41 b

### Solved Problems

*Example 17.1 Calculate the Van der Waals' constants for dry air, given that*

$$T_c = 132\text{K}, P_c = 37.2 \text{ atmospheres, } P \text{ per mole} = 82.07 \text{ cm}^3 \text{ atmos K}^{-1}.$$

**Soln.**

$$a = \frac{27}{64} \frac{R^2 T_c^2}{P_c}$$

$$b = \frac{RT_c}{8P_c}$$

$$= \frac{27}{64} \frac{(82.07)^2 (132)^2}{37.2}$$

$$= 13.31 \times 10^6 \text{ atmos.cm}^6$$

$$= \frac{82.07 \times 132}{8 \times 37.2}$$

$$= 36.41 \text{ cm}^3$$



**Example 17.2** Calculate the value of the critical temperature for carbon dioxide where  $a = 0.00874$  and  $b = 0.0023$  for 1 c.c. of the gas at N.T.P.

**Soln.**

$$a = 0.00874 \text{ atm.} = 0.00874 \times 1.01 \times 10^6 \text{ dynes/cm}^2$$

$$b = 0.0023 \text{ c.c., } R = 8.31 \times 10^7 \text{ ergs per gm-molecule per } ^\circ\text{C.}$$

$$\text{Volume of 1 gm-molecule at N.T.P} = 22.4 \text{ liters} = 22.4 \times 10^3 \text{ c.c.}$$

$$\therefore R \text{ for 1 c.c} = \frac{8.31 \times 10^7}{22.4 \times 1000} \text{ ergs.}$$

$$\therefore T_c = \frac{8a}{27bR}$$

$$= \frac{8 \times 0.00874 \times 1.01 \times 10^6 \times 22.4 \times 1000}{27 \times 0.0023 \times 8.31 \times 10^7}$$

$$= 306.8 \text{ K} = 33.8^\circ\text{C.}$$

**Example 17.3** For carbon dioxide it is found that if unit pressure be taken as the standard barometric pressure and unit volume as the volume at N.T.P., then  $a = 0.00874$  and  $b = 0.0023$ . Calculate  $T_c$  for carbon dioxide gas.

**Soln.**

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT$$

$$\left(1 + \frac{0.00874}{1^2}\right)(1 - 0.0023) = R \times 273$$

$$\therefore R = \frac{1.00874 \times 0.9977}{273}$$

$$\therefore T_c = \frac{8a}{27bR}$$

$$= \frac{8 \times 0.00874 \times 273}{27 \times 0.0023 \times 1.00874 \times 0.9977}$$

$$= 305.5 \text{ K} = 32.5^\circ\text{C}$$

**Example 17.4** Using the values of  $T_c = 5.3$ ,  $P_c = 2.25$  atm., calculate  $a$  and  $b$  for a gram-molecule of helium.

**Soln.**

$$\begin{aligned} a &= \frac{27}{64} \cdot \frac{R^2 T_c^2}{P_c} \\ &= \frac{27 \times (8.3 \times 10^7)^2 \times (5.3)^2}{64 \times 2.25 \times 1.01 \times 10^6} \\ &= 3.5 \times 10^4 \text{ atm.} \times \text{cm}^4 \\ b &= \frac{RT_c}{8P_c} = \frac{5.3 \times 8.3 \times 10^7}{2.25 \times 8 \times 1.01 \times 10^6} \\ &= 24 \text{ c.c.} \end{aligned}$$

**Example 17.5** For hydrogen gas the critical pressure is 12.8 atmospheres and the critical volume 69.68 c.c. per gm-mol. Calculate the values of (i) the constants of Van der Waals' equation for one gm-molecule of the gas, (ii) the critical temperature and (iii) the Boyle temperature. ( $R = 8.4 \times 10^7$  ergs/K, 1 atmosphere =  $1.01 \times 10^6$  dynes/cm<sup>2</sup>)

**Soln.**

(ii) According to Van der Waals' equation the critical coefficient of a gas is equal to  $\frac{8}{3}$ .

Hence

$$\begin{aligned} \frac{RT_c}{P_c V_c} &= \frac{8}{3}; \quad \text{or, } T_c = \frac{8P_c V_c}{3R} \\ T_c &= \frac{8 \times 12.8 \times 1.01 \times 10^6 \times 69.68}{3 \times 8.4} \\ &= 28.6 \text{ K} \end{aligned}$$

(iii) The Boyle temperature  $T_B$  is approximately equal to  $\frac{27}{64} T_c$ .

Hence

$$T_B = \frac{27}{8} T_c = \frac{27 \times 28.6}{8} = 96.5 \text{ K}$$

(i) The constants of Van der Waals' equation

$$\begin{aligned}
 a &= \frac{27}{64} \cdot \frac{R^2 T_c^2}{P_c} \\
 &= \frac{27}{64} \frac{(8.4 \times 10^7)^2 (28.6)^2}{12.8 \times 1.01 \times 10^6} \\
 &= 1883.4 \times 10^8 \text{ dynes/cm}^2 \\
 &= 1.865 \times 10^5 \text{ atmos.}
 \end{aligned}$$

$$\begin{aligned}
 b &= \frac{RT_c}{8P_c} \\
 &= \frac{8.4 \times 10^7 \times 28.6}{8 \times 12.8 \times 1.01 \times 10^6} \\
 &= 23.23 \text{ c.c}
 \end{aligned}$$

**Example 15.6** Calculate the values of  $\frac{RT_c}{P_c V_c}$  for helium and oxygen from the following data:

Substance	Molecular Weight	$T_c$ in $^{\circ}\text{C}$	$P_c$ in atmos	$\rho_c$ in gm/c.c.
Helium	4.00	-267.9	2.26	0.0693
Oxygen	32.00	-118.8	49.7	0.430

Given  $R = 8.2 \times 10^7 \text{ ergs/K}$

Soln.

(i) Helium :  $T_c = (-267.9 + 273)\text{K} = 5.1\text{K}$

$$P_c = 2.26 \times 1.01 \times 10^6 \text{ dynes/cm}^2$$

$$V_c = \frac{4}{0.0693} = 57.72 \text{ c.c}$$

$$\therefore \frac{RT_c}{P_c V_c} = \frac{8.2 \times 10^7 \times 5.1}{2.26 \times 1.01 \times 10^6 \times 57.72} \approx 3.4$$



(ii) Oxygen :  $T_c = (-118.8 + 273)^\circ\text{K}$   
 $= 154.2^\circ\text{K}$

$$P_c = 49.7 \times 1.01 \times 10^6 \text{ dynes/cm}^2$$

$$V_c = \frac{32}{0.430}$$

$$= 74.42 \text{ c.c.}$$

$$\therefore \frac{RT_c}{P_c V_c} = \frac{8.2 \times 10^7 \times 154.2}{49.7 \times 1.01 \times 10^6 \times 74.42}$$

$$= 3.4$$

**Example 17.7** Show that at critical temperature, the departure of Van der Waals' gas law from that of the perfect gas  $P_c V_c / T_c = R$  measures 62.5%.

**Soln.**

For a perfect gas:  $P_c V_c / T_c = R$ .

For a Van der Waals' gas :  $P_c V_c / T_c = \frac{8}{3}R$

$$\therefore \text{Required departure is : } \frac{8}{3}R - R = \frac{5}{3}R$$

$$\therefore \text{Percentage departure} = \frac{5R/3}{8R/3} \times 100$$

$$= 62.5\%$$

**Example 17.8** One mole of a gas, stored at 273 K, occupies a volume of 0.55 liter. Assuming it to be a Van der Waals' gas, calculate the pressure it exerts. Given:  $a = 0.37 \text{ Nm}^2 \text{ mol}^{-2}$ ,  $b = 43 \times 10^{-6} \text{ m}^3 \text{ mol}^{-1}$  and  $R = 8.31$

$\text{J mol}^{-1} \text{K}^{-1}$ .

**Soln.**

$$\text{For a Van der Waals' gas, } P = \frac{RT}{V-b} - \frac{a}{V^2}$$

Substituting the values of  $a$ ,  $b$ ,  $V$ ,  $T$  and  $R$

$$\begin{aligned}
 P &= \frac{8.31 \times 273}{(550 - 43) \times 10^{-6}} - \frac{0.37}{(550 \times 10^{-6})^2} \\
 &= \frac{8.31 \times 273}{507 \times 10^{-6}} - \frac{0.37}{(550 \times 10^{-6})^2} \\
 &= (4.47 \times 10^6 - 1.22 \times 10^6) \\
 &= 3.25 \times 10^6 \text{ Nm}^{-2}.
 \end{aligned}$$

**Example 17.9.** Calculate the critical temperature of helium from the following data : critical pressure = 2.26 atm., critical density = 0.069 gm/cm<sup>3</sup>,  $R = 8.31 \text{ J mol}^{-1} \text{ K}^{-1}$ .

**Soln.**

From the relation  $\frac{RT_C}{P_C V_C} = \frac{8}{3}$

we have  $T_C = \frac{8}{3} \frac{P_C V_C}{R}$

Now  $P_C = 2.26 \text{ atm.} = 2.26 \times 1.013 \times 10^5 \text{ Nm}^{-2}$

$\rho_C = 0.069 \text{ gm/cm}^{-3} = 69 \text{ Kg/m}^3$

$\therefore$  Critical volume for 1Kg of helium  
 $= (1/69) \text{ m}^3$ .

$\therefore V_C = (4/69) \text{ m}^3 \text{ Kmol}^{-1}$

and  $R = 8.31 \times 10^3 \text{ J K mol}^{-1} \text{ K}^{-1}$

$$\begin{aligned}
 \therefore T_C &= \frac{8 \times 2.26 \times 1.01 \times 10^5 \times (4/69)}{3 \times 8.31 \times 10^3} \\
 &= 4.26 \text{ K.}
 \end{aligned}$$

**Example 17.10.** The critical temperature and pressure of organ are  $-122^\circ\text{C}$  and 48 atmos respectively. Calculate the radius of an organ atom.

**Soln:** The Van der Waals' constant,

$$b = \frac{RT_c}{8P_c} = \frac{8.31 \times 151}{8 \times 48 \times 1.013 \times 10^5}$$

$$= 3.22 \times 10^{-5} \text{ m}^3 \text{ mol}^{-1}$$

But  $b = 4 \times$  volume occupied by all the molecules of 1 mole of gas (argon)

$$= 4 \times \frac{4}{3} \pi r^3 \times N_A$$

where  $r$  is the radius of argon atom and  $N_A =$  Avogadro number.

$$\therefore r^3 = \frac{3b}{16\pi N_A} = \frac{3 \times 3.22 \times 10^{-5}}{16 \times 3.14 \times 6.02 \times 10^{23}}$$

$$= 3.20 \times 10^{-30} \text{ m}^3$$

$$\therefore r = (3.20 \times 10^{-30})^{1/3} \text{ m}$$

$$= 1.475 \times 10^{-10} \text{ m}$$

$$= 1.475 \text{ \AA}$$

### EXERCISES

- [1] Describe Andrews' experiments on liquefaction of carbon dioxide. State and explain the results obtained by him. Hence show that the liquid and gaseous states are only distant stages of a long series of continuous changes.
- [2] "There exists a state of continuity of the liquid and gaseous states." Discuss Andrews' experimental isothermals for carbon dioxide and justify the above statement.
- [3] Deduce and explain Van der Waals' equation for a real gas.
- [4] Deduce Van der Waals' equation for a real gas. How far does this equation afford a satisfactory explanation of Andrews' experimental isothermals for carbon dioxide?
- [5] Discuss the experimental results of Andrews' on liquefaction of carbon dioxide. How far can they be explained from the standpoint of Van der Waals' equation?



- [6] Briefly describe the considerations which led Van der Waals' to his gas equation  $(P + \frac{a}{V^2})(V-b) = RT$ . Discuss how far this equation is in keeping with experimental facts.
- [7] What are the critical constants of a gas? Obtain expressions for the critical pressure and critical temperature in terms of the Van der Waals' constants ( $a$  and  $b$ ) and  $R$ .
- [8] What is meant by the critical temperature, critical pressure and critical volume of a gas? How can these quantities be determined? Discuss the behavior of a gas near the critical point.
- [9] Starting from Van der Waals' equation of state of a gas, derive the reduced equation of state of the gas. Show that if two gases have the same reduced pressure and volume, they also have the same reduced temperature.
- [10] Deduce Van der Waals' equation for a real gas and justify the different constants in it. What are the main defects of the equation?
- [11] Write short notes on:
- Andrews' experimental isothermals on carbon dioxide
  - Van der Waals' equation
  - Critical constants of a gas
  - Law of corresponding states
  - Reduced equation of state of a gas.
- [12] Given that the critical pressure, temperature and volume of a gas are 73 atmos,  $31^\circ\text{C}$  and 0.2066 of its volume at  $0^\circ\text{C}$  and 1 atm. respectively, calculate the constants of the Van der Waals' equation. ( $a = 24,230$  atms.,  $b = 3.505$  c.c./gm)
- [13] Calculate the values Van der Waals' equation for carbon dioxide, assuming the critical temperature to be  $31.1^\circ\text{C}$  and the critical pressure to be 73 atmospheres. Given,  $R = 0.003663$ , density of mercury = 13.6,  $g = 981$ ) ( $a = 70075 \times 10^{-9}$ ,  $b = 1.88 \times 10^{-9}$  c.c)
- [14] Calculate the values of the constants  $a$  and  $b$  of Van der Waals' equation in case of helium. For helium,  $T_c = -267.9^\circ\text{C}$ ,  $P_c = 2.26$  atms.,  $\rho_c = 0.0693$  gm/c.c. Molecular weight of helium = 4 and  $R = 8.2 \times 10^7$  ergs /K. ( $a = 4.81$ ,  $b = 1.41 \times 10^3$ )

- [15] Calculate the critical temperature and critical pressure of helium from the given data:  $a = 6.15 \times 10^{-5}$ ,  $b = 9.95 \times 10^{-4}$  per gm mole where units of pressure is the atmosphere and the unit of volume is the volume of the gas at N.T.P. ( $-268^{\circ}\text{C}$ ; 2.8 atmos)
- [16] Find the value of the critical temperature of  $\text{CO}_2$  assuming it to obey Van der Waals' equation. Van der Waals' constants for  $\text{CO}_2$  are :  $a = 8.74 \times 10^{-3}$  and  $b = 2.3 \times 10^{-3}$  where unit of pressure is atmosphere and the unit volume is the volume at N.T.P. ( $34.4^{\circ}\text{C}$ )
- [17] Calculate the values of Van der Waals' constants  $a$  and  $b$  for  $\text{CO}_2$  taking critical temperature =  $31.1^{\circ}\text{C}$  and critical pressure = 73 atmos,  $R = 3.663 \times 10^{-3}$ , density of mercury = 13.6 gm/cc and  $g = 981 \text{ cm/sec}^2$ . ( $a = 7.075 \times 10^{-9}$ ;  $b = 1.88 \times 10^{-9} \text{ c.c.}$ )
- [18] Calculate the critical temperature of  $\text{CO}_2$  for which the Van der Waals' constants are :  $a = 7.17 \times 10^{-3}$  and  $b = 1.91 \times 10^{-3}$ . Units of pressure and volume are atmosphere and gm.mol. of gas at N.T.P. ( $302 \text{ K}$ )
- [19] The critical temperature of  $\text{CO}_2$  is  $31^{\circ}\text{C}$  and the critical pressure is 73 atmos. Assuming that  $\text{CO}_2$  obeys Van der Waals' equation, estimate the diameter of its molecule. [ $3.23 \times 10^{-10} \text{ A}$ ]
- [20] For helium gas,  $b = 23.4 \text{ c.c./gm. mol}$ . Given the Avogadro's number  $N_A = 6.25 \times 10^{23}$ , calculate the diameter of the helium atom.



## CHAPTER XVIII

## TRANSMISSION OF HEAT

*Modes of transmission of heat – Thermal conductivity – Temperature gradient – Thermal resistance – Thermal diffusivity or thermometric conductivity – Rectilinear flow of heat along a bar – Comparison of thermal conductivities of different materials – Determination of thermal conductivity – Conduction of heat through composite blocks – Dimension of K – Solved problems – Exercises.*

## 18.1 Modes of transmission of heat

There are three distinct modes by which heat may be transferred or propagated from one place to another place. These are (i) conduction (ii) convection and (iii) radiation.

(i) **Conduction** : When a metal rod is heated at one end, the other end also becomes hot sooner or later. Thus heat has been transmitted from the hot end to the cold end. Such process of heat transference is termed conduction. For conduction to take place a material medium is necessary but there is no bodily motion of the material particles in the process. When heat is applied to one end of a metal rod, the molecules of the rod there vibrate more vigorously about their mean positions of rest and transfer the energy to the adjacent molecules by collision. These adjacent molecules, thus excited, transfer energy in similar fashion to the next layer of molecules and so on. Thus *conduction may be defined as the mode of transfer of heat in which heat energy travels from particle to particle in the direction of decreasing temperature without any bodily movement of the material particles from their normal positions.*

(ii) **Convection** : Convection is the mode of heat transference in which the material particles conveying the heat are themselves carried from one place to another place until the whole mass of the substance becomes uniformly heated. Transmission of heat in liquids and gases takes place in this manner by the actual motion of the heated particles.

(iii) **Radiation** : Transmission of heat either by conduction or convection requires a material medium. But there is yet another



mode of transference of heat in which no material medium is needed. *Radiation is that mode of transmission of heat in which the heat energy travels from the source of heat to its recipient without any material medium taking part in it.*

## 18.2 Thermal conductivity

Let a metal bar of uniform cross-section be heated at one end by steam, the other end being maintained at a lower temperature, say room temperature. Heat will flow or be conducted from the hotter to the colder end. This will be utilized in heating the bar, some portion, of course, being lost by radiation from the surfaces. Soon a stage will be reached when the metal bar will no longer utilize any heat for its warming; all the heat received from the source being lost to the surroundings by radiation. Thus *heat flowing in will be equal to the heat flowing out*. This state is known as the *steady state* of the metal bar, when temperature at every point of it shows a constant value, which gradually decreases from the hot to the cold end.

Let us consider a thin slice of the bar with parallel faces such that the direction of the flow of heat is normal to the faces (Fig. 18.1). There will be a gradual fall in temperature between the two faces. Let the left hand face of the slice be at a higher temperature  $\theta_1$ , the right hand face being at the lower temperature  $\theta_2$ . *It will be further assumed that there is no loss of heat from the side of the slab*. Let  $A$  and  $x$  be the area of cross-section and the thickness of the slab respectively. Then the amount of heat  $Q$  that flows from one side to the other in time  $t$  will depend,

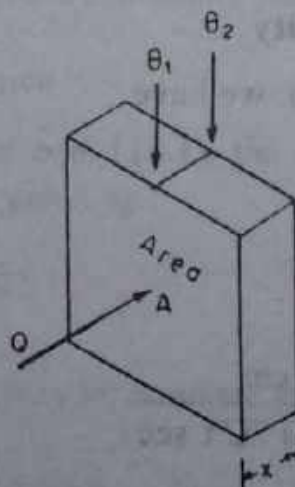


Fig. 18.1

- (i) directly on the face area  $A$
  - (ii) directly on the temperature difference,  $(\theta_1 - \theta_2)$
  - (iii) directly on time  $t$
  - (iv) inversely on the thickness of the slab
- and (v) of course on the nature of the material (some substances are better conductor of heat than others).

Hence

$$Q \propto \frac{A (\theta_1 - \theta_2) t}{x}$$

$$\text{or, } Q = K \frac{A (\theta_1 - \theta_2) t}{x} \quad (18.1)$$

where  $K$  is a constant depending on the nature of the material and is known as the *coefficient of thermal conductivity* or *coefficient of heat conduction* or simply as *thermal conductivity*.

If  $A = 1$  sq. cm,  $\theta_1 - \theta_2 = 1^\circ\text{C}$ ,

$x = 1$  cm and  $t = 1$  second in eqn. (18.1), then  $Q = K$ .

Hence *thermal conductivity of a material may be defined as the quantity of heat that flows in one second through a cm-cube of the material from the hot face to the cold face when there is a steady temperature difference of  $1^\circ\text{C}$  between the faces.*

#### units of thermal conductivity

Rearranging eqn. (18.1), we have

$$K = \frac{Q x}{A (\theta_1 - \theta_2) t}$$

Employing C.G.S. unit

$$\begin{aligned} K &= \frac{Q \text{ cal. } x \text{ cm}}{A \text{ cm}^2 (\theta_1 - \theta_2) ^\circ\text{C } t \text{ sec}} \\ &= \frac{Q x}{A (\theta_1 - \theta_2) t} \text{ cal. cm}^{-1} ^\circ\text{C}^{-1} \text{ sec}^{-1} \end{aligned}$$



In M.K.S. unit  $Q$  is in kilocalories,  $x$  in metres,  $A$  in  $\text{metre}^2$ ,  $(\theta_1 - \theta_2)$  in  $^\circ\text{C}$  and  $t$  in secs.; then

$$K = \frac{Q x}{A (\theta_1 - \theta_2) t} \text{ Kcal. m}^{-1} \text{ } ^\circ\text{C}^{-1} \text{ sec}^{-1}$$

It is expressed in SI units in watt per m per  $^\circ\text{C}$ .

### 18.3 Temperature gradient

The term  $\frac{\theta_1 - \theta_2}{x}$ , which represents the fall of temperature with distance in the direction of heat flow, is known as *temperature gradient*. Let the thickness of the slab be imagined to be diminished indefinitely such that the limiting value of  $\frac{\theta_1 - \theta_2}{x}$  is  $\frac{d\theta}{dx}$ . Indicating the fact that the temperature decreases as  $x$  increases by a negative sign, the temperature gradient becomes  $-\frac{d\theta}{dx}$ , eqn. (18.1) may then be written as

$$Q = -KA \frac{d\theta}{dx} t$$

$$\text{or, } K = - \frac{Q}{A \cdot \frac{d\theta}{dx} \cdot t}$$

Hence, the unit of thermal conductivity may also be written as *calorie per second per square centimetre per unit temperature gradient*.

### 18.4 Thermal resistance

Putting  $t = 1$  sec in eqn. (18.1), the rate of flow of heat or *heat current in cal. per sec*, is given by

$$q = \frac{KA(\theta_1 - \theta_2)}{x}$$

The above equation may be arranged as

$$q = \frac{(\theta_1 - \theta_2)}{x / KA} \text{ cal/sec} \quad (18.3)$$



Electric current in a circuit is given by

$$= \frac{\text{potential difference between two points}}{\text{resistance between these two points}}$$

$$= \frac{V}{R}$$

$$\text{Now } R = \rho \cdot \frac{l}{A} = \frac{l}{A\sigma}$$

where  $\sigma = \frac{1}{\rho}$  is the electric conductivity and  $l$  and  $A$  are the length and cross-section of the wire between the two points respectively.

Hence,

$$i = \frac{V}{l/\sigma A} \quad (18.4)$$

Comparing eqns. (18.3) and (18.4), it can be seen that the temperature difference  $(\theta_1 - \theta_2)$  corresponds to the potential difference  $V$  and  $x/AK$  corresponds to the electrical resistance. Thus in analogy with electrical resistance, the quantity  $x/AK$  is called the *thermal resistance*.

$\therefore$  heat current

$$= \frac{\text{temperature difference}}{\text{thermal resistance}}$$

$$= \frac{(\theta_1 - \theta_2)}{x/AK}$$

If in an electrical circuit, there are two or three resistances in series, then current is given by

$$i = \frac{V}{R_1 + R_2 + R_3}$$

$$= \frac{V}{(l_1/A_1\sigma_1) + (l_2/A_2\sigma_2) + (l_3/A_3\sigma_3)}$$

$$= \frac{V}{\sum l/A\sigma}$$

Similarly, if heat is conducted through two or three slabs of different materials in series, then equation for heat current becomes

$$q = \frac{\theta_1 - \theta_2}{(x_1 / A_1 K_1) + (x_2 / A_2 K_2) + (x_3 / A_3 K_3)}$$

$$= \frac{\theta_1 - \theta_2}{\sum x / AK}$$

where  $\theta_1$  and  $\theta_2$  are the temperatures of the outside surfaces.

### 18.5 Thermal diffusivity or thermometric conductivity

When a metal bar is heated, there is, at first, an unsteady flow of heat before steady state sets in. During this unsteady state, the rate at which the metal bar gets heated depends not only on the thermal conductivity ( $K$ ) of the material but also on its *thermal capacity per unit volume*. As the thermal capacity is different for different materials, the rate at which different materials gets heated when subjected to the same source of heat, is also different. The rate at which the temperature of any part of the bar changes is determined by a quantity, called *thermal diffusivity* by Lord Kelvin and *thermo-metric conductivity* by Maxwell. If  $s$  denotes the specific heat and  $\rho$  the density (mass per unit volume) of the material of the bar, then the thermal capacity per unit volume of the material is  $\rho s$ . The thermal diffusivity ( $h$ ) of the material is then given by

$$h = \frac{\text{thermal conductivity}}{\text{thermal capacity per unit volume}}$$

$$= \frac{K}{\rho s}$$

Thus the coefficient of thermal conductivity of a body divided by its thermal capacity per unit volume is called its diffusivity or thermometric conductivity.

*During unsteady state rate of flow of heat is proportional to the thermal diffusivity. Once the steady state is reached, heat flow is determined by thermal conductivity alone.*



### 18.6 Rectilinear flow of heat along a bar

Let us suppose that a long metal bar of uniform cross-section is heated steadily at one end M (Fig. 18.2) and the bar is so long that the other end N may be regarded to be at the same temperature as that of the surroundings.

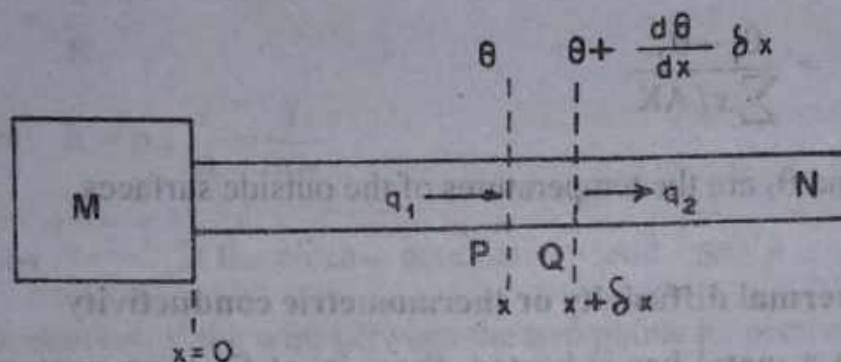


Fig. 18.2

Let the bar lie along the  $x$ -axis with the hot end at the origin ( $x = 0$ ). The length of the bar is supposed to be sufficiently large compared to its width and thickness so that *the sections perpendicular to the length of the bar at any point may be fairly taken as isothermal surfaces*. We shall now consider two such neighbouring planes at P and Q at distances  $x$  and  $x + \delta x$  from the hot end. Let  $\theta$  be the excess temperature of the layer at P over the surroundings, and  $\frac{d\theta}{dx}$  be the temperature gradient at the point. For the sake of convenience, we shall assume that  $\theta$  and  $x$  are increasing together. Then the excess temperature of the plane at Q over the surroundings will be  $\theta + \frac{d\theta}{dx} \cdot dx$  and the temperature gradient there is

$$\frac{d}{dx} \left( \theta + \frac{d\theta}{dx} \delta x \right).$$

Let  $A$  and  $K$  be the area of cross-section of the bar and the thermal conductivity of the material of the bar respectively. Then the rate of heat flow into the layer at P

$$q_1 = -KA \cdot \frac{d\theta}{dx}$$

and the rate of heat flowing out of the layer at Q



$$q_2 = -KA \frac{d}{dx} \left( \theta + \frac{d\theta}{dx} \delta x \right)$$

Hence, the gain of heat by the layer

$$\begin{aligned} q_1 - q_2 &= \left[ -KA \frac{d\theta}{dx} \right] - \left[ -KA \frac{d}{dx} \left( \theta + \frac{d\theta}{dx} \delta x \right) \right] \\ &= KA \frac{d^2\theta}{dx^2} \delta x \end{aligned}$$

Before the steady state is reached, any layer like P receives heat from the preceding layer by conduction. A part of this heat is absorbed in raising the temperature of the layer, a part is radiated from the surface and the rest is conducted to the next layer.

Thus out of the amount of heat gained by the layer a part is absorbed and the rest is radiated out of the surface.

If  $\rho$  and  $s$  are the density and specific heat of the material of the bar, and if  $\frac{d\theta}{dt}$  is the rate of increase of temperature of the material of the bar, then the quantity of heat required per second to raise the temperature of the element at the above rate

$$\begin{aligned} &= \text{mass} \times \text{specific heat} \times \text{rate of rise in temperature} \\ &= \text{volume} \times \text{density} \times \text{sp. ht.} \times \text{rate of rise in temperature} \\ &= A \cdot \delta x \times \rho \times s \times \frac{d\theta}{dt} \end{aligned}$$

By Newton's law of cooling, heat lost by radiation per second from the sides of the bar

$$\begin{aligned} &= \text{emissive power} \times \text{surface area} \times \text{average temperature excess over the surroundings} \\ &= E \cdot p \cdot \theta \end{aligned}$$

where

$E$  = emissive power or emissivity of the surface of the bar, defined as the quantity of heat radiated from the surface per second per unit area per unit temperature difference with the surroundings.

$p$  = the perimeter of the section of the bar,

$pdx$  = surface area of the section of the bar

and  $\theta$  = average excess temperature of the section of the bar over that of the surroundings.

Then

$$KA \frac{d^2\theta}{dx^2} \cdot \delta x = A \cdot \delta x \cdot \rho \cdot s \frac{d\theta}{dt} + E \cdot p \cdot \delta x \theta$$

$$\text{or, } \frac{d^2\theta}{dx^2} = \frac{\rho s}{K} \frac{d\theta}{dt} + \frac{Ep}{KA} \theta$$

$$\text{or, } \frac{d\theta}{dt} = \frac{K}{\rho s} \frac{d^2\theta}{dx^2} - \frac{Ep}{\rho As} \theta$$

$$\text{or, } \frac{d\theta}{dt} = h \cdot \frac{d^2\theta}{dx^2} - \mu \theta \quad (18.6)$$

where  $h = \frac{K}{\rho s}$  is the diffusivity

$$\text{and } \mu = \frac{Ep}{\rho As}$$

Eqn. (18.6) is the standard (Fourier) equation for one dimensional flow of heat in the variable state.

The loss of heat by radiation can be minimized (but it can never be eliminated) by *lagging* the bar with appropriate materials. Neglecting the heat lost by radiation from the sides of the bar,  $\mu = 0$  and eqn. (18.6) becomes

$$\frac{d\theta}{dt} = h \cdot \frac{d^2\theta}{dx^2} \quad (18.7)$$

**steady state**

When the steady state is reached there is no change in temperature with time, i.e.,  $\frac{d\theta}{dt} = 0$ . Eqn. (18.6), therefore, becomes

$$h \frac{d^2\theta}{dx^2} = \mu\theta$$

$$\text{or, } \frac{d^2\theta}{dx^2} = \frac{\mu}{h} \theta = m^2\theta \quad (18.8)$$

$$\text{where } m^2 = \frac{\mu}{h} = \frac{E_p}{\rho A s} \cdot \frac{\rho s}{K} = \frac{E_p}{AK}$$

Eqn. (18.8) represents the heat flow in the steady state when the heat loss due to radiation is not negligible.

To solve this equation let us assume a trial solution  $\theta = e^{nx}$ .

$$\text{Then } \frac{d\theta}{dx} = ne^{nx} \text{ and } \frac{d^2\theta}{dx^2} = n^2e^{nx}$$

Substituting this in the equation of steady heat flow (eqn. 18.8)

$$\frac{d^2\theta}{dx^2} = n^2e^{nx} = m^2e^{nx}$$

$$\text{or, } n^2 = m^2$$

$$\therefore n = \pm m$$

Therefore the complete solution is

$$\theta = Ae^{mx} + Be^{-mx}$$

where A and B are constants which can be evaluated from the boundary conditions.

Let the excess temperature over the surroundings at the hot end be  $\theta_0^\circ\text{C}$  and that at the cold end be zero (which is true when the bar is sufficiently long i.e.,  $x = \infty$ ).

Thus we have

$$\text{at the cold end, } x = \infty \theta = 0$$

$$\therefore 0 = Ae^{m\infty} + Be^{-m\infty} = Ae^{m\infty}$$

But  $e^{m\infty}$  is not equal to zero; hence



$$A = 0$$

Again at the hot end,  $x = 0$  and  $\theta = \theta_0$

$$\therefore \theta_0 = 0 + Be^0 = B$$

Hence the complete solution for the steady state heat flow becomes

$$\theta = \theta_0 e^{-mx} \quad (18.9)$$

The above equation gives the excess temperature at any point  $x$  along the bar over the surroundings, after the steady state is reached.

### 18.7 Comparison of thermal conductivities of different materials (Ingen Hausz experiment)

Conductivities of different metals can be compared by the Ingen Hausz experiment by applying the principle discussed above (Art. 18.6). In the experiment several rods of different metals of the same length and same cross-section are polished and coated with a thin uniform layer of wax to give the same surface conditions. All these metal rods are introduced into the holes on one side of a metal trough containing water (Fig. 18.3). The water is then heated by burner or by passing electric current through resistance coils immersed in it. Heat is conducted along each rod, therefore, increasing the temperature of the rods. After steady state is attained,

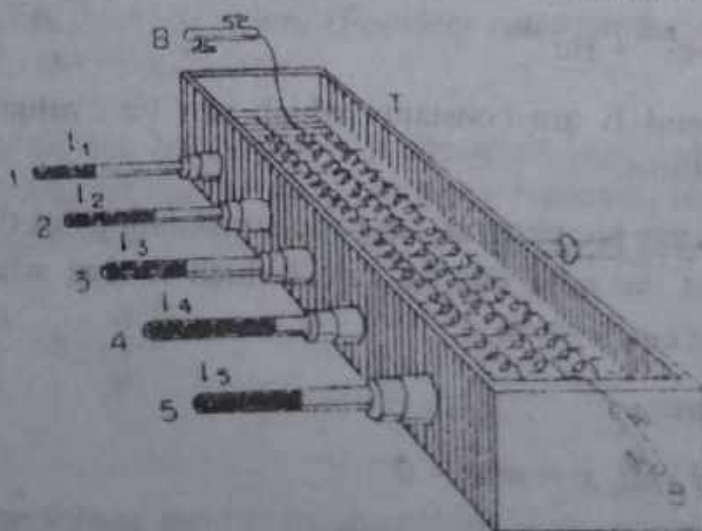


Fig. 18.3

wax will be found to have melted to different lengths in different rods depending upon the thermal conductivities of the materials of the rods. The temperature excess at points upto which wax has melted along the length of the rods must be same and is equal to the melting point of wax. Let  $l_1, l_2, l_3, \dots$  be the lengths upto which wax melts in different rods and  $K_1, K_2, K_3, \dots$  be the corresponding thermal conductivities of the material of these rods. Then if  $\theta_0$  and  $\theta_m$  be the temperature of the hot water bath and melting points of wax respectively, measured above the temperature of the surrounding medium, we have

$$\theta_m = \theta_0 e^{-m_1 l_1} = \theta_0 e^{-m_2 l_2} = \theta_0 e^{-m_3 l_3} = \dots$$

$$\text{or, } m_1 l_1 = m_2 l_2 = m_3 l_3 = \dots$$

As the rods are all similar,  $E, p$  and  $A$  are same for all of them. Hence from the relation

$$m = \sqrt{\frac{Ep}{KA}}, \text{ we have}$$

$$l_1 \sqrt{\frac{Ep}{K_1 A}} = l_2 \sqrt{\frac{Ep}{K_2 A}} = l_3 \sqrt{\frac{Ep}{K_3 A}} = \dots$$

$$\text{or, } \frac{l_1}{\sqrt{K_1}} = \frac{l_2}{\sqrt{K_2}} = \frac{l_3}{\sqrt{K_3}} = \dots$$

$$\text{or, } \frac{l_1^2}{K_1} = \frac{l_2^2}{K_2} = \frac{l_3^2}{K_3}$$

$$\text{or, } K_1 : K_2 : K_3 \dots = l_1^2 : l_2^2 : l_3^2$$

Hence the ratio of the conductivities can be determined. Considering any two rods, we get

$$\frac{K_1}{K_2} = \frac{l_1^2}{l_2^2} \quad (18.10)$$

Therefore, if  $K_1$  is known,  $K_2$  can be determined.

### 18.8 Determination of thermal conductivity

There is no single method for determining the thermal conductivity of different materials. The actual methods are different depending on whether the material is a good conductor, a partial conductor or a bad conductor. The methods also differ depending on whether the material is supplied in the form of a solid bar, or a tube, or in the form of powder.

#### (i) determination of the thermal conductivity of a good conductor (Searle's method)

G.F.C. Searle of Cambridge university devised the following method for determining the thermal conductivity of a good conductor like copper, brass, etc., supplied in the form of a bar or a rod. The experimental arrangement of the method, otherwise known as *guard-ring method* is shown in Fig. 18.4.

Searle's apparatus consists of a solid cylinder AB of the experimental material about 20 cm in length and 5 cm in diameter. The end A is soldered to a cylindrical chamber C through which steam is passed. A copper pipe is coiled near the end B of the rod. Water at room temperature enters through one end of the pipe and comes out through

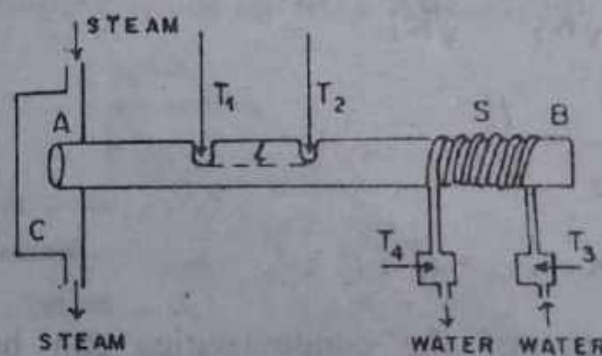


Fig. 18.4

the other end at a higher temperature. Two suitably placed thermometers  $T_3$  and  $T_4$  record the temperatures of incoming and outgoing water respectively. Two other thermometers  $T_1$  and  $T_2$  are



placed with their bulbs inside two cavities in the central region of the rod at a known distance (about 9 cms) apart. The cavities are filled with mercury to ensure good contact.

The cylindrical rod whose thermal conductivity is to be determined is wrapped with some non-conducting material such as cotton or flannel and is fitted in a wooden box lined with some insulating material and with provisions to fit the thermometers and the inlet and outlet tubes.

Steam is passed into the chamber and a gradual rise of temperature is shown by the four thermometers. After some time a steady state is reached when the respective reading of all the four thermometers become stationary. Under this condition heat flowing in is equal to the heat flowing out. Heat transmitted from the hot end A to the cooler end B is absorbed by water circulating through the spiral tube S.

Let  $\theta_1$ ,  $\theta_2$ ,  $\theta_3$  and  $\theta_4$  be the readings of the four thermometers  $T_1$ ,  $T_2$ ,  $T_3$  and  $T_4$  respectively. If  $l$  represents the distance between the thermometers  $T_1$  and  $T_2$  and  $d$  is the mean diameter of the rod, then the rate of flow of heat in the steady state is given by

$$\frac{dQ}{dt} = \frac{KA(\theta_1 - \theta_2)}{l}$$

where  $K$  is the thermal conductivity of the material of the rod and  $A = \pi \left(\frac{d}{2}\right)^2$  is the area of cross-section of the rod.

If in the steady state the mass of hot water collected per second be  $m$  and  $\theta_3$  and  $\theta_4$  denote the temperatures of the incoming and outgoing water as recorded by the thermometers  $T_3$  and  $T_4$  respectively, then the heat absorbed (or carried away) by water per second

$$\begin{aligned} &= m.s. (\theta_4 - \theta_3) \\ &= m. (\theta_4 - \theta_3) \quad (s = 1 \text{ for water}). \end{aligned}$$

Thus

$$\frac{KA(\theta_1 - \theta_2)}{l} = m. (\theta_4 - \theta_3)$$

$$\text{or, } K = \frac{m(\theta_4 - \theta_3) l}{A(\theta_1 - \theta_2)} \quad (18.11)$$

Since  $m$ ,  $l$ ,  $A$ ,  $\theta_1$ ,  $\theta_2$ ,  $\theta_3$  and  $\theta_4$  are known, the thermal conductivity  $K$  can be calculated.

*Note :* If the specimen be in the form of a rectangular bar its thickness  $b$  and width  $a$  are to be measured. The product  $a \times b$  is to be taken for the area of cross-section of the bar.

**(ii) determination of the thermal conductivity of a bad conductor (Lees and Chorlton's method).**

Wood, cork, cardboard, rubber, glass, clay, asbestos, ebonite, mica and all other non-metallic bodies are very bad conductors of heat. In determining the conductivity of such a poor conductor one must remember that a thin layer or slab of the material must be used. The difficulty then arises in maintaining the face at uniform temperature and in measuring that temperature. Lees and Chorlton overcame this difficulty by placing a slab of a good conductor, such as brass or copper, of exactly the same diameter as the specimen under test on each side of it (the poor conductor).

Fig. 18.5 shows the arrangement due to Lees and Chorlton. A brass disc  $A$  of about 10 cm in diameter is supported by means of strings from a large ring on retort stand. The experimental material in the form of a thin circular slab ( $S$ ) of uniform thickness and having the same diameter as  $A$ , is placed on it ( $A$ ). A steam chamber  $C$  is placed on the slab  $S$ . The steam chamber consists of two parts – the top part is a hollow brass cylinder  $C$  of the same diameter as  $S$ , having side tubes for the entry and exit of steam while the bottom part  $B$  is a thick metal plate having the same surface area as that of  $A$ . The surfaces of  $A$  and  $B$  are nickel plated to obtain uniform and good emissivity. Two thermometers  $T_1$  and  $T_2$  are introduced into two holes drilled in the base  $B$  and in  $A$ . The whole system is suspended from a retort stand by strings attached to the lowest brass block  $A$ .



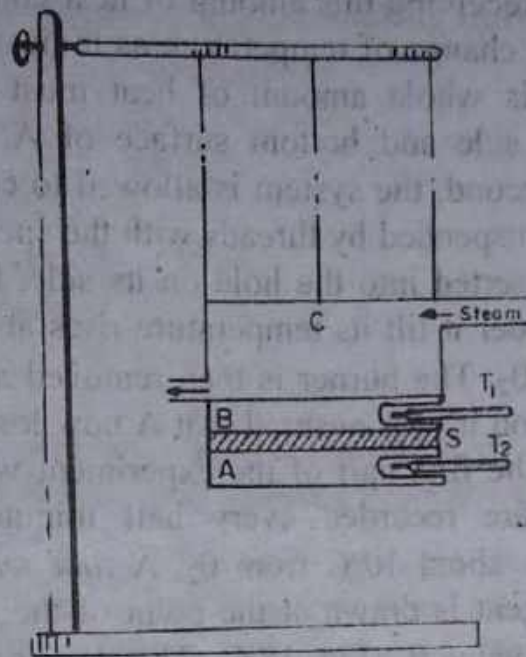


Fig. 18.5

When steam is passed through C, A is warmed up by the heat conducted by S. (Since the slab is sufficiently thin, the loss of heat from the edge of the slab is small enough to be neglected in calculation. It can, therefore, be safely assumed that the flow of heat across S will be vertically downwards). After sometime a steady state is attained; the readings of the thermometers  $T_1$  and  $T_2$  become steady and they are noted. At this stage the rate of heat loss by the block B is equal to the rate of flow of heat through the specimen S. The rate of flow of heat through the specimen in this steady state is given by

$$\frac{dQ}{dt} = \frac{KA(\theta_1 - \theta_2)}{d}$$

where

$\theta_1$  = temperature of B in steady state as recorded by  $T_1$

$\theta_2$  = temperature of A in steady state as recorded by  $T_2$

A = area of cross-section of the slab S

K = thermal conductivity of the slab S

d = thickness of the slab.



The disc A is receiving this amount of heat continuously; but there is no corresponding change of temperature as indicated by  $T_2$ . Hence, in the steady state this whole amount of heat must have been lost by radiation from the side and bottom surface of A. To determine this radiation loss per second, the system is allowed to cool and dismantled. The disc A is then suspended by threads with the specimen S over it and a thermometer is inserted into the hole on its side. It is then heated by moving a burner under it till its temperature rises about  $10^\circ\text{C}$  above the steady temperature  $\theta_2$ . The burner is then removed and A is allowed to cool. By keeping S on it, it is ensured that A now loses heat in the same surroundings as in the first part of the experiment when it gained heat. The temperatures are recorded every half minute or so until the temperature falls by about  $10^\circ\text{C}$  from  $\theta_2$ . A *time-temperature* curve is now plotted. A tangent is drawn at the point of the curve at which the temperature has the value  $\theta_2$  (Fig. 18.6). The slope of this tangent given the rate of cooling,  $\frac{d\theta}{dt}$  at  $\theta_2$ . If  $m$  and  $s$  be the mass and specific heat of A, then the heat lost per second from A is

$$\frac{dQ}{dt} = ms \frac{d\theta}{dt}$$

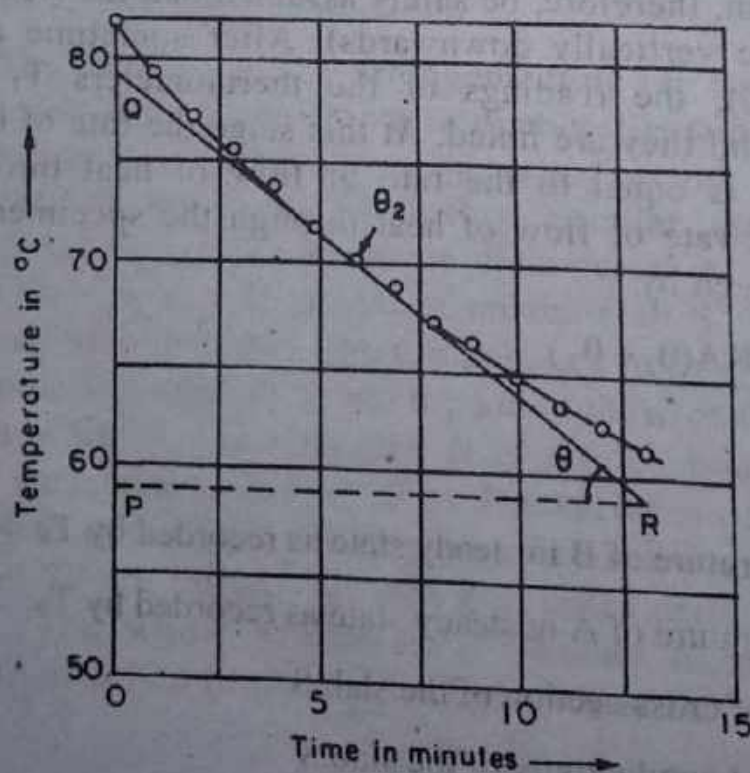


Fig. 18.6

Since the conditions of the surrounding during steady state and during the cooling are the same, we have

$$\frac{KA(\theta_1 - \theta_2)}{d} = ms \frac{d\theta}{dt}$$

or,  $K = \frac{ms \frac{d\theta}{dt} \cdot d}{A(\theta_1 - \theta_2)} \quad (18.12)$

**(iii) determination of thermal conductivity by the method of radial flow of heat (Nusselt's spherical shell method)**

This method has been found suitable for determination of thermal conductivities of those substances which are available in powdered or fibrous form, such as sand, asbestos, clay, cork, charcoal, etc. The experimental arrangement is shown in Fig. 18.7. A and B are two concentric hollow spheres of copper of radii  $r_1$  and  $r_2$  respectively. The spheres are split into two halves, so that they can be put together or separated when required. The source of heat, generally an electrically heated coil, is placed at the centre of the spheres. As the heat will flow radially from the surfaces of the spheres in all directions, concentric isothermal surface can be described about the common centre of the heater and the spheres.

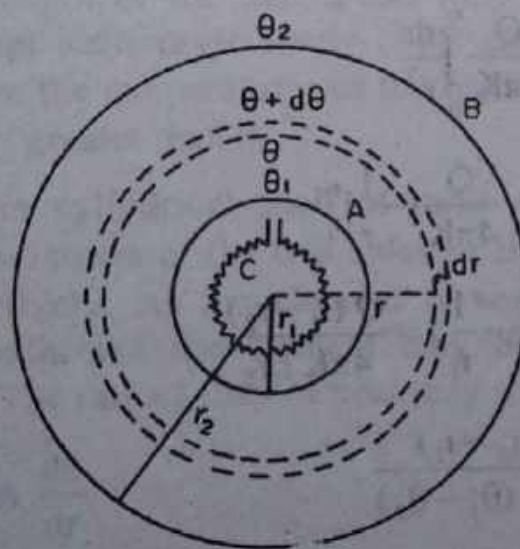


Fig. 18.7



Let the substance whose thermal conductivity is to be determined be placed in the space between A and B. When steady state is attained, the temperatures at different points in the substance under test are measured by means of thermo-couples suitably placed at those points. Let  $\theta_1$  and  $\theta_2$  be the steady temperatures on the surfaces of A and B respectively.

Consider an element of shell of thickness  $dr$  at a distance  $r$  from the centre. Let  $\theta$  and  $\theta + d\theta$  be the temperatures of the inner and outer surfaces of this shell respectively. Then the area through which there is a radial flow of heat  $= 4\pi r^2$  and the amount of heat that flows across it per second is

$$Q = -KA \frac{d\theta}{dr} = -K \cdot 4\pi r^2 \cdot \frac{d\theta}{dr}$$

where  $K$  is the thermal conductivity of the substance.

Rearranging

$$d\theta = -\frac{Q}{4\pi K} \cdot \frac{dr}{r^2}$$

In the steady state  $Q$  remains constant for all such shells of different radii. Therefore, integrating for the whole thickness of the material, we get,

$$\int_{\theta_1}^{\theta_2} d\theta = -\frac{Q}{4\pi K} \int_{r_1}^{r_2} \frac{dr}{r^2}$$

$$\begin{aligned} \text{or, } \theta_2 - \theta_1 &= -\frac{Q}{4\pi K} \left[ -\frac{1}{r} \right]_{r_1}^{r_2} \\ &= \frac{Q}{4\pi K} \left( \frac{1}{r_2} - \frac{1}{r_1} \right) = \frac{Q (r_1 - r_2)}{4\pi K r_1 r_2} \end{aligned}$$

$$\therefore K = \frac{Q (r_2 - r_1)}{4\pi r_1 r_2 (\theta_1 - \theta_2)} \quad (18.13)$$

*(iv) determination of thermal conductivity by radial flow of heat through a cylindrical shell*



When some hot liquid flows through a cylindrical tube, heat flows in all directions along the radius of the section of the cylinder, through the wall of the cylinder, the inner wall being at higher temperature than the outer wall (Fig. 18.8). Such a flow of heat through the wall of a tube through which hot liquid is passing is called *radial flow of heat*.

Compared to the other type of radial flow as discussed in the previous method, here we have two coaxial cylinders instead of two spheres. The hot liquid may be replaced by an electrically heated coil lying along the axis of the cylinder.

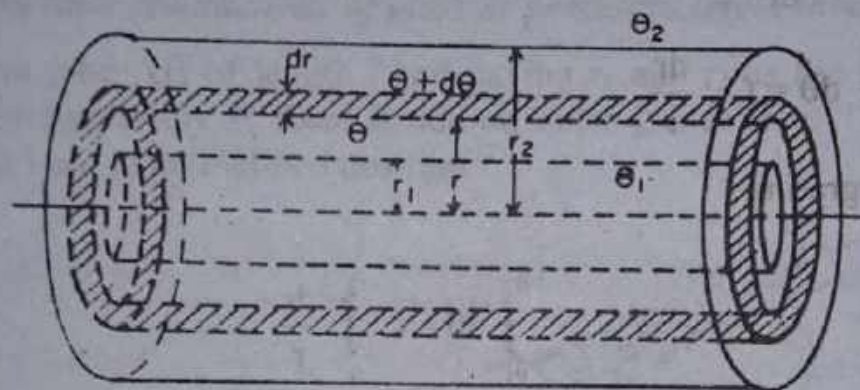


Fig. 18.8

Let  $l$  be the length of the cylindrical tube, and  $r_1$  and  $r_2$  its internal and external radii respectively (Fig. 18.8). In the steady state let  $\theta_1$  and  $\theta_2$  be the temperatures of the inside and outside wall of the tube,  $\theta_1$  being greater than  $\theta_2$ .

Consider a thin cylindrical shell of radii  $r$  and  $r + dr$ , the corresponding temperatures of the two sides of the element being  $\theta$  and  $\theta - d\theta$  respectively. As the element is very thin heat flows normally to the walls and the element may be considered as a parallel sided slab. The rate of flow of heat may then be given by

$$\frac{dQ}{dt} = -KA \frac{d\theta}{dr}$$

where  $K$  = thermal conductivity of the material of the tube,  $A$  = area of the walls of the element considered ( $= 2\pi rl$ ).

$$\begin{aligned}\therefore \frac{dQ}{dt} &= -K \cdot 2\pi r l \cdot \frac{d\theta}{dr} \\ &= -2\pi K l \cdot r \cdot \frac{d\theta}{dr} \quad (18.14)\end{aligned}$$

In the steady state  $\frac{dQ}{dt}$  is the same for all values of  $r$ . This means that since  $K$  and  $l$  and constant,  $r \cdot \frac{d\theta}{dr}$  is also constant.

Let  $r \cdot \frac{d\theta}{dr} = C$ , where  $C$  is a constant.

$$\text{or, } d\theta = C \cdot \frac{dr}{r}$$

Integrating

$$\int_{\theta_1}^{\theta_2} d\theta = C \int_{r_1}^{r_2} \frac{dr}{r}$$

$$\text{or, } \theta_2 - \theta_1 = C \log_e \frac{r_2}{r_1}$$

$$\text{Hence } \theta_1 - \theta_2 = C \log_e \frac{r_1}{r_2}$$

$$\text{or, } C = \frac{\theta_1 - \theta_2}{\log_e (r_1 / r_2)}$$

Putting this value of  $C$ , i.e.,  $r \frac{d\theta}{dr}$  in the relation (18.14), we get

$$\frac{dQ}{dt} = -2\pi K l \frac{\theta_1 - \theta_2}{\log_e r_1 - \log_e r_2}$$

$$= 2\pi K l \frac{\theta_1 - \theta_2}{\log_e r_2 - \log_e r_1}$$



$$\text{or, } K = \frac{dQ}{dt} \cdot \frac{\log_e(r_2/r_1)}{2\pi l (\theta_1 - \theta_2)} \quad (18.15)$$

By determining rate of flow of heat  $\frac{dQ}{dt}$  experimentally, the thermal conductivity  $K$  can be determined from a knowledge of  $r_1$ ,  $r_2$ ,  $\theta_1$ ,  $\theta_2$  and  $l$ .

The method is generally adopted for determining the thermal conductivities of bad conductors like porcelain, glass, rubber, etc., which are available in the form of hollow cylindrical pipes or tubes.

**(v) thermal conductivity of glass or porcelain (rigid tube)**

A glass tube AB of length  $l$  and having  $r_1$  and  $r_2$  as the inner and outer radii respectively is fixed inside an outer jacket (Fig. 18.9). The apparatus is kept in an inclined position.

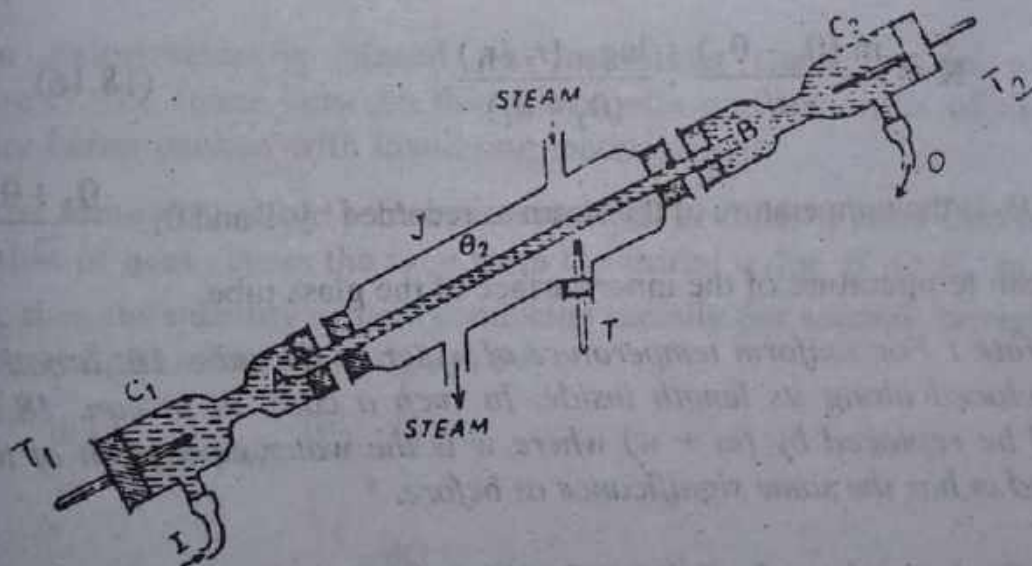


Fig. 18.9

Steam from a boiler (not shown) is allowed to pass continuously through the outer jacket  $J$  and the temperature of the flowing steam can be determined directly by the thermometer  $T$  which is inserted into the jacket. At the two ends of  $AB$  are fitted two chambers  $C_1$  and  $C_2$  —  $C_1$  being connected to an inlet tube  $I$  through which water is passed at a constant rate. This water flows through  $AB$  to pass out finally through the outlet tube  $O$  fitted to the chamber  $C_2$ . The temperatures of the



incoming and outgoing water can be recorded by two thermometers  $T_1$  and  $T_2$ , fitted into  $C_1$  and  $C_2$  respectively. The outgoing water from O can be collected into a beaker.

The rate of flow of water is adjusted with the help of a pinch cock (not shown) so that the two thermometers  $T_1$  and  $T_2$  show a temperature difference of say 5 to 10°C. When the two thermometers show constant temperature, water flowing through the glass tube for a known time  $t$  seconds is collected in a beaker. The water is weighed and the mass  $m$  of water that flows per second through AB is computed.

If the temperatures of the incoming and outgoing water as recorded by  $T_1$  and  $T_2$  be  $\theta_3$  and  $\theta_4$  respectively, then the constant rate of flow of heat per second is  $\frac{dQ}{dt} = ms(\theta_4 - \theta_3) = m(\theta_4 - \theta_3)$  ( $s = 1$ )

Substituting this value of  $\frac{dQ}{dt}$  in eqn. (18.15) for cylindrical heat flow, we get

$$K = \frac{m(\theta_4 - \theta_3)}{2\pi l} \cdot \frac{\log_e(r_2/r_1)}{(\theta_2 - \theta_1)} \quad (18.16)$$

where  $\theta_2$  is the temperature of the steam as recorded by T and  $\theta_1 = \frac{\theta_3 + \theta_4}{2}$ , the mean temperature of the inner surface of the glass tube.

*Note :* For uniform temperature of water in the tube AB, a coil is often placed along its length inside. In such a case,  $m$  in eqn. 18.16 should be replaced by  $(m + w)$  where  $w$  is the water equivalent of the coil and  $m$  has the same significance as before.

#### (vi) thermal conductivity of flexible tubes (e.g. India rubber)

A known quantity of water is taken in a calorimeter C (Fig. 18.10). A rubber tubing whose inner and outer radii are  $r_1$  and  $r_2$  respectively is taken and a measured length  $l$  of the tube is immersed into water of known mass  $m$  in the calorimeter of known water equivalent  $w$ . One end of the tube is connected to a boiler B where steam is continuously generated while the free end of the tube passes out of the calorimeter where condensed steam may be collected in a beaker A.

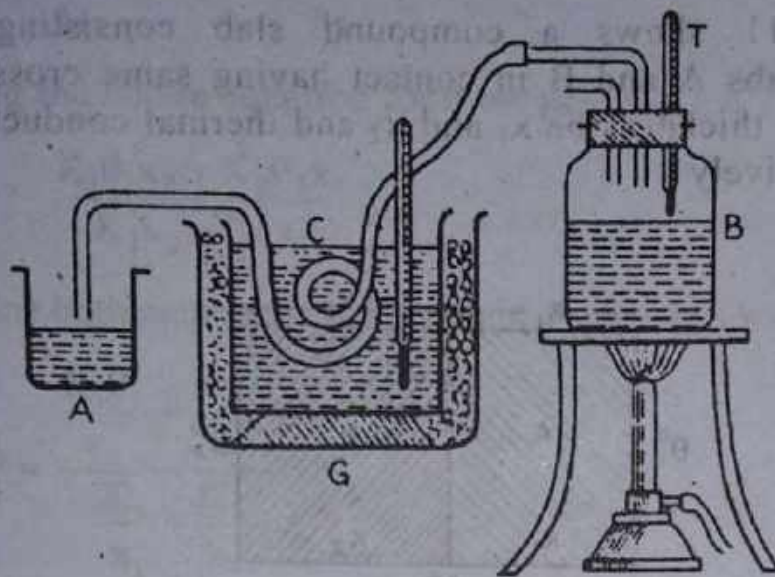


Fig. 18.10

The calorimeter is placed on insulating stand inside an enclosure G, the space between the calorimeter and the walls of the enclosure being packed with insulating material.

If the temperature of the calorimeter and its content rises due to conduction of heat across the tube from the initial value  $\theta'_1$  to  $\theta'_2$  in  $t$  seconds, then the quantity of heat conducted radially per second through the immersed part of the tube is given by

$$\frac{dQ}{dt} = (m + w) \frac{(\theta'_2 - \theta'_1)}{t}$$

Substituting this value of  $\frac{dQ}{dt}$  in eqn. (18.15), we get

$$K = \frac{(m + w) (\theta'_2 - \theta'_1)}{t} \cdot \frac{\ln (r_2 / r_1)}{2\pi l (\theta_2 - \frac{\theta'_1 + \theta'_2}{2})} \quad (18.17)$$

where  $\theta_2$  is the temperature of steam as recorded by thermometer T in the boiler and  $\theta_1 = (\theta'_1 + \theta'_2)/2$  the mean temperature of the outer surface of the tube.



### 18.9 Conduction of heat through composite blocks

Fig. 18.11 shows a compound slab consisting of two rectangular slabs A and B in contact having same cross-sectional area. Let their thickness be  $x_1$  and  $x_2$  and thermal conductivities  $K_1$  and  $K_2$  respectively.

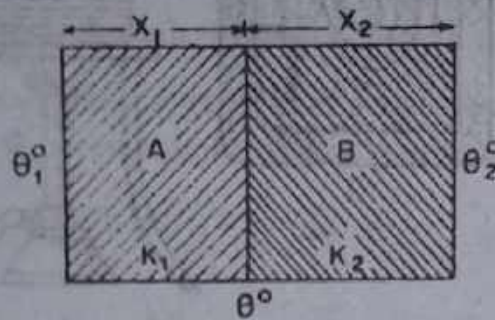


Fig. 18.11

Let the temperatures of the outer faces of A and B be  $\theta_1$  and  $\theta_2$  respectively,  $\theta_1$  being greater than  $\theta_2$ . Suppose the temperature of the interface (*i.e.*, surfaces in contact) in the steady state be  $\theta$ .

The rate of flow of heat through section A is given by

$$\frac{dQ_1}{dt} = \frac{K_1 A (\theta_1 - \theta)}{x_1} \quad (18.18)$$

and that through section B is

$$\frac{dQ_2}{dt} = \frac{K_2 A (\theta - \theta_2)}{x_2} \quad (18.19)$$

In the steady state, the same amount of heat flows through every section. Hence the rate of heat transfer through section B is the same as that through section A. Hence

$$\begin{aligned} \frac{dQ_1}{dt} &= \frac{dQ_2}{dt} \\ \text{or, } \frac{K_1 A (\theta_1 - \theta)}{x_1} &= \frac{K_2 A (\theta - \theta_2)}{x_2} \end{aligned}$$



$$\text{or, } \frac{K_1 (\theta_1 - \theta)}{x_1} = \frac{K_2 (\theta - \theta_2)}{x_2}$$

Solving the above equation for  $\theta$  we get

$$\theta = \frac{K_1 \theta_1 x_2 + K_2 \theta_2 x_1}{K_1 x_2 + K_2 x_1} \quad (18.20)$$

Dividing both numerator and denominator by  $x_1 x_2$ , we have

$$\theta = \frac{\frac{K_1 \theta_1}{x_1} + \frac{K_2 \theta_2}{x_2}}{\frac{K_1}{x_1} + \frac{K_2}{x_2}} \text{ } ^\circ\text{C} \quad (18.21)$$

Substituting the value of  $q$  as given by eqn. (18.20) in eqn. (18.18) or eqn. (18.17), we get

$$\begin{aligned} \frac{dQ}{dt} &= \frac{K_1 A (\theta_1 - \frac{K_1 \theta_1 x_2 + K_2 \theta_2 x_1}{K_1 x_2 + K_2 x_1})}{x_1} \\ &= \frac{K_1 K_2 A x_1 (\theta_1 - \theta_2)}{x_1 (K_1 x_2 + K_2 x_1)} \\ &= \frac{A (\theta_1 - \theta_2)}{\frac{x_1 K_1 x_2}{K_1 K_2 x_1} + \frac{K_2 x_1^2}{K_1 K_2 x_1}} \\ &= \frac{A (\theta_1 - \theta_2)}{\frac{x_2}{K_2} + \frac{x_1}{K_1}} \quad (18.22) \end{aligned}$$

$$\begin{aligned} &= \frac{\theta_1 - \theta_2}{\frac{x_1}{K_1 A} + \frac{x_2}{K_2 A}} \\ &= \frac{\theta_1 - \theta_2}{\sum \frac{x}{KA}} \quad (18.23) \end{aligned}$$

where the summation extends over all the slabs.

Thus we have,

*thermal current (rate of heat flow at steady state)*

$$= \frac{\text{temperature difference}}{\text{thermal resistance}}$$

where  $\sum \frac{x}{KA}$  denotes the thermal resistance of the composite slab.

**Note :** If the composite slab be replaced by a single slab of thickness  $(x_1 + x_2)$  of some material of conductivity  $K$ , so that the same rate of flow of heat is maintained when the exposed faces of this slab is maintained at  $\theta_1$  and  $\theta_2$  as before, then we have

$$\frac{dQ}{dt} = \frac{A (\theta_1 - \theta_2)}{(x_1 + x_2)/K} \quad (18.24)$$

From equations (18.22) and (18.24) it can be seen that

$$\frac{x_1 + x_2}{K} = \frac{x_1}{K_1} + \frac{x_2}{K_2}$$

Here  $K$  is called the *equivalent or overall thermal conductivity* of the composite slab having total thickness  $x_1 + x_2$ .

### 18.10 Wiedemann – Franz Law

Wiedemann and Franz observed that all good conductors of electricity are also good conductors of heat. Based on experimental results, they, in 1853, gave a purely empirical law that the ratio of the thermal to electrical conductivities at a particular temperature is the same for all metals. Lorentz extended it in 1872 and showed that the ratio is proportional to the absolute temperature. Combining the two :

*the ratio of the thermal to electrical conductivity for all metals has the same value at a given temperature and the ratio is directly proportional to the absolute temperature.* This statement has come to be known as the Wiedemann – Franz law.

Drude attempted to put forward an explanation of the above law on the basis of classical electron theory. He assumed that



(i) free electrons are responsible for both thermal and electrical conductivities. Inside a metal the free electrons behave like a gas molecule – the electron gas – moving freely in metal lattices.

(ii) the velocity distribution of the free electrons obey the Maxwellian distribution law and their energy obeys the law of equipartition of energy.

Let us consider the conduction of electricity in a metallic wire. If  $X$  be the electric intensity along the wire, then the force experienced by every electron in the metal is

$$F = -eX$$

where  $-e$  is the electronic charge.

Hence the acceleration acquired by the electron as a result of this force is  $f = -\frac{eX}{m}$  where  $m$  is the mass of the electron. As a result

the velocity of the electron increases between collisions; but the electron parts with its acquired velocity when it collides with another atom. Thus the velocity at the start of the path is zero, that at the end is  $ft$  where  $t$  is the time required to traverse the distance between successive collisions.

If  $\lambda$  is the mean free path, i.e., the average distance traversed between successive collisions and  $C$  the mean velocity of the electrons corresponding to the absolute temperature  $T$ , then  $t = \frac{\lambda}{C}$ .

So, the average drift velocity of the electron

$$\begin{aligned} v_d &= \frac{1}{2} (0 + ft) = \frac{1}{2} \left( 0 + f \frac{\lambda}{C} \right) \\ &= \frac{1}{2} \cdot \frac{eX}{m} \cdot \frac{\lambda}{C} \end{aligned}$$

If  $n$  is the number of electrons per unit volume, the charge density  $\rho = -ne$

$\therefore$  the current density  $J = \rho v_d = -nev_d$

Current flowing through an area  $ds$  is

$$I = \int J \cdot ds = -ne \int v_d \cdot ds$$



$$\begin{aligned}
 &= -ne \int v_d ds \cos\theta \\
 &= -nev_d \int ds' \quad (ds' = ds \cos\theta) \\
 &= -nevs ds'
 \end{aligned}$$

So the current per unit area perpendicular to the flow is given by

$$i = \frac{I}{s'} = -nev_d = \frac{1}{2} \frac{ne^2 X}{m} \cdot \frac{\lambda}{c}$$

$\therefore$  Electrical conductivity  $\sigma$  of unit length and unit cross-sectional area is given by

$$s = \frac{i}{X} = \frac{1}{2} \frac{ne^2 \lambda}{mc} \quad (i)$$

But according to law of equipartition of energy,

$$\frac{1}{2} mc^2 = \frac{3}{2} KT$$

$$\therefore s = \frac{ne^2 \lambda c}{6KT} \quad (ii)$$

But the thermal conductivity,  $K$ , for the electron gas can be expressed in heat unit as

$$K = \frac{1}{3} \frac{nc\lambda}{J} \frac{dE}{dT}$$

where  $J$  is the Joule's constant.

But, if the electron-gas is assumed to possess only translational energy,  $E = \frac{3}{2} KT$ .

$$\therefore \frac{dE}{dT} = \frac{3}{2} K.$$

$$\therefore K = \frac{1}{3} \frac{nc\lambda}{J} \cdot \frac{3}{2} K = \frac{1}{2} \frac{nc\lambda}{J} K \quad (iii)$$

From (ii) and (iii)

$$\frac{K}{\sigma} = \frac{3}{J} \left( \frac{K}{e} \right)^2 T = \text{const.} \times T \quad (\text{iv})$$

Eqn. (iv) is the Wiedemann – Franz law

The law was put to stringent experimental tests by many workers, notably by Jaeger and Diesselhorst, Lees, Meissner and Onnes, and Holst. Their results can be summarised as follows:

(i) At ordinary temperature, between  $-100^{\circ}\text{C}$  to  $+100^{\circ}\text{C}$ , the law is obeyed with considerable accuracy by a large number of metals. But at lower temperatures, the ratio  $K/\sigma$  decreases rapidly and the value tends to zero at absolute zero.

(ii) As the temperatures of the metal is decreased, the thermal and electrical conductivities of the metal increase. But the increase in the electrical conductivity is higher and its value tends to infinity at absolute zero for pure metals (super conductivity). Near the absolute zero,  $K$  tends to become constant.

### 18.11 Some practical applications of heat conduction

Metals are good conductors of heat and wood, brick, glass, granite, wool, cotton, felt, rubber, ebonite are bad conductors of heat.

(1) Because brick is a bad conductor of heat, thick brick walls are used in the construction of a cold storage, so that heat from outside cannot flow inside the cold storage.

(2) Quilts and bed clothings filled with cotton are used in winter. Air layers in the pores of the cotton are bad conductors of heat. Therefore the flow of heat to the outside is prevented.

(3) The windows are provided with double doors, both in cold countries and hot countries. Air in between the two doors forms a non-conducting layer. In the cold countries, this non-conducting layer does not allow heat to flow from inside to outside while in hot countries, it does not allow heat to flow from outside to inside.

(4) Because metals are good conductor of heat, sauce pans, hot water buckets, kettles and other utensils are made of metals. They are usually provided with wooden or ebonite handles so that heat from the utensil is not conducted to the hand.



(5) Ice box has a double wall made of tin or iron. The space between the two walls is packed with cork or felt. This is done because cork and felt are poor conductors of heat and prevent the flow of heat from outside into the box.

(6) Davy's safety lamp : The high conductivity of a metal and the low conductivity of a gas has been utilized in the construction of Davy's safety lamp used in mines. This lamp consists of an oil lamp, the flame of which is enclosed within a cylindrical wire gauge of close mesh working as the chimney. When the gas is lit in the lamp, the heat is conducted away by the gauge so rapidly that even if the lamp is surrounded by an explosive gas, the temperature at any point in the gauge does not reach the ignition point. The temperature outside the wire gauge remains lower than the ignition temperature.

### 18.12 Convection

Suspend a hot piece of metal in air. The air immediately around it is heated by conduction and radiation. The heat causes an expansion and a decrease in density of the air. As a result the hot air rises up and cold air comes from below and the sides flow in to take its place. This continues until the sheet of metal is cooled to room temperature. Such air movement is termed *natural convection*. If some external agency, such as a fan or pump, is used to maintain the air flow mentioned above, then the process is called *forced convection*.

A detailed study of the phenomenon of convection is beyond the scope of this book. Some practical applications of convection in science and technology and some natural phenomenon where convection plays an important role will be illustrated below.

#### Applications in technology

(a) **Ventillation** : The object of ventillation is to establish convection currents between the outside air and the air inside the room. Two things are necessary for proper ventillation of a room – an outlet for the warm and impure air near the top of the room, and an inlet for the cold pure air near the bottom of the room. Air inside the room gets warmer due to respiration of people in the room. Warm air containing more  $\text{CO}_2$  and water vapour has less density and moves upwards. Fresh cold air from outside enters at the bottom of the room through the doors and



windows. The impure air moves outside through the ventilators. Thus a convection current of air or draught is set up inside the room.

**(b) Hot water supply in buildings and central heating :** In cold countries, the temperature in winter falls below  $0^{\circ}\text{C}$ . The rooms of a building are kept warm by a central heating system based on the phenomenon of convection.

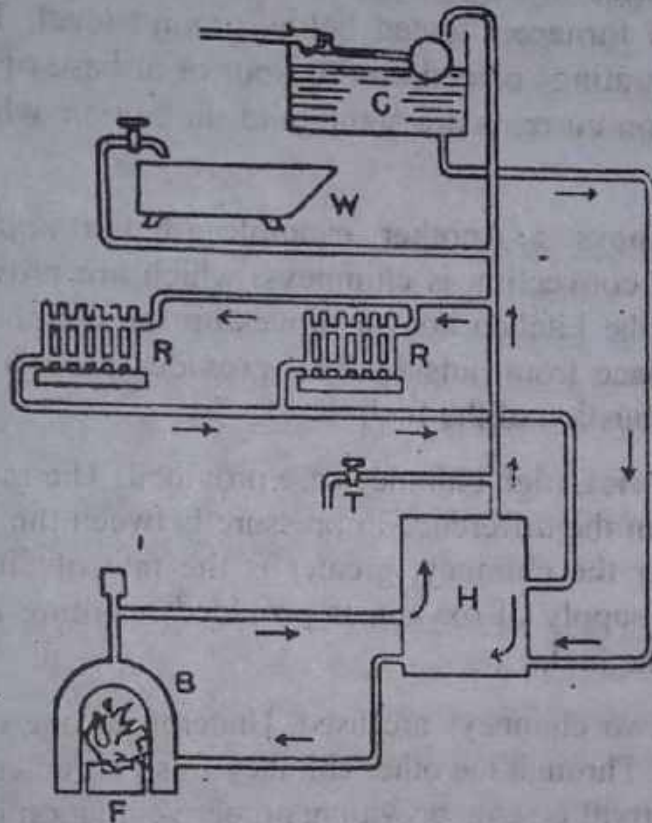


Fig. 18.12

In the central heating system, the boiler B is situated at the basement – the lowest level of the building and a supply cistern C is placed near the roof – the highest level of the building (Fig. 18.12). The rooms are fitted with horizontal pipes bent a number of times into zig-zag forms which are called *radiators* or more appropriately *convectors* (R. R in the figure). The water in the boiler is heated by the furnace F, and since the density of hot water is less than that of cold water, the hot water rises and passes through the radiators of different rooms. Radiators get heated and radiate heat to the room. After losing heat to the radiator, water becomes cold and returns back to the boiler. Hot water also reaches the cold water tank (cistern) at the top of the building. The hot water flows from the boiler to the cold water tank and the cold water

flows from the tank to the boiler to be heated in turn. Thus a continuous circulation due to convection occurs and the hot water continues to circulate throughout the whole system of pipes and the building is kept warm continuously at a constant temperature.

It may be pointed out that steam or hot air is used instead of hot water in central heating of large buildings. In the case of heating by hot air, the air is passed through a chamber which is heated either by steam, or directly by a furnace situated below ground level. The hot air then issues through gratings placed on the floor or at base of the walls of the room. Convection currents are setup, and the hot air which rises warms the rooms.

**(c) Chimneys :** Another example of the application of the phenomenon of convection is chimneys, which are provided in kitchen and factory. In the kitchen hot air moves up the chimney and fresh air enters the fireplace from outside. This provides enough oxygen for the continuous combustion of the fuel.

In the factories, high chimneys are provided. The rate of circulation of air depends on the difference in pressure between the two ends of the chimney. Higher the chimney, greater is the rate of circulation of air. Thus sufficient supply of oxygen is provided resulting in the complete combustion of the fuel.

In mines, two chimneys are used. Underneath one of the chimneys fire is produced. Through the other chimney fresh air enters into the mine. A convection current is setup providing proper ventilation in the mine.

The principle of a chimney is also used in oil lamps and oil stoves. When the wick is lighted, air and kerosene oil burn above the wick. Heated air moves up and fresh air from the holes provided at the bottom of the chimney enters the inside. The phenomenon is continuous and the convection currents set up help in supplying enough oxygen for combustion. If holes are not provided, the flame gets extinguished quickly.

2. **Applications in science :** Newton's law of cooling (see Art. 15.1)
3. **Natural phenomena**

**(a) Land and Sea Breezes :** Land has a lower specific heat and a greater absorbing power than water. Hence, near the sea, land becomes more heated than the sea water by the sun's heat during the day time.



Consequently in the evening the air above the land becomes more heated and rises up. Cooler air from the surface of the sea moves towards the sea shore to take its place by convection causing what is known as *sea breeze*.

Since good absorbers are also good radiators, during the night the land loses more heat than the sea. Because of its lower specific heat, the temperature of the land will become lower than that of the sea in the early hours of the morning. The air over the surface of water is warmer and moves upward. Cooler air from the land moves towards the sea by convection, causing what is called *land breeze*.

(b) **Trade winds** : The surface of the earth gets heated more at the equator than at the poles. Warm air at the equator moves up and to replace it, cold air from the poles move towards the equator. In the northern hemisphere, it is coming from the north. But due to the rotation of the earth from west to east, the wind appears to come from north-east. Similarly, in the southern hemisphere, the wind appears to come from south-west. These winds are called trade winds because they were used by traders for sailing their vessels in ancient days.

(c) **Ocean currents** : Water in the ocean gets heated more at the equator than at the poles. Heated water at the equator expands and the level of water rises in the equator. This hot water is called the hot current or *gulf stream*. On the other hand, due to convection, the cold water from the pole moves towards the equator beneath the ocean and this current is called the *under current*.

### 18.13 Dimension of K

Eqn. (18.2) can be rearranged as

$$K = - \frac{Q}{A \cdot \frac{d\theta}{dx} \cdot t} = - \frac{Q \cdot dx}{A \cdot d\theta \cdot t}$$

Q represents energy and its dimensions are

$$[Q] = [ML^2T^{-2}]$$

$$[dx] = [L]$$

$$[A] = [L^2]$$

$$[d\theta] = [\theta]$$



$$[t] = [T]$$

$$\begin{aligned} \therefore K &= \frac{[ML^2 T^{-2}][L]}{[L^2][\theta][T]} \\ &= MLT^{-3} \theta^{-1} \end{aligned}$$

**Example 18.1.** The opposite faces of a metal plate of 0.2 cm thickness are at a difference of temperature of  $100^\circ\text{C}$ . If the area of the plate is  $200 \text{ cm}^2$ , find the quantity of heat that will flow through the plate in one minute.  $K = 0.2$  CGS units.

**Soln.**

$$Q = \frac{KA(\theta_1 - \theta_2)t}{d}$$

$$\theta_1 - \theta_2 = 100^\circ\text{C}, K = 0.2$$

$$t = 60 \text{ seconds}, d = 0.2 \text{ cm}$$

$$A = 200 \text{ cm}^2$$

$$\therefore Q = \frac{0.2 \times 200 \times 100 \times 60}{0.2}$$

$$= 12 \times 10^5 \text{ cal.}$$

**Example 18.2.** An ice box is built of wood 1.75 cm thick, lined inside with cork 3 cm thick. If the temperature of the inner surface of the cork is  $0^\circ\text{C}$  and that of the outer surface of wood is  $12^\circ\text{C}$ , what is the temperature of the interface? Thermal conductivities of wood and cork are 0.0006 and 0.00012 CGS units respectively.

**Soln.**

Let  $\theta$  be the temperature of the interface after steady state is reached. Then

$$K_1 = 0.0006, K_2 = 0.00012$$

$$d_1 = 1.75 \text{ cm}, d_2 = 3 \text{ cm}, A_1 = A_2$$

$$\frac{K_1 A_1 (12 - \theta)}{d_1} = \frac{K_2 A_2 (\theta - 0)}{d_2}$$

$$\text{or, } \frac{0.0006 (12 - \theta)}{1.75} = \frac{0.00012 (\theta)}{3}$$

$$\text{or, } \theta = 10.74^\circ\text{C.}$$

**Example 18.3.** A room has a wall 6 metres by 3 metres made of bricks 24 cm thick lined with plaster 1 cm thick with a glass window of area  $2\text{ m}^2$  and thickness 0.5 cm. Assuming that the temperature of the inside surfaces exceeds that of the outside surfaces by  $10^\circ\text{C}$ , find the heat conducted per second through (a) the glass and (b) the plaster lined brick. Assume the thermal conductivities of brick, plaster and glass to be  $1.4 \times 10^{-3}$ ,  $1.0 \times 10^{-3}$  and  $2.0 \times 10^{-3} \text{ cal cm}^{-1} \text{ s}^{-1} \text{ }^\circ\text{C}^{-1}$  respectively.

**Soln.**

Rate of flow of heat ( $t = 1 \text{ sec}$ ) through glass

$$q_1 = \frac{KA(\theta_1 - \theta_2)}{x}$$

$$K = 2.0 \times 10^{-3}$$

$$A = 2 \text{ m}^2 = 2 \times 10^4 \text{ cm}^2$$

$$\theta_1 - \theta_2 = 10^\circ\text{C}$$

$$x = 5 \times 10^{-1} \text{ cm}$$

$$\therefore q_1 = \frac{2.0 \times 10^{-3} \times 2 \times 10^4 \times 10}{5 \times 10^{-1}}$$

$$= 800 \text{ cal / sec.}$$

Rate of heat flow through the plaster lined brick

$$q_2 = \frac{\theta_1 - \theta_2}{\frac{x_1}{K_1 A} + \frac{x_2}{K_2 A}}$$

$$\theta_1 - \theta_2 = 10^\circ\text{C}$$

$$A = (6 \times 3) \text{ m}^2 - 2 \text{ m}^2$$

$$= 16 \text{ m}^2 = 16 \times 10^4 \text{ cm}^2$$

$$x_1 = 24 \text{ cm}$$

$$K_1 = 1.4 \times 10^{-3}$$

$$x_2 = 1 \text{ cm}$$

$$K_2 = 1.0 \times 10^{-3}$$

$$\begin{aligned} \therefore q_2 &= \frac{10}{\frac{24}{1.4 \times 10^{-3} \times 16 \times 10^4} + \frac{1}{1.0 \times 10^{-3} \times 16 \times 10^4}} \\ &= \frac{10}{\frac{24}{224} + \frac{1}{160}} = \frac{10}{\frac{127}{1120}} \\ &= \frac{1120 \times 10}{127} \simeq 88 \text{ cal/sec.} \end{aligned}$$

**Example 18.4.** Water is boiled in a rectangular steel tank, the bottom of which is 2mm thick. If the water level falls at the rate of 0.01 m in 5 minutes, what is the temperature of the lower surface of the tank? Thermal conductivity of steel = 0.012 SI units and latent heat of steam = 540 kcal/kg.

**Soln.**

The flow of heat across the bottom of the tank in 5 minutes is

$$Q = \frac{KA(\theta_1 - \theta_2)t}{x} \text{ kcal.}$$

Here

$\theta_1$  = temperature of the lower surface of the tank bottom

$\theta_2$  = temperature of the upper surface of the tank bottom

= normal boiling temperature of water

= 100°C.

A = surface area of the tank bottom in  $\text{m}^2$



$$x = \text{thickness of the tank bottom} \\ = 2 \times 10^{-3} \text{ m.}$$

Now mass of water evaporated in 5 minutes

$$= \text{volume} \times \text{density}$$

$$= (A \times dx) \times \rho$$

Here,  $dx$  = fall of water level in 5 minutes

$$= 0.01 \text{ m}$$

$\rho$  = density of water

$$= 10^3 \text{ kgm/m}^3$$

$\therefore$  mass of water evaporated in 5 minutes

$$= A \times 0.01 \text{ m}^3 \times 10^3 \text{ kgm/m}^3$$

$$= 10 A \text{ kgm}$$

Heat required to evaporate this amount of water

$$= 10 A \times 540 \text{ kcal}$$

$$= 5400 A \text{ kcal.}$$

This should be the heat conducted from the lower surface to the upper surface of the tank bottom in 5 minutes.

Hence

$$5400 A = \frac{KA(\theta_1 - \theta_2)t}{x} \\ = \frac{0.012 \times A(\theta_1 - 100) \times 5 \times 60}{2 \times 10^{-3}}$$

$$\text{or, } \theta_1 = 103^\circ\text{C.}$$

*Example 18.5.* Heat is conducted through a slab of two different layers of different materials of thermal conductivities 0.02 and 0.03 SI units respectively. The thickness of each layer is 1 cm. If the temperatures of the two outer layers are 373K and 273K, find the temperature gradients of the layers.

**Soln.**

Let the temperature of the interface be  $\theta$  K. Under steady conditions, the rate of heat flow through the layers is the same.

Rate of heat flow through the first slab

$$\frac{dQ_1}{dt} = \frac{K_1 A (\theta_1 - \theta)}{x_1} \text{ kcal/sec.}$$

where  $A$  is the surface area of the slab. Rate of heat flow through the second slab

$$\frac{dQ_2}{dt} = \frac{K_2 A (\theta - \theta_2)}{x_2} \text{ kcal/sec.}$$

Since  $\frac{dQ_1}{dt} = \frac{dQ_2}{dt}$

$$\frac{K_1 A (\theta_1 - \theta)}{x_1} = \frac{K_2 A (\theta - \theta_2)}{x_2}$$

or,  $\frac{0.02A (373 - \theta)}{1 \times 10^{-2}} = \frac{0.02A (\theta - 273)}{1 \times 10^{-2}}$

or,  $\theta = 313\text{K.}$

Drop in temperature in the first slab

$$= 373 - 313 = 60\text{K.}$$

Thickness of the slab  $= 1 \times 10^{-2} \text{ m.}$

Hence temperature gradient in the first slab

$$= \frac{d\theta}{dx} = \frac{60}{1 \times 10^{-2}} = 6000\text{K/m,}$$

Fall in temperature in the second slab

$$= 313 - 273 = 40\text{K.}$$

Hence temperature gradient  $= \frac{40}{1 \times 10^{-2}} = 4000\text{K/m.}$

*Example 18.6. In an Ingen Hausz experiment, wax melts over 10 cm of copper rod and over 4 cm of iron rod. If the conductivity of copper is 0.90 CGS units, what is the conductivity of iron?*

**Soln.** We have (see Art. 18.7)

$$l_1 = 10 \text{ cm}, l_2 = 4 \text{ cm}, K_1 = 0.90$$

$$\text{Now } \frac{K_2}{K_1} = \frac{l_2^2}{l_1^2}$$

$$\text{so, } K_2 = \frac{l_2^2}{l_1^2} \times K_1 = \frac{16}{100} \times 0.90 = 0.144.$$

**Example 18.7.** A wire of resistivity  $2 \times 10^{-4}$  ohms per  $\text{cm}^2$  and 1 mm in diameter carries a current of 10 amps. If it is covered uniformly with a cylindrical layer of insulating material having coefficient of thermal conductivity of  $6 \times 10^{-4}$  calories  $\text{cm}^{-1} \text{sec}^{-1} \text{°C}^{-1}$  and a diameter of 1 cm, what is the temperature difference between the inner and outer surfaces of the insulator? 1 cal = 4.2 watt-secs.

**Soln.** We know that

$$\theta_1 - \theta_2 = \frac{Q \log_e (r_2 / r_1)}{2\pi K l t} \quad (\text{see Art. 18.8})$$

Here, we have  $l = 1 \text{ cm}$  and  $t = 1 \text{ sec}$ .

$$Q = 0.24 i^2 R = 0.24 i^2 \cdot \rho \frac{l}{A}$$

$$= \frac{0.24 \times 10^2 \times 2 \times 10^{-4} \times 1}{\pi (0.05)^2}$$

$$= \frac{24 \times 2}{\pi \times 25}$$

$$\therefore \theta_1 - \theta_2 = \frac{Q \log_e (r_2 / r_1)}{2\pi K}$$

$$= \frac{24 \times 2}{\pi \times 25} \times \frac{\ln (.50/.05)}{2\pi \times 6 \times 10^{-4}}$$

$$= \frac{4 \times 2.303}{\pi^2 \times 25 \times 10^{-4}} = 373.4^\circ \text{C}.$$



**Example 18.8.** Steam at  $100^{\circ}\text{C}$  passes through a rubber tube of inner and outer diameters of 1 cm and 3 cm respectively, the tube being immersed in ice. Find the temperature existing half-way through the thickness of the wall of the tube.

**Soln.** We have, from Art. 18.8,

$$\frac{2\pi/K_1}{q} (\theta_1 - \theta_2) = \log_e \frac{r_2}{r_1}$$

Here  $r_1 = 0.5 \text{ cm}$ ;  $r_2 = 1.5 \text{ cm}$

$\theta_1 = 100^{\circ}\text{C}$ ;  $\theta_2 = 0^{\circ}\text{C}$

$$\therefore \ln \frac{1.5}{0.5} = \frac{2\pi/K}{q} (100 - 0)$$

$$\text{or, } \ln 3 = \frac{2\pi/K}{q} \times 100 \quad (i)$$

when only half the thickness of the tube is considered, then

$r_1 = 0.5 \text{ cm}$ ;  $r_2 = 0.5 + 0.5 = 1 \text{ cm}$

$\theta_1 = 100^{\circ}\text{C}$ ;  $\theta$  = temperature half-way through

$$\therefore \ln \frac{1.0}{0.5} = \frac{2\pi/K}{q} (100 - \theta)$$

$$\text{or, } \ln 2 = \frac{2\pi/K}{q} (100 - \theta)$$

Dividing eqn. (ii) by (i), we get

$$\frac{100 - \theta}{100} = \frac{\ln 2}{\ln 3} = \frac{0.693}{1.098}$$

$$\text{or, } \theta = 100 - 63.1 = 36.9^{\circ}\text{C}.$$

**Example 18.9.** In an experiment to determine the thermal conductivity of cardboard by the Lees and Chorlton's method, the following observations were made:

steady temp. of the upper disc =  $99.5^{\circ}\text{C}$

steady temp. of the lower disc =  $83.5^{\circ}\text{C}$

time taken by the lower disc to cool from  $86^{\circ}\text{C}$  to  $81^{\circ}\text{C}$  = 3.5 min.

thickness of lower disc, which is copper = 1 cm

thickness of card board = 4.5 mm

Calculate the thermal conductivity of the cardboard. [ density of copper =  $8.9 \text{ gm/cm}^3$ ; sp. ht. of copper = 0.09]

**Soln.** Heat conducted per second through cardboard disc is

$$\frac{KA(99.5 - 83.5)}{0.45} \text{ cal/sec.}$$

Rate of loss of heat per second at the mean temp. of  $83.5^{\circ}\text{C}$  through the lower copper disc

= rate of cooling  $\times$  mass  $\times$  specific heat

= rate of cooling  $\times$  volume  $\times$  density  $\times$  specific heat

$$= \left[ \frac{86 - 81}{210} \right] \times [A \times 1 \times 8.9] \times 0.09$$

$$= \frac{0.8 A}{42} \text{ cal/sec}$$

Equating the two rates of loss of heat, we get

$$\frac{0.8 A}{42} = \frac{KA \times 16}{0.45}$$

or,  $K = 0.00054 \text{ C.G.S. units.}$

**Example 18.10.** A large cylindrical block of copper 0.2 m long has one end in steam at  $373\text{K}$ . Assuming that there is a layer of water 0.01 cm thick at each end of the block, find the temperature inside the metal at each end and also the temperature gradient in the bar. Thermal conductivities of copper and water are 0.91 and 0.00014 SI units respectively.

**Soln.** The thickness of the water film can be replaced by an equivalent thickness of copper. The equivalent copper thickness of one water film is

$$l = 0.01 \times \frac{0.091}{0.00014} = 6.5 \text{ cm} = 0.065 \text{ m.}$$

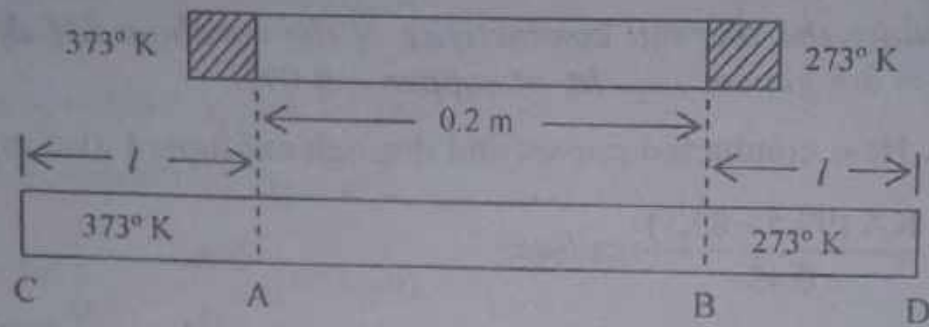


Fig. 18.13

The copper bar may, therefore, be considered to be of length  $(0.2 + 2l)$  m with the ends at 373K and 273K.

$$\begin{aligned} \text{Temp. gradient} &= \frac{(373 - 273)}{0.2 + 2 \times 0.065} \\ &= 303 \text{ K/m} \end{aligned}$$

$$\begin{aligned} \text{Temp. drop over } l \text{ on the hot end side} \\ &= 303 \times 0.065 = 19.7 \text{ K.} \end{aligned}$$

$$\begin{aligned} \text{Temp. inside the metal on the hot end side} \\ &= 373 - 19.7 = 353.3 \text{ K} \end{aligned}$$

$$\begin{aligned} \text{Similarly temp. inside the metal on the cold end side} \\ &= 273 + 19.7 = 292.7 \text{ K.} \end{aligned}$$

**Example 18.11.** Calculate the difference in temperature between the inner and outer surfaces of an annular cylinder of aluminium of length 5 cm when it is being heated internally by an axial coil delivering 10 watt power. Given thermal conductivity of aluminium = 0.5 C.G.S. unit, inner diameter = 3 cm, outer diameter = 6 cm.  $J = 4.2 \text{ J/cal.}$



**Soln:** From the expression for radial flow of heat through a cylindrical shell, we have

$$K = \frac{dQ}{dt} \frac{\log_e (r_2/r_1)}{2\pi l (\theta_1 - \theta_2)}$$

$$\text{or, } \theta_1 - \theta_2 = \frac{\ln (r_2/r_1)}{2\pi/K} \frac{dQ}{dt}$$

$$\text{Here } \frac{dQ}{dt} = 10 \text{ watt} = 10 \text{ J/s} = (10/4.2) \text{ cal/s,}$$

$$r_2/r_1 = 3/1.5 = 2; l = 5 \text{ cm and } K = 0.5$$

$$\begin{aligned} \therefore \theta_1 - \theta_2 &= \frac{10 \times \ln 2}{4.2 \times 2 \times 3.14 \times 5 \times 0.5} \\ &= \frac{10 \times 0.6923}{4.2 \times 3.14 \times 5} = 0.105^\circ\text{C.} \end{aligned}$$

**Example 18.12.** A composite slab is made of two parallel layers of different materials A and B in contact. Their conductivities are 70 W/m/K and 200 W/m/K respectively and thickness 0.045m and 0.025m respectively. Find the temperature of the interface of A and B, when their outer surfaces are maintained at 373 K and 273 K respectively.

**Soln:** Let  $\theta$  be the temperature of the interface at the steady state. In this state, the rate of flow of heat per unit area of B will be the same as that across A.

$$\therefore Q = K_1 \frac{373 - \theta}{x_1} = K_2 \frac{\theta - 273}{x_2}$$

$$\therefore \frac{70 \times (373 - \theta)}{0.045} = \frac{200 \times (\theta - 273)}{0.025}$$

$$\text{or, } 70 \times (373 - \theta) \times 0.025 = 200 \times (\theta - 273) \times 0.045$$

$$\text{or, } 7 \times 5 \times (373 - \theta) = 45 \times 4 \times (\theta - 273)$$

$$\text{or, } 215 \theta = 62195$$

$$\therefore \theta \simeq 289 \text{ K.}$$

**Example 18.13.** A lagged bar is made in two parts, each having the same cross-sectional area  $4 \times 10^{-2} \text{ m}^2$ . The first part is 0.15 m long and has a thermal conductivity of  $385 \text{ W m}^{-1} \text{ deg}^{-1}$ , while the second part is 0.05 m long and has a thermal conductivity of  $100 \text{ W m}^{-1} \text{ deg}^{-1}$ . Find the temperature of the joint and the quantity of heat conducted per second along the bar when one end is at  $100^\circ\text{C}$  and the other end at  $0^\circ\text{C}$ .

**Soln:** If the temperature of the joint is  $\theta$ , then the rate of flow of heat down the first part is

$$\left(\frac{dQ}{dt}\right)_1 = \frac{K(\theta_1 - \theta)A}{l_1} = \frac{385 \times (100 - \theta) \times 4 \times 10^{-2}}{0.15}$$

The rate of flow of heat flowing down the second part is

$$\left(\frac{dQ}{dt}\right)_2 = \frac{K(\theta - \theta_2)A}{l_2} = \frac{100 \times (\theta - 0) \times 4 \times 10^{-2}}{0.05}$$

In the steady state  $\left(\frac{dQ}{dt}\right)_1 = \left(\frac{dQ}{dt}\right)_2$

$$\therefore \frac{385 \times (100 - \theta) \times 4 \times 10^{-2}}{0.15} = \frac{100 \times (\theta - 0) \times 4 \times 10^{-2}}{0.05}$$

On simplification,

$$\theta = 56.2^\circ\text{C}.$$

Substituting this value of  $\theta$  in either equation,

$$\frac{dQ}{dt} = \frac{100 \times 4 \times 10^{-2} \times 56.2}{0.05} = 4496 \text{ watt.}$$

**Example 18.14.** A spherical hot water tank, fitted with an electrical heater, has an internal radius of 0.2 m and has a wall 0.05 m thick made of a poor thermal conductor ( $K = 0.84 \text{ W m}^{-1} \text{ deg}^{-1}$ ). If the temperature of the outside of the wall is  $15^\circ\text{C}$  when the water is at  $95^\circ\text{C}$ , calculate the power which must be dissipated in the heater so as to maintain the water temperature at  $95^\circ\text{C}$ .

**Soln:** To maintain the water temperature constant, heat must be supplied at the rate at which it is conducted away through the wall.

From relation  $\frac{dQ}{dt} = 4\pi K \frac{(\theta_1 - \theta_2)}{r_2 - r_1} r_1 r_2$ , the rate of heat flowing away through the wall

$$= \frac{4\pi \times 0.84 \times (95 - 15) \times 0.2 \times 0.25}{0.25 - 0.2}$$

$$= 4\pi \times 0.84 \times 80$$

$$= 844.5 \text{ watt.}$$

This is the rate at which power must be dissipated in the heater.

**Example 18.15.** Two thin concentric spherical shells of radius 5 cm respectively have their annular cavity filled with charcoal. When energy is supplied at the steady rate of 10.8 W to a heater at the centre, a temperature difference of 50°C is set up between the spheres. Find the thermal conductivity of charcoal.

**Soln:** From the relation

$$\frac{dQ}{dt} = 4\pi K \frac{\theta_1 - \theta_2}{r_2 - r_1} r_1 r_2$$

we have

$$K = \frac{(r_2 - r_1)}{4\pi(\theta_1 - \theta_2)r_1 r_2} \cdot \frac{dQ}{dt}$$

Substituting the values of  $r_1$ ,  $r_2$ ,  $\theta_1$ ,  $\theta_2$  and  $\frac{dQ}{dt}$ , we get

$$\begin{aligned} K &= \frac{10.8 \times (15 - 5)}{4 \times 3.14 \times 50 \times 15 \times 5} \\ &= 0.0023 \text{ W cm}^{-1} \text{ deg}^{-1}. \end{aligned}$$



## EXERCISES

- [1] Define thermal conductivity and thermometric conductivity of a substance.

Find the dimension of  $K$  – the thermal conductivity of a substance.

'Thermal conductivity of iron = 0.175 C.G.S. units'. Explain.

- [2] Define and explain the coefficient of thermal conductivity. What is its unit? How can you measure the thermal conductivity of copper in the laboratory?

- [3] Distinguish between thermal conductivity and thermometric conductivity. Bring out the connection between the two. Describe Searle's method of determining the thermal conductivity of a solid.

- [4] Describe a simple experiment for the rough comparison of thermal conductivities of several metal rods. Deduce the formula you would use.

- [5] Discuss the rectilinear flow of heat along a long metal bar of uniform cross-section and obtain an expression for the temperature at any point of the bar in the steady state.

- [6] Show that in the steady state of a metal bar heated at one end

$$\frac{d^2\theta}{dx^2} = -\mu^2\theta$$

where the symbols have their usual significance.

- [7] Describe Ingen Hausz's experiment and prove from the mathematical theory that the conductivities of the different metals vary as the square of the length upto which the wax melts.

- [8] Describe Lees and Chorlton's method of determining the thermal conductivity of a bad conductor. Develop the formula you would use.

- [9] Explain the terms thermal conductivity and thermometric conductivity or diffusivity. Describe an experiment to determine thermal conductivity of a poor conductor in the form of a circular disc.

- [10] Describe and explain the cylindrical shell method of determining the conductivity of a solid.

- [11] Describe and give the theory of a method useful for a practical determination of the thermal conductivity of a liquid.

- [12] Distinguish between conductivity and diffusivity of heat. When steam is passed through a circular tube of length  $l$ , having internal

and external diameters  $a$  and  $b$  respectively, prove that the radial flow of heat outwards is given by

$$2\pi K / (\theta_2 - \theta_1) / \log_e \frac{b}{a}$$

where  $\theta_1$  and  $\theta_2$  are the steady temperatures of the inner and outer walls of the tube respectively.

- [13] Obtain an expression for the rate of flow of heat through a composite slab and find an expression for the interface temperature.
- [14] Describe a method for determining the thermal conductivity of poor solid conductors (i) in the form of a rigid hollow cylindrical tube and (ii) in the form of flexible hollow cylindrical tube.
- [15] State and obtain Wiedemann – Franz law. How far does the law agree with experimental results?
- [16] A boiler is made of a copper plate 2.4mm thick coated inside with a layer of tin 0.2mm thick. Surface area exposed to hot gases at 973K is  $0.01\text{m}^2$ . Calculate the maximum amount of steam that could be raised per hour at atmospheric pressure. Conductivities of Cu and tin are 0.09 and 0.015 SI units respectively. Latent heat of steam at normal pressure is 540 Kcal/kg. [1000 kg].
- [17] Two bars of nickel and copper of the same diameter and cross-sectional area are coated with wax, and one end of each bar is introduced in a bath of oil maintained at a high temperature. When a steady state has been reached it is found that wax has been melted upto a length of 2 cm in the nickel bar and upto a length of 5 cm in the copper bar. Calculate the ratio of the diffusivities. Assume the densities of nickel and copper to be equal. (sp. ht. of nickel = 0.106; sp. ht. of copper = 0.091). [0.137].
- [18] A closed metal vessel having ten square metres as total wall area, and whose sides are 0.5 cm thick, is filled with melting ice, and is kept surrounded by water at  $100^\circ\text{C}$ . How much ice will be melted in an hour? (Thermal conductivity of the metal = 0.02 C.G.S. unit, latent heat of fusion of ice = 80 cal/gm). [ $1.8 \times 10^8$  gms].
- [19] A boiler is made of iron plates, 1.2 cm thick. If the temperature of the outer surface be  $120^\circ\text{C}$  and that of the inner  $100^\circ\text{C}$ , calculate the mass of water evaporated per hour. Assume the area of the heating surface to be 50,000 sq. cm. thermal conductivity of iron = 0.2 C.G.S. unit and latent heat of water = 540cal/gm. [ $1.11 \times 10^3$  Kg].

- [20] It is required to maintain a steady flow of heat at the rate of 3000 calories per second through a metal plate with a cross-section of 60 sq. cms. and thickness 0.5 cm. What should be the temperature difference between the faces? [ $16.34^{\circ}\text{C}$  app.].
- [21] A uniform composite bar 50 cm long consists of 20 cm of aluminium (thermal conductivity  $0.5 \text{ cal cm}^{-1} \text{ s}^{-1} ^{\circ}\text{C}^{-1}$ ) in good thermal contact with 30 cm of copper (thermal conductivity  $0.9 \text{ cal cm}^{-1} \text{ s}^{-1} ^{\circ}\text{C}^{-1}$ ). If the aluminium end is maintained at  $0^{\circ}\text{C}$  and copper end at  $100^{\circ}\text{C}$ , calculate the temperature of the aluminium – copper junction, assuming no heat is lost from the side of the bar. [ $54.5^{\circ}\text{C}$ ].
- [22] A sheet of rubber and a sheet of cardboard, each 2 mm thick are pressed together and their outer faces are maintained respectively at  $0^{\circ}\text{C}$  and  $25^{\circ}\text{C}$ . If the thermal conductivities of rubber and cardboard are  $3.1 \times 10^{-4}$  and  $1.2 \times 10^{-4} \text{ cal cm}^{-1} \text{ s}^{-1} ^{\circ}\text{C}^{-1}$  respectively, find the quantity of heat which flows in one hour across a piece of the composite sheet of area  $100 \text{ cm}^2$ . [ $3906 \text{ cal/hr}$ ].
- [23] A uniform bar of copper 20 cm long is lagged by heat-insulating material. One end is kept in steam and the other in ice. What is the temperature gradient along the bar if there is a layer of water 0.01 cm thick at each end? Thermal conductivity of copper =  $1.0 \text{ cal cm}^{-1} \text{ s}^{-1} ^{\circ}\text{C}^{-1}$  and of water =  $0.0015 \text{ cal cm}^{-1} \text{ s}^{-1} ^{\circ}\text{C}^{-1}$ . [ $3.0^{\circ}\text{C/cm}$ ].
- [24] Assuming that the effective conducting surface of a refrigerator is  $6 \text{ m}^2$ , its wall thickness 5 cm, and that the conductivity of the material is  $10^{-4} \text{ cal cm}^{-1} \text{ s}^{-1} ^{\circ}\text{C}^{-1}$ , calculate the minimum wattage of the motor required to maintain the interior at  $0^{\circ}\text{C}$  when the external temperature is  $15^{\circ}\text{C}$ . [ $75.6 \text{ W}$ ].
- [25] Find how much steam per minute is generated in a boiler made of boiler plate 0.5 cm thick if the area of the walls of the fire chamber is 2 sq. metres, the mean temperature of the plate faces being  $200^{\circ}\text{C}$  and  $120^{\circ}\text{C}$  respectively. Latent heat of steam at  $120^{\circ}\text{C}$  = 522 and conductivity of steel plate = 0.164. [ $60, 321 \text{ gm}$ ].
- [26] A copper tube has an internal radius of 2 cm and an external radius of 2.5 cm. The inside of the tube is maintained at  $100^{\circ}\text{C}$  and the outside at  $20^{\circ}\text{C}$ . Calculate the quantity of heat conducted through unit length of the tube per second. Thermal conductivity of copper = 0.91 C.G.S units. [ $2049 \text{ cal}$ ].



## CHAPTER XIX

## RADIATION

*Introduction-Thermal radiation-Emission and absorption of radiation-Prevost's theory of exchanges-Black body-Some fundamental definitions-Kirchhoff's law-Energy density and pressure of diffuse radiation-Stefan-Boltzmann's law-Wien's law-Rayleigh-Jeans' law-Planck's law of radiation-Deduction from Planck's law-Comparison of Wien's, Rayleigh-Jeans' and Planck's radiation laws-Solar constant-Temperature of the sun-Solved problems-Exercises.*

## 19.1 Introduction

Heat can be transferred from a hotter body to a colder body even in vacuum where no conduction or convection can take place. The process by which heat energy can be transferred in vacuum is known as radiation. The radiant energy in its process of transference does not however exhibit itself unless it falls on matter. When it falls on matter, it gets absorbed and is converted into heat which can be detected by suitable devices *e.g.*, thermopiles.

Study of heat rays has shown them to be electromagnetic in nature *i.e.*, similar to light. For instance, like light they travel in straight line with the same speed  $c (= 3 \times 10^8 \text{ m/s})$  and obey the laws of optics. For example they form shadows when obstructed according to the laws of geometrical optics. They carry energy, exert pressure and under proper condition, they may also give rise to light. In fact the most common sources of light are heated solids and gases through which an electric discharge is passing. The tungsten filament of an incandescent lamp and the familiar neon sign are examples in each category.

The spectral distribution of heat rays varies according to the emitter and, for temperatures less than about  $550^\circ\text{C}$ , they are practically entirely infra-red (IR) with wavelength within the range of  $0.7 \mu\text{m}$  and  $100 \mu\text{m}$ . At higher temperatures, some of the radiations are of shorter wavelength and the emitter becomes incandescent. Thus light is a thermal radiation of shorter wavelength. Radiation of still shorter wavelengths is called ultra-violet (UV); still shorter are X-rays,  $\gamma$ -rays etc. In short, the radiant energy or radiation is a more general term for light and can be of any wavelength from 0 to  $\infty$ .

It is possible to stimulate a substance to emit electromagnetic radiations in a number of different ways. For example, (i) an electric conductor carrying higher frequency current emits radio waves (radiation from an antenna) (2) a gas under electric discharge may emit visible or violet radiation (3) a metal target bombarded with high speed electrons emits X-rays (4) radioactive atoms emit  $\gamma$ -rays (5) a substance exposed to radiations from an external source may emit fluorescent radiation (fluorescence) and (vi) a solid or liquid emits radiation when heated to a high temperature. The radiations generated from processes (i) to (v) are non-thermal in character. The radiation involved in physical process (vi) is called thermal radiation as the emission is by virtue of its temperature. However, all the radiations arising out of stimulation under these different physical (i-vi) are electromagnetic radiations differing only in frequencies. Fig. 19.1 shows the tremendous range of electromagnetic radiation.

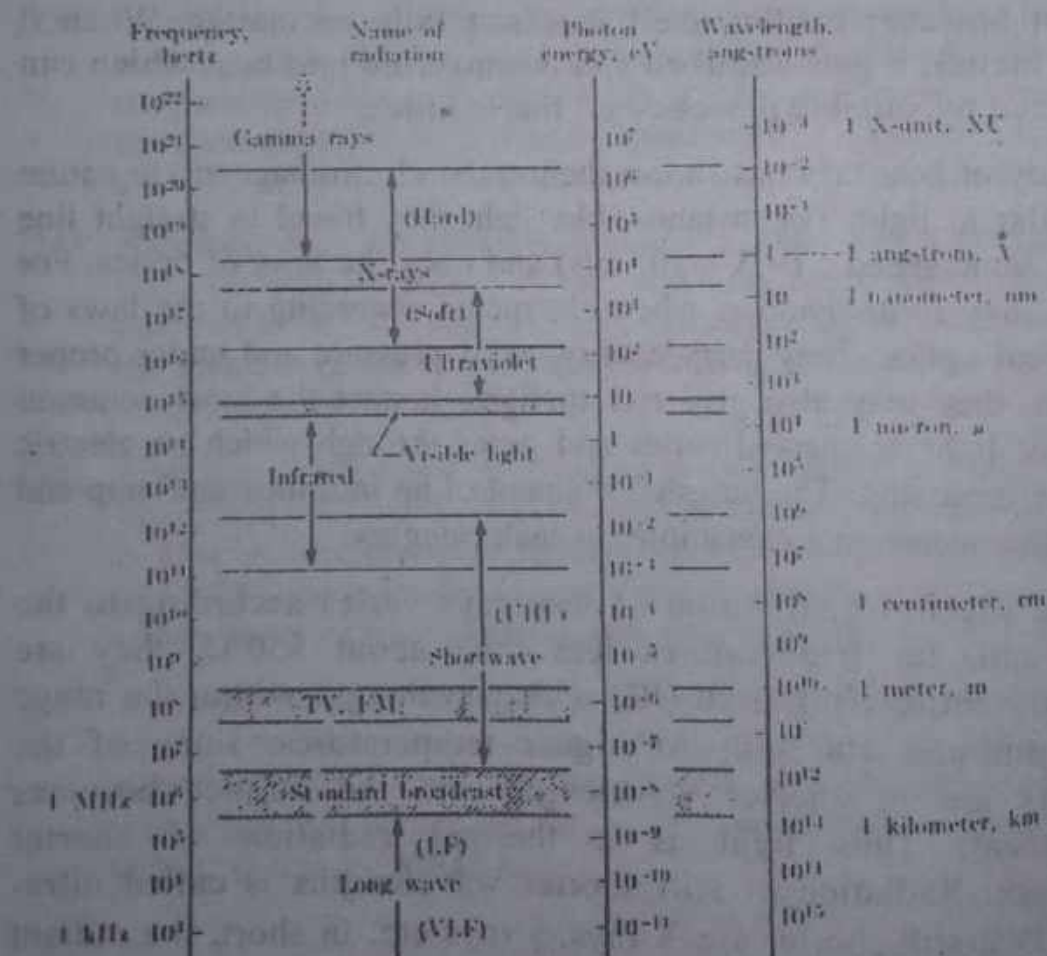


Fig. 19.1

In this chapter, thermal radiation from bodies will be considered only. The character (intensity and quality) of this radiation depends only on the temperature of the body and its surface, and restricts ourselves to aspects conventionally studied under heat. It will be shown how these studies led, in 1900, to the birth of quantum physics.

## 19.2 Thermal radiation

A body emits visible radiation if it is hot enough. A close relation between radiation and temperature is further implied by the fact that a white hot body is hotter than a red hot one. This matter may be further explored by passing the radiation from a hot body through some dispersive instrument such as the grating spectrometer and measuring the radiant energy as a function of the corresponding wavelength of the radiation. If the measured energy is plotted against wavelength, a graph similar to the dashed curve (No 1) in Fig. 19.2 will be obtained. The ordinate of this curve is called *monochromatic emittance*,  $W_\lambda$ , which is the amount of energy radiated

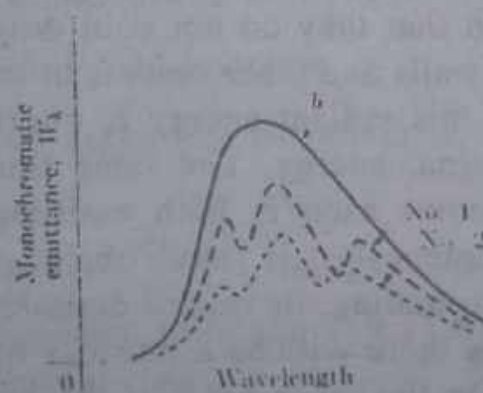


Fig. 19.2

per unit time per unit area of emitter in a wavelength range  $d\lambda$  while the abscissa is the wavelength. If the same experiment is repeated for another body (No. 2) of a different material but at the same temperature the dotted curve might be obtained. As can be seen from the figure that at most wavelengths the first body is a more efficient emitter than the second at the given temperature. Although the two curves differ, they have the same general character. They come to their highest points at about the same wavelength. If the emission



from a great variety of substances – all at the same temperature, are studied, a great variety of emission curve will be obtained, but none of these would ever have a greater monochromatic emittance,  $W_{\lambda b}$ , than the envelop curve, shown as a solid line in Fig 19.2. It appears that this curve may have a significance which does not depend on the nature of the emitting material. Let us attempt to find or make an emitter which has an emission curve identical to the solid curve of Fig. 19.2.

### 19.3 Emission and absorption of radiation

If the surfaces of all bodies are continually emitting radiant energy, then why it is that all bodies do not eventually radiate away all their internal energy and cool down to a temperature of absolute zero. The answer is that they would do so if energy were not supplied to them in some form. For example, in the case of the filament of an electric lamp, energy is supplied electrically to make up for the energy radiated. As soon as the energy supply is cut off, the filament do, in fact, cool down very quickly to room temperature. The reason that they do not cool down further is that their surroundings (the walls and other objects in the room) are also radiating, and some of this radiant energy is intercepted, absorbed and converted into internal energy. The same thing is true of all other objects in the room- each is both emitting and absorbing radiant energy simultaneously. If any object is at a higher temperature than its surrounding, its rate of emission will exceed its rate of absorption. Thus there will be a net loss of energy and the body will cool down. On the other hand, if the body is at a lower temperature than its surrounding, its rate of absorption will be larger than its rate of emission and its temperature will rise. When the body is at the same temperature as its surrounding, the two rates become equal; there is no net loss or gain of energy and hence, no change in temperature. The body is said to be in *thermal equilibrium* with its surrounding. Fig. 19.2 shows that the emittance of a surface  $W$  is different at different wavelengths. From what has been discussed in the paragraph above implies that at a given wavelength the emittance should be greater for higher temperature. To simplify our next discussion let us consider an infinitesimal band of wavelengths and several opaque bodies in thermal equilibrium with

each other and their surroundings. As the bodies are opaque, they will not transmit radiation and therefore, in general, part of the incident radiation will be absorbed and the remainder reflected. If  $a$  be the *absorptance* i.e., the fraction absorbed and  $r$  be the *reflectance* i.e., the fraction reflected, then  $a+r$  must be unity. Since the surfaces may be different

$$a_1 + r_1 = 1, \quad a_2 + r_2 = 1, \quad \text{etc.}$$

If the bodies are thought of as tiny specks near each other but too small to cast shadows on each other, then they will be *bathed* in radiation of uniform intensity,  $I$ , since they are in thermal equilibrium with their surroundings. The total radiation emitted from body No. 1 in time  $\Delta t$  is  $W_1 \Delta A_1 \Delta t$  where  $W_1$  and  $\Delta A_1$  are the emittance (radiation emitted per unit time per unit area of the surface) and area of the body respectively. The absorption by the same body in the same time is  $a_1 I \Delta A_1 \Delta t$ . For the condition of thermal equilibrium to exist these must be equal. Therefore we have

$$W_1 \Delta A_1 \Delta t = a_1 I \Delta A_1 \Delta t \quad (i)$$

Similarly for body No. 2 we have

$$W_2 \Delta A_2 \Delta t = a_2 I \Delta A_2 \Delta t, \text{ etc.} \quad (ii)$$

Dividing eqn. (i) by (ii), we obtain

$$\frac{W_1}{W_2} = \frac{a_1}{a_2} \quad \text{or,} \quad \frac{W_1}{a_1} = \frac{W_2}{a_2} \quad (19.1)$$

It becomes evident from eqn. (19.1) that  $W/a$  for any substance must be a constant (which may, of course, still depend on wavelength and temperature).

Thus it has been proved that a body or a surface which is a good emitter (high value of  $W$ ) must also be a good absorber (high value of  $a$ ) and conversely. If a perfect absorber could be found then the best possible emitter would necessarily have been found, the graph of which is shown as  $b$  in Fig. 19.2.

#### 19.4 Prevost's theory of exchange

Prior to Prevost's theory of exchanges in 1792, ideas regarding radiant energy were much confused. People used to talk of 'hot radiations',

'cold radiations' etc. For instance, it was said that a block of ice produces a sensation of cold because ice emits cold radiations. All these confusions were set at naught by Prevost when he put forward his exchange theory. According to the exchange theory, all bodies whether hot or cold lose heat by radiation to their surroundings and also receive heat from their surroundings in the same process simultaneously *i.e.*, there is a continuous exchange of heat between all bodies irrespective of their temperatures. The rate at which heat is emitted by a body increases with its temperature but is independent of its neighbouring objects. When two bodies at unequal temperatures are placed near each other, both emit as well as receive heat. But the cold body which is at a lower temperature emits less heat but receives more heat. Similarly, the hot body radiates more but receives less heat from the cold body. Thus the cold body has a *net gain* of heat while the hot body suffers a *net loss* of heat energy. The result is that the temperature of the cold body increases while the temperature of the hot body decreases. This is the reason why when we stand near a block of ice, we get the feel of cold sensation. However, this exchange of heat between the bodies goes on till both the bodies attain the same temperature. But it should be clearly understood that even now they are mutually exchanging heat but now they emit and absorb the same amount of heat *i.e.*, a dynamic equilibrium has been set up between the two resulting in no further change in their temperatures.

### 19.5 Black body

In general, when radiation falls on matter, it may be partly reflected, partly absorbed and partly transmitted. If  $r$  be the *reflectance*,  $a$  the *absorptance* and  $t$  the *transmittance* *i.e.*, the fraction transmitted, then

$$r + a + t = 1 \quad (19.2)$$

The values of  $r$ ,  $a$  and  $t$  may be different for different wavelengths. Thus, instead of  $r$ ,  $r_\lambda$ , is used to denote the reflectance of a substance for the wavelength  $\lambda$ , the corresponding symbols for absorptance and transmittance being  $a_\lambda$  and  $t_\lambda$  respectively.

For bodies having  $a = 0$  and  $t = 0$ , it follows from eqn. (19.2) that  $r = 1$ . Such bodies are called perfectly white bodies. The nearest



approach to a perfectly white body, so far as visible radiations are concerned, is a piece of chalk. Perfectly white body is an ideal conception.

If however  $r = 0$  and  $t = 0$ , the absorption coefficient  $a = 1$ . Such bodies are called perfectly black body, the nearest approach to which is lamp black (96-98 %). A perfectly black body can neither reflect nor transmit light that falls on it. It thus appears black. Perfectly black body is also an ideal conception.

A *perfectly black body* has been defined in the paragraph above as one that absorbs all radiations incident on it – it does neither transmit nor reflect any radiation. When heated to high temperature, such a body emits radiations of all wavelengths and such radiations are called *total radiation*.

It was shown by Kirchhoff that any enclosure whose walls are impervious to any type of radiation and is maintained at a constant temperature behaves as a perfect black body. The quality of radiation emitted by it is that of total radiation, *i.e.*, they depend on the temperature of the enclosure and are independent of the nature of the material of the walls of the enclosure.

Any speck of matter when placed inside it will, in the steady state, attain the temperature of the enclosure and emit black radiation characteristic of that temperature.

*It may therefore be noted that radiation in equilibrium with matter is black radiation.* For emitting black radiation a body should not necessarily be black; blackness only hastens to attain the equilibrium state.

No actual surface is perfectly black, the closest approach being lamp black (96-98 %) which reflects about 2 % of the incident radiation. Perfectly black body is an ideal conception. However a body showing close approximation to a perfectly black body can be constructed.

### Fery's black body

Designed by Fery, it consists of a double walled (hollow) spherical metallic shell M. The inner surface is coated with lamp black while the outer surface F is nickel polished. (Fig. 19.3).

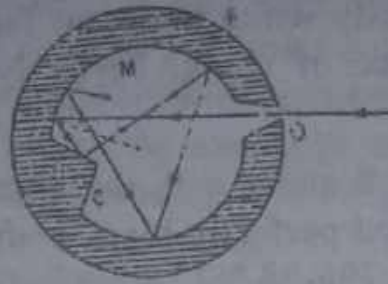


Fig. 19.3

It has a small opening at O just opposite to which there is a small conical projection C in the inner wall. When any radiation enters the enclosure through the hole, it suffers multiple reflections on the inner walls until they are finally absorbed. The projection C prevents any direct reflection of rays back through the opening. Thus practically no radiation comes out through O again. Hence the opening behaves like a perfect absorber. Conversely, the radiation emitted by the walls when the chamber is heated will come out through the opening and will be of the same nature as emitted by a perfect black body. *It should be remembered that only the hole and not the walls of the body, acts as the black body radiator.* The lampblack hastens to attain the constancy of temperature and the outer polish makes the enclosure impervious to heat.

### 19.6 Some fundamental definitions

Before discussing the laws governing the emission and absorption of radiation by a body, let us define some important quantities that enter into the laws.

#### Emissive power

The *emissive power* of a surface for a given wavelength  $\lambda$  represented by  $e_\lambda$  is defined as the energy emitted per square centimeter per second in the form of radiation of wavelength in the range of  $\lambda$  to  $\lambda + d\lambda$ . Then the total emissive power  $e$  of the surface, defined as the total energy emitted per square centimeter per second, will be given by  $e = \int e_\lambda d\lambda$ . In the special case of a black body radiation, the emissive power for a wavelength  $\lambda$  will be denoted by  $E_\lambda$  where  $E_\lambda$  depends on the temperature of the body and wavelength concerned.

### Absorptive power

The *absorptive power* of a surface for a given wavelength is defined as the fraction of the incident energy absorbed between the wavelengths  $\lambda$  and  $\lambda + d\lambda$  and is denoted by  $a_\lambda$ . Then the total absorptive power  $a$  of the body, defined as the fraction of the incident energy absorbed, will be given by  $a = \int a_\lambda d\lambda$ .

### Reflecting power

The *reflecting power (or coefficient of reflection)* is defined as the fraction of incident energy reflected by the surface. If  $Q$  be heat energy falling per second per sq. cm. on a body and  $Q'$  be the energy reflected by it, then reflecting power  $= \frac{Q'}{Q}$ .

### Transmitting power

The *transmitting power (or coefficient of transmission)* is defined as the fraction of incident energy transmitted through it.

If  $r$  represents the fraction of incident radiation reflected,  $a$  the fraction absorbed and  $t$  the fraction transmitted, then the sum of these coefficients must be unity, i.e.,  $r + a + t = 1$ .

### Surface emissivity or coefficient of emission

Surface emissivity of a surface is defined as the amount of heat radiated by unit area of the surface per second when the temperature difference between the surface and the surrounding is  $1^\circ\text{C}$ . If  $E$  and  $E'$  be the emissivities of a surface with and without lampblack respectively, under identical conditions of temperature, then the emissivity of the surface is given by  $e = \frac{E'}{E}$ .

## 19.7 Kirchhoff's law

In 1859, Kirchhoff deduced from experimental observations an important law which states that

*"the ratio of the emissive power to the absorptive power for radiation of a given wavelength ( $\lambda$ ) is the same (constant) for all*



bodies at the same temperature and is equal to the emissive power of a perfectly black body at that temperature."

Mathematically,

$$\frac{e_{\lambda}}{a_{\lambda}} = \text{constant} = E_{\lambda} \quad (19.3)$$

where,

$e_{\lambda}$  = emissive power of a body for radiation of wavelength  $\lambda$

$a_{\lambda}$  = the absorptive power of the body for the same radiation

and  $E_{\lambda}$  = emissive power of a perfectly black body for the above radiation

Eqn. 19.3 is the mathematical form of Kirchhoff's law.

#### Derivation of Kirchhoff's law:

Let us consider an enclosure at a uniform temperature  $T$ , the walls of the enclosure being opaque to radiations of all wavelengths. The enclosure is insulated thermally from the surroundings and filled with temperature radiation emitted by the walls.

First of all let us find the total emission from a small surface element  $dA$  of a black body on one side of it. The amount of radiation emitted per second in a solid angle  $dw$  in a direction making an angle  $\theta$  with the normal to the surface is, from Lambert's cosine law, equal to

$$dA e_{\lambda} d_{\lambda} dw \cos \theta \quad (i)$$

Now, in terms of spherical polar coordinates  $(r, \theta, \phi)$ , an elementary conical pencil emanating in the direction lying between  $\theta$  and  $\theta + d\theta$  and  $\phi$  and  $\phi + d\phi$  (Fig. 19.4) encloses a solid angle given by

$$\begin{aligned} dw &= \frac{\text{area}}{r^2} = \frac{r \sin \theta d\phi r d\theta}{r^2} \\ &= \sin \theta d\theta d\phi \end{aligned}$$

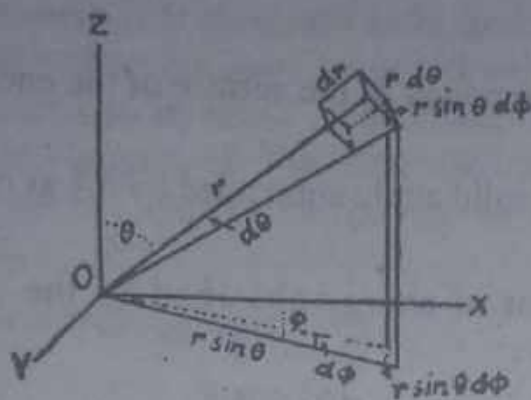


Fig. 19.4

Therefore, the amount of energy emitted by the elemental area  $dA$  in the direction between  $\theta$ ,  $\theta + d\theta$  and  $\phi$ ,  $\phi + d\phi$  is

$$dA e_{\lambda} d\lambda \cos\theta \sin\theta d\theta d\phi \quad (\text{ii})$$

Hence the total amount of energy emitted by  $dA$  for all wavelengths and for all possible values of  $\theta$  and  $\phi$  is

$$\begin{aligned} dA \int_0^{\infty} e_{\lambda} d\lambda \int_0^{\pi/2} \sin\theta \cos\theta d\theta \int_0^{2\pi} d\phi \\ = \pi dA \int_0^{\infty} e_{\lambda} d\lambda \end{aligned}$$

since the limits of integration extend for all wavelengths from 0 to  $\infty$  and the value of the second integral is  $\frac{1}{2}$  and that of the third  $2\pi$ .

$\therefore$  For the whole body the emission is given by

$$\pi \left( \sum dA \right) \int_0^{\infty} e_{\lambda} d\lambda \quad (\text{iii})$$

Let us now find the mutual radiation between  $dA$  and an elemental area  $dA'$  of the enclosure at a distance  $r$  from it. Let  $dA$  and  $dA'$  be so oriented that the normals to the surfaces make angle  $\theta$  and  $\theta'$  respectively with  $r$ . Then the amount of energy emitted by the elemental area  $dA'$  of the enclosure in the direction  $dA$  of the body is

$$dQ_{\lambda} = E_{\lambda} d\lambda dA' \cos\theta' \frac{dA \cos\theta}{r^2}$$

where

$E_\lambda$  is the emissive power of the surface of the enclosure

$\frac{dA \cos \theta}{r^2}$  is the solid angle subtended by  $dA$  at  $dA'$

Then the amount of energy absorbed by the elemental area  $dA$  of the body is

$$a_\lambda dQ_\lambda = a_\lambda E_\lambda d\lambda dA \cos \theta \frac{dA' \cos \theta'}{r^2}$$

$$= a_\lambda E_\lambda d\lambda dA \cos \theta dw$$

$$= a_\lambda E_\lambda d\lambda dA \cos \theta (\sin \theta d\theta d\phi)$$

where  $a_\lambda$  is the absorptive power of  $dA$

$$\text{and } dw = \frac{dA' \cos \theta'}{r^2} = \text{solid angle subtended by } dA' \text{ at } dA$$

$$= \sin \theta d\theta d\phi$$

Therefore, the total energy absorbed by  $dA$  from the total surface of the enclosure for all wavelengths is

$$dA \int_0^\infty a_\lambda E_\lambda d\lambda \int_0^{\pi/2} \cos \theta \sin \theta d\theta \int_0^{2\pi} d\phi$$

$$= \pi dA \int_0^\infty a_\lambda E_\lambda d\lambda$$

Thus, for the whole body the absorption is

$$\pi \left( \sum dA \right) \int_0^\infty a_\lambda E_\lambda d\lambda \quad (\text{iv})$$

In the equilibrium state, we must equate (iii) and (iv). Therefore,

$$\pi \left( \sum dA \right) \int_0^\infty e_\lambda d\lambda = \pi \left( \sum dA \right) \int_0^\infty E_\lambda a_\lambda d\lambda$$

$$\text{or, } \int_0^\infty e_\lambda d\lambda = \int_0^\infty E_\lambda a_\lambda d\lambda$$

The equality must hold for each portion of the spectrum i.e.,



any value of  $\lambda$ , because if it does not hold good for any portion of the spectrum *i.e.*, the absorbed part be not equal to the emitted part, there will be a continuous increase or decrease of radiant energy for that particular wavelength at the expense of energy of other wavelengths. In other words, the emissivity  $E_\lambda$  will vary with time, which is impossible.

Hence for each wavelength, we have

$$e_\lambda = a_\lambda E_\lambda \quad (19.4)$$

$$\text{or, } \frac{e_\lambda}{a_\lambda} = E_\lambda$$

If the same body  $dA$  be placed inside another enclosure of emissivity  $E'_\lambda$  at the same temperature  $T$ , but having different shape and nature of the walls, then it can be similarly shown that

$$e_\lambda = a_\lambda E'_\lambda \quad (\text{iii})$$

$$\text{or, } \frac{e_\lambda}{a_\lambda} = E'_\lambda$$

Since  $e_\lambda$ ,  $a_\lambda$  depend on the nature of the body and its temperature only and not upon its surroundings, hence

$$E_\lambda = E'_\lambda$$

If a black body *i.e.*, having  $a_\lambda = 1$  be placed inside the enclosure, then we have from eqn. (19.4)  $E_\lambda = e_\lambda$ , that is  $E_\lambda$  becomes equal to the emissive power of the black body, which is a constant at a particular temperature. Thus *radiation in any hollow enclosure is independent of the nature and shape of the walls and is identical with black body radiation at the same temperature*

So from eqn. (19.4) it immediately follows that at any temperature the ratio of the emissive to the absorptive power of a body is constant and is equal to the emissive power of a perfectly black body. This is called *Kirchhoff's law*.

### Importance of Kirchhoff's law : Applications

The importance of Kirchhoff's law can hardly be overestimated. In a sense, it opened up two entirely new branches of science, *viz.* *astrophysics* (physics of the sun and stars) and *spectroscopy*.

In 1680 Newton demonstrated by passing sunlight through a prism that it can be split into a spectrum of seven colours of the rainbow. In 1801, than a hundred year later, Fraunhofer repeated the experiment with better instruments and found, to his surprise, that the solar spectrum is not continuous but is crossed by a large number of dark lines. He noticed 500 of them (present number is about 20,000) and designated the prominent ones by the letters A, B, C, D ... But neither Fraunhofer nor his contemporaries could account for the origin of the dark lines. But he realized their great importance, measured them and catalogued their wavelengths. He also investigated the light from stars and found similar dark lines in their spectra. Subsequently, other physicists, notably, Fizeau observed that if the solar spectrum is examined side by side with the spectrum of a sodium flame, the two yellow lines of the sodium spectrum appear exactly at the place of Fraunhofer D-lines in the solar spectrum. Similar is the case with hydrogen spectrum. Although these facts paved the way for a possible solution, the dark lines of Fraunhofer remained unexplained, until Kirchhoff completely solved the problem on the basis of his law.

Kirchhoff supposed that the central core (photosphere) of the sun consists of a glowing mass that emits a continuous spectrum without the dark lines. But this light has to pass through a relatively cooler mantle surrounding the core (chromosphere). The mantle however is hot enough to have all the elements like  $H_2$ ,  $N_2$ ,  $O_2$ , Na, Cu, etc. in it in the gaseous form. Every substance emits radiation i.e., light when heated. Every substance has again got the power of absorbing light. Thus according to Kirchhoff, sodium can emit the D - lines when it is excited, hence it can also absorb the same light when white light falls upon it and allows other light to pass through it unaffected. The gases in the outer cooler mantle round the sun, therefore, deprive the continuous spectrum from the central core, of those particular lines they themselves can emit, and give rise to the dark lines. The D - lines (absorption spectrum of sodium) in the Fraunhofer spectrum prove that there is sodium in the sun's atmosphere. Similarly the other dark lines testify to the presence of their respective elements in the sun's atmosphere. The presence of dark lines in the stellar spectra is similarly accounted for. It is the analysis of such dark lines that one is able to tell of what elements the cooler atmosphere surrounding the sun and stars are composed of.

Kirchhoff's discovery is of still far-reaching consequence than the mere success in explaining the Fraunhofer dark lines. It clearly asserts that *every different* type of atom, when properly excited, emits light of a definite wavelength which is characteristic *of the atom*. So each atom can be identified by a particular line it emits. Thus the branch of *spectrum analysis* that aims in identifying elements by their characteristic lines was born. It is thrilling to note that as many as 40 new elements were thus added to the list already known.

### 19.8 Energy density and pressure of diffuse radiation

Let us find, in this section, an expression for the energy density of diffuse radiation, e.g., the radiation inside a uniformly heated enclosure of any shape at a certain temperature.

Let us consider an elemental surface  $dA$  drawn anywhere in a uniform temperature enclosure and let an isotropic radiation be incident on this area at an angle  $\theta$  with the normal (Fig 19.5). The amount of radiation passing through this area in a direction normal to the area is proportional to  $dA \cos \theta$ . Also the amount of radiation

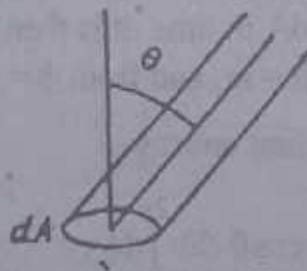


Fig. 19.5

propagated into a small solid angle  $dw$  is proportional to  $dw$ . Again, the radiation passing through any area is obviously proportional to time  $dt$ . Therefore, the amount of radiant energy passing through the area  $dA$  into the solid angle  $dw$  in time  $dt$  is given by

$$K dA \cos \theta dw dt \quad (i)$$

where  $K$  is a constant known as the *intensity of radiation*.



Let us suppose that the small solid angle lies between the direction  $\theta$  and  $\theta + d\theta$  and  $\phi$  and  $\phi + d\phi$ , where  $\phi$  signifies the direction of the radiation in azimuth. Then the solid angle  $dw$  is given by

$$dw = \sin \theta d\theta d\phi$$

The amount of radiant energy passing into the solid angle  $dw$  through the area  $dA$  in time  $dt$  is, therefore, given by

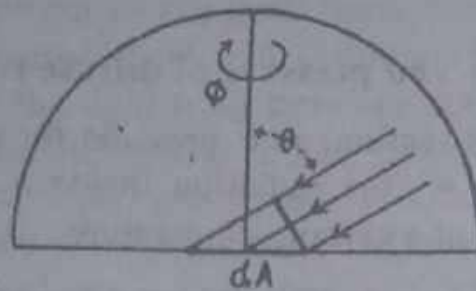


Fig. 19.6

$$KdA \cos \theta \sin \theta d\theta d\phi dt. \quad (ii)$$

Let us now suppose that the area  $dA$  is situated at the centre of a hemisphere of radius  $r$  (Fig. 19.6). The total amount of radiant energy passing downwards through the area  $dA$  in time  $dt$  is then obtained by integrating the expression (ii) from  $\theta = 0$  to  $\theta = \pi/2$  and from  $\phi = 0$  to  $\phi = 2\pi$ . Or,

the total amount of radiant energy

$$\begin{aligned} &= KdA dt \int_0^{\pi/2} \sin \theta \cos \theta d\theta \int_0^{2\pi} d\phi \\ &= \frac{1}{2} KdA dt \int_0^{\pi/2} 2 \sin \theta \cos \theta d\theta \int_0^{2\pi} d\phi \\ &= \frac{1}{2} KdA dt \int_0^{\pi/2} \sin 2\theta d\theta \int_0^{2\pi} d\phi \\ &= \frac{1}{2} KdA dt \left[ -\frac{\cos 2\theta}{2} \right]_0^{\pi/2} [\phi]_0^{2\pi} \\ &= \pi K dA dt \end{aligned} \quad (iii)$$

Now the radiation in a uniform enclosure at a constant temperature is the *full radiation* or *black body* radiation. Expression (iii) must, therefore, be the same as the amount of energy emitted from an area  $dA$  of a *full radiator* or a *black body* in time  $dt$ . Hence, the energy emitted by unit area of a black body in unit time, *i.e.*, the emissive power of the black body is  $\pi K$ .

Let us now consider a small element of volume  $v$  inside the enclosure (Fig. 19.7). From some point  $O$  inside  $v$  draw a sphere of radius  $r$ , which is large compared with the linear dimensions of  $v$ . Each ray of radiation that passes through the volume  $v$  may be supposed to come from some point on the surface of this sphere. Consider the radiation which passes through  $v$  coming from an infinitesimal area  $dA$  on the sphere. With a point on  $dA$  as the vertex and the radial line as axis, an elementary cone may be described. If this cone has a cross section  $\alpha$  at a distance  $r$  from the spherical surface, the solid angle is given by

$$dw = \frac{\alpha}{r^2}$$

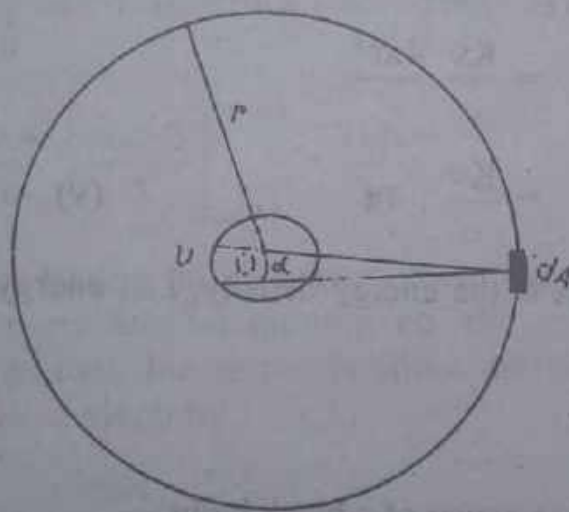


Fig. 19.7

If  $l$  is the length of the infinitesimal cone cut off by the volume  $v$ , a ray will take the time  $\lambda/c$  to travel through  $v$ , where  $c$  is the velocity of light. Hence in the expression (i), putting  $dt = \lambda/c$ ,  $dw = \alpha/r^2$  and  $\cos \theta = 1$ , since the linear dimensions of  $v$  are small compared with  $r$ , the amount of radiation coming from  $dA$  and contained within the infinitesimal cone cut off by  $v$  is given by

$$K dA \cdot \frac{\alpha}{r^2} \cdot \frac{l}{c}$$

$K$  being the specific intensity of radiation, *i.e.*, the amount of radiation proceeding in given direction per second per unit area per unit solid angle.

The total amount of radiation coming from  $dA$  and contained in the volume  $v$  is obtained by summing over the whole solid angle subtended by  $v$  at  $dA$  and is given by

$$K \frac{dA}{r^2} \sum \alpha l = \frac{K dA}{r^2 c} \cdot v \quad (\because \sum \alpha l = v) \quad (iv)$$

Summing the expression (iv) again over all elements  $dA$  of the sphere, the total amount of radiant energy contained in the volume  $v$  due to the entire enclosure at any instant is

$$\begin{aligned} \sum \frac{K dA}{r^2 c} \cdot v &= \frac{K v}{c} \sum \frac{dA}{r^2} \\ &= \frac{K v}{c} \cdot \frac{4\pi r^2}{r^2} \\ &= \frac{K v}{c} \cdot 4\pi \quad (v) \end{aligned}$$

Therefore, if  $E$  is the energy density, *i.e.*, energy per unit volume, we have

$$E = \frac{4\pi K}{c} \quad (19.5)$$

and the emissive power of a black body

$$= \pi K = \frac{1}{4} Ec.$$

Radiation being identical with light, it is expected that it must exert a small but finite pressure on surfaces on which it is incident. Let us suppose that instead of volume  $v$  we have unit area of a perfectly absorbing surface at the center of the sphere. We shall find the pressure exerted on one side of the surface in the uniform temperature enclosure.



The amount of radiation coming from an annulus of width  $d\theta$  on the sphere in unit time on the unit area of the surface is given by

$$2\pi K \cos\theta \sin\theta d\theta.$$

If uninterrupted, this radiation will travel a distance in unit time where  $c$  is the velocity of light. Hence the corresponding energy density is given by

$$\frac{2\pi K \cos\theta \sin\theta d\theta}{c}$$

The normal pressure is then given by

$$\frac{2\pi K \cos\theta \sin\theta d\theta}{c} \times \cos\theta = \frac{2\pi K \cos^2\theta \sin\theta d\theta}{c}$$

since the radiation is incident at an angle  $\theta$  with the normal to the unit area. Hence the total pressure due to the incident radiation is given by

$$\begin{aligned} \frac{2\pi K}{c} \int_0^{\pi/2} \cos^2\theta \sin\theta d\theta &= -\frac{2\pi K}{c} \int_1^0 \cos^2\theta d(\cos\theta) \\ &= -\frac{2\pi K}{c} \left[ \frac{\cos^3\theta}{3} \right]_{\cos\theta=1}^{\cos\theta=0} = \frac{2\pi K}{3c} \end{aligned} \quad (19.6)$$

Since the radiation moving away from the unit area considered is exactly the same as that moving up, the reaction produces the same pressure as that due to the incident radiation. Hence, the total pressure  $P$  is, then, given by

$$P = \frac{2\pi K}{3c} \times 2 = \frac{4\pi K}{3c} = \frac{E}{3} \quad (19.5)$$

$$\text{since } E = 4\pi K/C$$

### 19.9 Stefan-Boltzmann's law: Total energy of radiation

The dependence of total radiation from a radiator on its temperature was first studied experimentally by Stefan. According to him, the total radiation is proportional to the fourth power of the absolute temperature of the radiator. Boltzmann on the supposition

that the black body radiation behaves as a perfect gas, applied the laws of thermodynamics to this radiation and deduced the Stefan law theoretically. The law is, therefore, generally referred to as Stefan-Boltzmann law. It can be stated as follows:

*'If a black body at an absolute temperature  $T$  be surrounded by another black body at absolute temperature  $T_0$ , the amount of radiation emitted per second per unit area of the former is*

$$E = \sigma (T^4 - T_0^4)$$

*where  $\sigma$  is known as Stefan's constant.*

### Derivation

Let  $E$  be the energy density of radiation in an enclosure at constant absolute temperature  $T$ ,  $V$  be the volume of the enclosure and  $p$  the pressure of radiation. Then the total energy of radiation is  $U = EV$  and  $p$  and  $E$  are functions of  $T$ . Let us suppose that a small quantity of heat  $dQ$  flows into the enclosure from outside and at the same time the volume of the enclosure increases by  $dV$ . Then heat  $dQ$  is spent up in two ways : (i) in changing the energy content of the enclosure and (ii) in doing external work when the enclosure expands by  $dV$  against the pressure  $p$ . Which is equal to  $p dV$ . Then, from the first law of thermodynamics, we can write

$$dQ = dU + pdV$$

$$= d(EV) + \frac{E}{3}dV$$

$$= EdV + VdE + \frac{E}{3}dV$$

$$= VdE + \frac{4}{3}EdV \quad (i)$$

The change in entropy in the process, as given by the second law of thermodynamics, is

$$dQ = T dS \quad (ii)$$

$$\text{or, } dS = \frac{dQ}{T} = \frac{V}{T}dE + \frac{4}{3} \frac{E}{T}dV \quad (iii)$$

But since  $S = f(E, V)$

$$dS = \left( \frac{\partial S}{\partial E} \right)_V dE + \left( \frac{\partial S}{\partial V} \right)_E dV \quad (vi)$$

From (iii) and (iv), we thus have

$$\left( \frac{\partial S}{\partial E} \right)_V = \frac{V}{T} \text{ and } \left( \frac{\partial S}{\partial V} \right)_E = \frac{4}{3} \frac{E}{T} \quad (v)$$

Again, since  $dS$  is a perfect differential, we have

$$\left( \frac{\partial^2 S}{\partial V \partial E} \right) = \left( \frac{\partial^2 S}{\partial E \partial V} \right)$$

$$\therefore \text{From (v)} \quad \frac{\partial}{\partial V} \left( \frac{V}{T} \right) = \frac{4}{3} \frac{\partial}{\partial E} \left( \frac{E}{T} \right) \quad (vi)$$

Now, the temperature  $T$  is independent of  $V$  and is a function of  $E$  alone. So from (vi)

$$\frac{1}{T} = \frac{4}{3} \left( \frac{1}{T} - \frac{E}{T^2} \frac{dT}{dE} \right)$$

$$\text{or, } 1 = \frac{4}{3} - \frac{4}{3} \frac{E}{T} \cdot \frac{dT}{dE}$$

$$\text{or, } \frac{dE}{E} = 4 \frac{dT}{T}$$

Integrating,

$$\text{Log}_e E = 4 \log_e T + \log_e \alpha \text{ (constant)}$$

$$\text{or, } E = \alpha T^4 \quad (19.8)$$

where  $\alpha$  is called the *total density radiation constant*.

Thus *the energy density* of full radiation (black body radiation) is proportional to the fourth power of the *absolute temperature*.

Substituting the value of  $E$  from eqn. (19.5), we get

$$\frac{4\pi K}{c} = \alpha T^4$$



$$\text{or, } \pi K = \frac{\alpha c}{4} \cdot T^4 = \sigma T^4 \quad (19.9)$$

Now  $\pi K$  is the emissivity of the body.

This is **Stefan- Boltzmann law** and may be stated as follows:

*The total energy emitted per second per unit area (emissive power) of a black- body is proportional to the fourth power of the absolute temperature of the body.*

$\sigma$  is called the Stefan (or Stefan- Boltzmann) constant and its value is found experimentally to be

$$\begin{aligned} \sigma &= 5.735 \times 10^{-5} \text{ ergs cm}^{-2} \text{ deg}^{-4} \text{ sec}^{-1} \\ &= (5.6696 \pm 0.0010) \times 10^{-8} \text{ W. m}^{-2} \cdot \text{K}^{-4} \end{aligned}$$

Eqn. (19.9) refers to emission only. If the black - body be surrounded by a perfectly black surface at temperature  $T_0$ , the latter will also emit radiation equal to  $\sigma T_0^4$  which will be incident on the first body and the gain of heat absorption is  $\sigma T_0^4$  and so the net loss of energy per sec per unit area is given by

$$E = \sigma (T^4 - T_0^4) \quad (19.10)$$

Expression (19.10) may be stated as follows:

*If a black body at absolute temperature  $T$  be surrounded by another black body at absolute temperature  $T_0$ , the amount of energy  $E$  lost per sec per sq. cm. of the body is given by  $E = \sigma (T^4 - T_0^4)$ . This is another statement of Stefan- Boltzmann law.*

If the body has an emissivity  $e$ , then

$$E = e\sigma [T_1^4 - T_2^4]$$

### 19.10 Energy distribution in black body radiation: Wien's law

Radiation emitted by a black body is not confined to a single wavelength but is spread over a continuous spectrum. Stefan- Boltzmann law tells us how the total energy of radiation is related to the temperature of the source of radiant energy. It tells us nothing about how this total energy is distributed amongst the different wavelengths. The attempts to find an answer to this question

constitute a long and painstaking history of the growth and development of physics, culminating into the revolutionary quantum theory of radiation of M. Planck.

Considered from the point of view of quantum theory, the problem appears to be rather easy. But before the advent of the quantum theory, scientists like Wien, Rayleigh, Jeans, and Planck were not aware of the mechanism of emission and absorption of radiation. In fact, this mechanism came to be known only through Planck's investigation with black body radiation. They had at their disposal only the laws of classical thermodynamics and electromagnetism. It is both interesting as well as instructive to follow how a complete solution of the problem was arrived at, step by step, with much toil and labour.

The first step in this direction was taken by Wien in 1893. He showed that the spectral distribution of energy emitted by a black body at a temperature  $T$  can be expressed as

$$E_{\lambda}d\lambda = C\lambda^{-5} f(\lambda T) d\lambda$$

where  $E_{\lambda}d\lambda$  is the energy density of radiation between the wavelengths  $\lambda$  and  $\lambda + d\lambda$ ,  $C$  is a constant and  $f(\lambda T)$  is a function of the product  $\lambda T$ .

The above relation is known as **Wien's law of energy distribution**.

It was also shown by Wien that if radiation of a particular wavelength at a certain temperature is adiabatically changed to another wavelength, then the temperature changes in the inverse ratio, that is,

$$\lambda_1 T_1 = \lambda_2 T_2 = \lambda_3 T_3 = \dots\dots\dots$$

or,  $\lambda T = \text{constant}$ .

This relation is known as **Wien's displacement law**.

#### Derivation of Wien's law:

We shall first derive Wien's displacement law purely from thermodynamic consideration, and then Wien's law of spectral distribution of energy.

Let us consider an imaginary experiment performed with a spherical enclosure filled with black radiation characterized by its thermodynamic temperature  $T$ . The walls of the enclosure are perfectly but diffusely (angle of incidence  $\neq$  angle of reflection) reflecting, capable of moving slowly outward. Let the enclosure be allowed to expand adiabatically by supposing its walls moving out slowly with a small uniform velocity  $u$  ( $\ll c$ ). Due to the reversible adiabatic expansion, the temperature  $T$  will fall to say  $T'$ . However as can be proved by the application of thermodynamics, the quality of radiation will remain unaffected, *i.e.*, the radiation will remain a black radiation characteristic of the lower temperature.

Although the nature of the radiation remains unchanged, the wavelength of the radiation will change due to *Doppler shift* on each reflection at the moving wall. Let  $\lambda$  and  $\lambda'$  be the wavelengths of a particular wave before and after each reflection on the wall respectively. Then by Larmor's theorem, we have

$$\frac{\lambda'}{\lambda} = \frac{c+u}{c-u} = 1 + \frac{2u}{c} \quad \text{approximately}$$

$$\text{or, } \lambda' = \lambda \left(1 + \frac{2u}{c}\right)$$

if the reflecting surface is receding from radiation which is incident normally

[According to Larmor's original equation,  $\frac{\lambda'}{\lambda} = \frac{c-u}{c+u}$ , where the reflecting surface is approaching the incident radiation. But here the walls are receding from radiation with velocity  $u$ . Thus, the sign of  $u$  being reversed, we get

$$\begin{aligned} \frac{\lambda'}{\lambda} &= \frac{c+u}{c-u} = \frac{1 + \frac{u}{c}}{1 - \frac{u}{c}} \\ &= \left(1 + \frac{u}{c}\right) \left(1 - \frac{u}{c}\right)^{-1} \end{aligned}$$



$$= \left(1 + \frac{u}{c}\right) \left(1 + \frac{u}{c}\right)^{-1}$$

$$= 1 + \frac{u}{c}, \text{ neglecting } \frac{u^2}{c^2} \Bigg]$$

If the radiation falls at an angle  $\alpha$  with the normal to the receding surface, then the resolved part  $u \cos \alpha$  of the velocity  $u$  along the direction in which the ray is travelling is only effective in changing the wavelength. So, for oblique incidence, we have

$$\lambda' = \lambda \left(1 + \frac{2u \cos \alpha}{c}\right)$$

For a narrow beam incident on the boundary of the spherical enclosure at B, the angle of reflection, however, will be different from  $\alpha$ , since the walls reflect *diffusely*.

Let us confine our attention to a single ray. If the ray starts from the point A (Fig. 19.8) at time  $t = 0$  and  $s_1, s_2, s_3, \dots, s_n$  is the lengths of the chords over which it travels between successive reflections in unit time, then we have

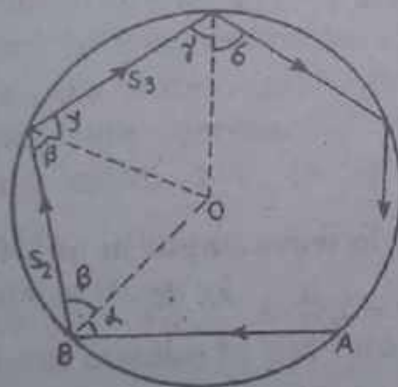


Fig. 19.8

$$s_1 + s_2 + \dots + s_n = \sum_{i=1}^n s_i = c$$

From the figure

$$\cos \alpha = \frac{s_1}{2r}, \cos \beta = \frac{s_2}{2r} \text{ and so on,}$$

So after the first reflection the wavelength of the radiation given by

$$\begin{aligned} \lambda' &= \lambda \left( 1 + \frac{2u}{c} \cos \alpha \right) = \lambda \left( 1 + \frac{2u}{c} \cdot \frac{s_1}{2r} \right) \\ &= \lambda \left( 1 + \frac{u}{c} \cdot \frac{s_1}{r} \right) \end{aligned}$$

After  $n$  successive reflections, the changed wavelength (or final wavelength) is given by

$$\begin{aligned} \lambda_n &= \lambda_f = \lambda \left( 1 + \frac{u}{c} \cdot \frac{s_1}{r} \right) \left( 1 + \frac{u}{c} \cdot \frac{s_2}{r} \right) \dots \left( 1 + \frac{u}{c} \cdot \frac{s_n}{r} \right) \\ &= \lambda \left\{ 1 + \frac{u}{cr} (s_1 + s_2 + \dots + s_n) \right\} \\ &= \lambda \left( 1 + \frac{u}{cr} \sum_{i=1}^n s_i \right) \\ &= \lambda \left( 1 + \frac{u}{cr} \cdot c \right) \\ &= \lambda \left( 1 + \frac{u}{r} \right) \end{aligned}$$

Hence the change in wavelength in unit time is given by

$$\lambda_f - \lambda = \frac{d\lambda}{dt} = \lambda \frac{u}{r} = \frac{\lambda}{r} \cdot \frac{dr}{dt} \quad (\because u = dr / dt)$$

$$\text{or, } \frac{d\lambda}{\lambda} = \frac{dr}{r} \quad (i)$$

Integrating,

$$\log_e \lambda = \log_e r + C \quad (C = \text{constant of integration})$$

$$= \log_e r + \log_e K \quad (K = \text{another constant})$$

$$\text{or, } \log_e \frac{\lambda}{r} = \log_e K$$

$$\text{or, } \frac{\lambda}{r} = K = \text{constant.}$$

$$\therefore \frac{\lambda}{r} = \frac{\lambda'}{r'} \quad (\text{ii})$$

where  $\lambda$  and  $\lambda'$  are the wavelengths of the particular ray we are considering corresponding to radii  $r$  and  $r'$  respectively.

For a ray whose wave-length is very near to  $\lambda$ , say  $\lambda + d\lambda$  we can write

$$\frac{\lambda + d\lambda}{r} = \frac{\lambda' + d\lambda'}{r'}$$

$$\text{or, } \frac{\lambda}{r} + \frac{d\lambda}{r} = \frac{\lambda'}{r'} + \frac{d\lambda'}{r'}$$

$$\therefore \frac{d\lambda}{r} = \frac{d\lambda'}{r'} \quad (\text{iii})$$

Now the given amount of energy, we started with, will now be distributed over a larger volume. Moreover, a part of it is spent in doing external work *i.e.*, the work done by the radiation pressure on the walls of the enclosure during its expansion. Hence the temperature of the enclosure will change. The process being adiabatic

$$dQ = dU + pdV = 0$$

where  $dU$  is the total change in the internal energy and  $pdV$  is the external work done by the radiant energy.

$$\text{But } U = E.V \quad \text{and } p = \frac{1}{3} E$$

where  $V$  is the volume of the enclosure and  $E$  is the energy density inside the enclosure.

Thus from above.



$$d(EV) + \frac{1}{3} E dV = 0$$

$$\text{or, } E dV + V dE + \frac{1}{3} E dV = 0$$

$$\text{or, } \frac{4}{3} E dV + V dE = 0$$

$$\text{or, } \frac{4}{3} \cdot \frac{dV}{V} = -\frac{dE}{E}$$

Integrating

$$\frac{4}{3} \log_e V = -\log_e E + C_1 \quad (C_1 = \text{constant of integration})$$

$$\text{or, } \frac{4}{3} \log_e V + \log_e E = \log_e K_1 \quad (K_1 = \text{another constant})$$

$$\text{or, } E V^{4/3} = K_1 = \text{constant}$$

$$\text{or, } E V^{4/3} = E' V'^{4/3} \quad (\text{iv})$$

where  $E'$  is the energy density in the volume  $V'$ . But we know that  $E = \sigma T^4$  and  $E' = \sigma T'^4$ . Therefore, from eqn. (iv) we have

$$\sigma T^4 V^{4/3} = \sigma T'^4 V'^{4/3}$$

$$\text{or, } VT^3 = V'T'^3 \quad (\text{v})$$

Again, since we are considering radiation inside a spherical enclosure,  $V = \frac{4}{3}\pi r^3$  and  $V' = \frac{4}{3}\pi r'^3$ . Eqns. (iv) and (v) may therefore, be written as

$$Er^4 = E'r'^4 \quad (\text{vi})$$

$$\text{and } Tr = T'r' \quad (\text{vii})$$

From eqns. (ii) and (vii) we have

$$\frac{\lambda}{\lambda'} = \frac{r}{r'} \quad \text{and} \quad \frac{T'}{T} = \frac{r}{r'}$$

Hence combining eqns. (ii) and (vii) we have

$$\frac{\lambda}{\lambda'} = \frac{T'}{T}$$

$$\text{or, } \lambda T = \lambda' T'$$

$$\text{or, } \lambda T = \text{constant.} \quad (19.11)$$

Thus, if radiation of a particular wavelength at a given temperature is adiabatically changed to another wavelength, the temperature changes in the inverse ratio.

Equation (19.11) is **Wien's displacement law** of radiation.

### Wien's distribution law

We shall first prove that  $E_\lambda \lambda^5 = \text{const.}$  For this let us isolate waves of lengths between  $\lambda$  and  $\lambda + d\lambda$  in the spherical chamber and subject only these waves to an adiabatic expansion. The work done for the expansion to take place is  $\frac{1}{3} E_\lambda d\lambda \Delta V$  where  $\Delta V$  is the expansion in volume. This must be equal to the decrease in the total energy content, that is  $-\Delta (E_\lambda d\lambda V)$ .

$$\begin{aligned} \therefore \frac{1}{3} E_\lambda d\lambda \Delta V &= -\Delta (E_\lambda d\lambda V) \\ &= -(\Delta E_\lambda) d\lambda \cdot V - E_\lambda \Delta (d\lambda \cdot V) \\ &= -\Delta E_\lambda d\lambda V - E_\lambda \Delta (d\lambda) V - E_\lambda d\lambda \Delta V \end{aligned}$$

Dividing throughout by  $E_\lambda V d\lambda$ , we have

$$\begin{aligned} \frac{1}{3} \frac{\Delta V}{V} &= -\frac{\Delta E_\lambda}{E_\lambda} - \frac{\Delta (d\lambda)}{d\lambda} - \frac{\Delta V}{V} \\ \text{or, } \frac{4}{3} \frac{\Delta V}{V} &= -\frac{\Delta E_\lambda}{E_\lambda} - \frac{\Delta \lambda}{\lambda} \quad (\text{ix}) \end{aligned}$$

It can be proved that  $d\lambda$  changes in the same way as  $\lambda$ .

$$\text{So, } \frac{\Delta (d\lambda)}{d\lambda} = \frac{\Delta \lambda}{\lambda}$$

Again,

$$V = \frac{1}{3} \pi r^3 \quad \text{so that} \quad \Delta V = 4\pi r^2 \Delta r$$

$$\therefore \frac{\Delta r}{r} = \frac{1}{3} \frac{\Delta V}{V} = \frac{\Delta \lambda}{\lambda} \quad (\text{from eqn. i}) \quad (x)$$

Substituting this value in eqn. (ix)

$$4 \frac{\Delta \lambda}{\lambda} = - \frac{\Delta E_\lambda}{E_\lambda} - \frac{\Delta \lambda}{\lambda}$$

$$\text{or, } 5 \frac{\Delta \lambda}{\lambda} + \frac{\Delta E_\lambda}{E_\lambda} = 0 \quad (xi)$$

Integrating eqn. (xi), we get

$$E_\lambda \lambda^5 = \text{const} = E'_\lambda \lambda'^5 \quad (19.12)$$

where  $E'_\lambda$  is the energy density of the radiation of wavelength  $\lambda'$  to which  $\lambda$  is transformed as a result of adiabatic expansion. This will correspond to equilibrium radiation density corresponding to temperature  $T' = \lambda T / \lambda'$  (eqn. 19.11)

Plainly,  $E_\lambda$  is a function of  $T$ . So the constant in eqn. (19.12) must involve  $T$  and must remain constant throughout the adiabatic change of wavelength. It has already been proved that for an adiabatic change  $\lambda T = \text{constant}$  (eqn. 19.11). So this constant must be a function of the product  $\lambda T$ .

$$\therefore E_\lambda d\lambda = \frac{c}{\lambda^5} f(\lambda T) d\lambda \quad (19.13)$$

Eqn. (19.13) is the common expression for **Wien's distribution law**.

Eqn. (19.13) can be expressed in another form as shown below:

$$\begin{aligned} \therefore E_\lambda d\lambda &= \frac{c}{\lambda^5} f(\lambda T) d\lambda \\ &= \frac{cT^5}{\lambda^5 T^5} f(\lambda T) d\lambda \\ &= cT^5 F(\lambda T) d\lambda \end{aligned} \quad (19.14)$$



where the function  $F(\lambda T) = (\lambda T)^{-5} f(\lambda T)$

From eqn. (19.14) we have

$$\frac{E_\lambda}{E_{\lambda'}} = \left(\frac{T}{T'}\right)^5$$

Eqn. (19.14) thus leads to an important information - *the radiation density increases directly as the fifth power of absolute temperature.*

Fig. 19.9 shows a plot of  $E_\lambda$  against  $\lambda$  at different temperatures. As can be seen from the graph, if the distribution at a certain temperature say  $T$  be given, that at another higher temperature  $T'$  can be obtained graphically, simply by shortening the abscissa in the ratio  $(T/T')$  and enlarging the corresponding ordinate in the ratio  $(T'/T)^5$ . Consequently, the curve becomes higher, but more closed up and the total area representing the intensity is changed in the ratio  $(T'/T)^4$ . The wavelength interval  $(d\lambda)$  is also shortened in the ratio

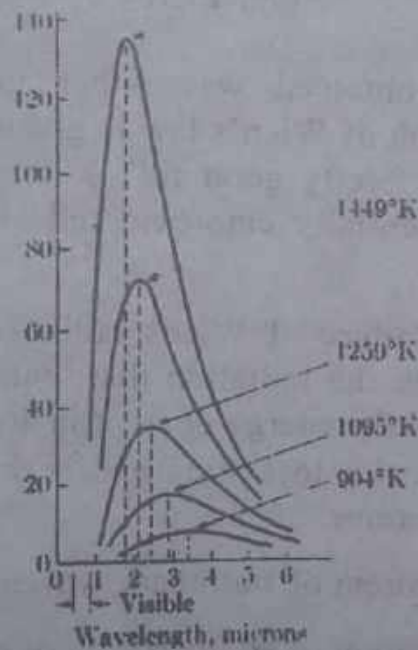


Fig. 19.9

ratio  $(T/T')$ . If  $\lambda_m$  and  $\lambda'_m$  are the wavelengths interval corresponding to the points of maximum emission at  $T$  and  $T'$  respectively, then

$$\lambda_m T = \lambda'_m T' = \text{constant.}$$

### 19.11 Rayleigh- Jeans' law

Wien's distribution law is essentially empirical and contains two constants whose values can be adjusted. Wien chose these values

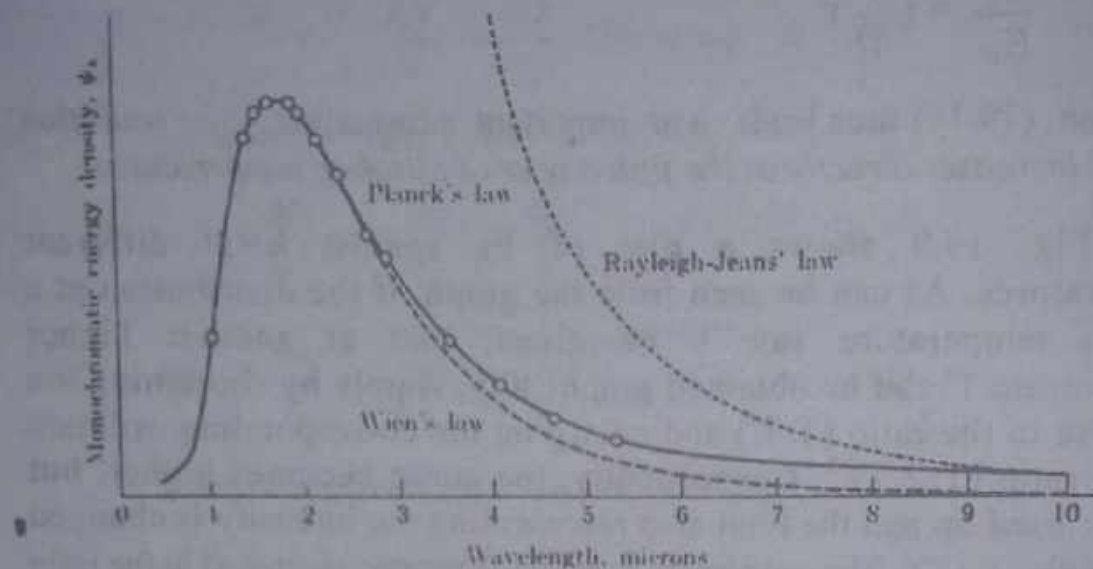


Fig. 19.10

so that the fit he obtained was rather good except at long wavelengths. The graph of Wien's law is given by the dashed curve in Fig. 19.10. But a "pretty good fit" is not good enough, and a formula which is essentially empirical tells us nothing about the nature of the radiation.

Because of the failure of Wiens radiation law, Rayleigh and Jeans set out to derive the radiation distribution law in a rigorous manner. The law gives the energy content in an isothermal enclosure in a wavelength interval  $\lambda$  to  $\lambda + d\lambda$  and is derived on the basis of the following two theorems:

- (i) the theorem of stationary waves in a hollow space
- and (ii) the theorem of equipartition of energy.

The black body chamber is filled with diffuse radiation of all frequencies between 0 and  $\infty$ . This radiation is composed of electromagnetic waves in space. They are reflected time and again from the walls forming standing waves in space of the enclosure. A standing wave of a given frequency interacts with oscillating atoms in the wall. Energy can flow in either direction from the wall to the

field or conversely. The temperature of the field is therefore the same as the temperature of the walls. To derive the law giving the energy content in an isothermal enclosure in a wavelength interval  $\lambda$  and  $\lambda + d\lambda$ , one needs to know how many *different* standing waves, called modes of radiation, there are in this wavelength interval. Each mode is then assumed to have an average energy of  $kT$ .

A stretched string can vibrate in one segment giving out the fundamental, or in two, three or more segments giving out the overtones. Each of these stationary vibrations is independent in that the energy of each can be varied independent of others. Rayleigh applied this to the waves of radiant energy.

Let two parallel boundaries be separated by a distance  $a$  and let a plane wave of wavelength  $\lambda$  move between these boundaries at right angles to them. Stationary waves with characteristic nodes  $N$ 's and antinodes  $A$ 's will be produced by waves reflected from the boundary (Fig. 19.11).

Let the incident and reflected waves be represented by

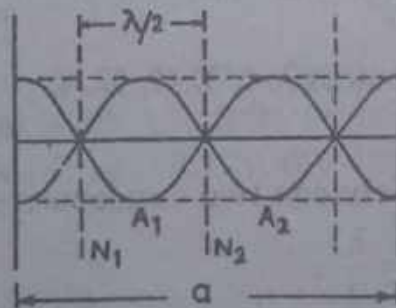


Fig. 19.11

$$y_1 = A \sin \frac{2\pi}{\lambda} (ct - x)$$

$$\text{and } y_2 = A \sin \frac{2\pi}{\lambda} (ct + x)$$

where  $c$  is the velocity of the wave.

The resultant displacement  $y$  of the particle due to interference of the two wave trains is, therefore,



$$\begin{aligned}
 y &= y_1 + y_2 \\
 &= A \sin \frac{2\pi}{\lambda} (ct - x) + A \sin \frac{2\pi}{\lambda} (ct + x) \\
 &= 2A \cos \frac{2\pi x}{\lambda} \sin \frac{2\pi ct}{\lambda} \\
 &= C \sin \frac{2\pi ct}{\lambda} \quad (i)
 \end{aligned}$$

where  $C = 2A \cos \frac{2\pi x}{\lambda}$  and is the resultant amplitude of vibration. As can be seen,  $C$  is a function of  $x$  but is independent of  $t$ .

From (i)  $y = 0$  for all  $t$ , if  $\cos \frac{2\pi x}{\lambda} = 0$ ;

$$\text{or, } \frac{2\pi x}{\lambda} = (2r + 1) \frac{\pi}{2}$$

$$\text{or, } x = (2r + 1) \frac{\lambda}{4}, r = 0, 1, 2, \dots$$

Thus particles at distances  $\lambda/4, 3\lambda/4, 5\lambda/4, \dots$  etc. are permanently at rest – the nodes.

$\therefore$  The distance between two consecutive nodal points i.e., the nodal separation  $= \frac{2\lambda}{4} = \frac{\lambda}{2}$ .

If there are  $n_1$  nodal points, then  $n_1 (\lambda / 2) = a$ .

If the walls are assumed to be perfectly reflective, there would be no exchange of energy between the wall and the electromagnetic field. However, in diffuse radiation, the wave meets the reflecting surface at some angle, say,  $\theta$ .

$$\text{Then } n_1 \frac{\lambda}{2} = a \cos \theta.$$

So, if  $\theta_1, \theta_2$  and  $\theta_3$  be the angles which the normal to the plane wave front makes with the normals to the three pairs of faces of a cube of side  $a$ , then

$$N_1 \frac{\lambda}{2} = la, \quad n_2 \frac{\lambda}{2} = ma, \quad n_3 \frac{\lambda}{2} = na \quad (\text{ii})$$

where  $l$ ,  $m$  and  $n$  are the direction cosines and  $n_1$ ,  $n_2$ , and  $n_3$  are the nodal planes parallel to the first, second and the third pair of faces respectively.

From (ii) we have

$$\begin{aligned} n_1^2 + n_2^2 + n_3^2 &= \frac{4}{\lambda^2} (l^2 + m^2 + n^2) a^2 \\ &= \frac{4a^2}{\lambda^2} \quad (\because l^2 + m^2 + n^2 = 1). \end{aligned}$$

$$\therefore (n_1^2 + n_2^2 + n_3^2)^{1/2} = \frac{2a}{\lambda}$$

Thus, there will be a system of stationary or proper vibrations with wavelength  $\lambda$  corresponding to each set of whole numbers  $n_1$ ,  $n_2$  and  $n_3$ .

So, if a cubic lattice is constructed in the positive octant with three rectangular axes with summits as points with whole number (positive) co-ordinates (Fig. 19.12), the distance of one such points

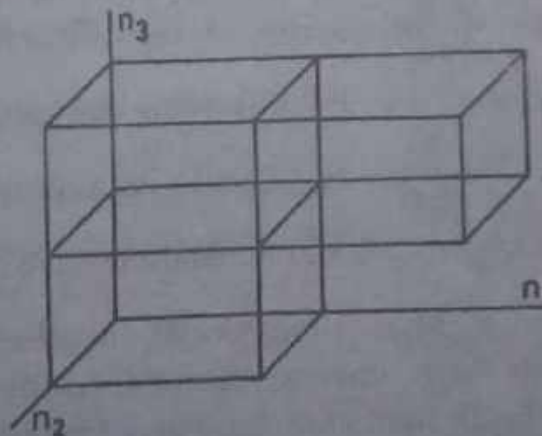


Fig. 19.12

$(n_1, n_2, \text{ and } n_3)$  from the origin is given by

$$r = (n_1^2 + n_2^2 + n_3^2)^{1/2} = \frac{2a}{\lambda} = \frac{2a\nu}{c}$$

where  $\nu$  and  $c$  are the frequency and velocity of the waves respectively.

Let us now find out the number of proper vibrations between frequencies  $\nu$  and  $\nu + d\nu$ . This number will be equal to the number of points in the lattice between the spheres of radii  $r (= 2a\nu / c)$  and  $r + dr [= 2a(\nu + d\nu) / c]$ . Since there is on an average only one point in unit volume, the number sought  $dn$  is equal to the volume cut out from the positive octant by the spheres .

$$\therefore dn = \frac{1}{8} 4\pi r^2 dr = 4\pi a^3 \nu^2 d\nu / c^3 = 4\pi V \nu^2 d\nu / c^3 \quad (\text{iii})$$

where  $V$  is the volume of the cube.

Hence, the possible modes of vibration per unit volume are  $4\pi \nu^2 d\nu / c^3$ .

The wave motion is supposed to be electromagnetic radiation in vacuum, and since each ray can be regarded as a superposition of two rays polarized at right angles to each other, expression (iii) should be doubled. The number of proper vibrations per unit volume between the frequencies  $\nu$  and  $\nu + d\nu$  is  $8\pi \nu^2 d\nu / c^3$ .

Once the number of proper vibrations per unit volume has been established, the distribution of energy can be derived, by the principle of equipartition of energy. A one dimensional harmonic oscillator has one degree of freedom having a kinetic energy  $\frac{1}{2} kT$ . Since the average potential energy equals its average kinetic energy ( $\frac{1}{2} kT$ ), the total average energy is  $kT$ . A standing wave of electromagnetic radiation represents a one dimensional oscillator and should therefore have an average energy of  $kT$ . To derive the energy content in an isothermal enclosure between frequencies  $\nu$  and  $\nu + d\nu$ , the number of proper vibrations per unit volume between these frequencies should be multiplied by  $kT$ . The energy density between the frequencies  $\nu$  and  $\nu + d\nu$  is, therefore,



$$E_\nu d_\nu = \frac{8\pi\nu^2 d_\nu}{c^3} kT \quad (19.15)$$

In terms of wavelength, expression (i $\nu$ ) becomes

$$E_\lambda d_\lambda = \frac{8\pi kT}{\lambda^4} d\lambda \quad (19.16)$$

This is Rayleigh-Jean's law. The law, along with Wien's law, is plotted in Fig. 19.10 for comparison with the experimental data. At first glance, Rayleigh-Jean's law appears vastly inferior to Wien's law. Although it fits well for long wavelengths, at short wavelengths or high frequencies it heads towards infinity in what has been dramatically called the "ultraviolet catastrophe". Theoretically, however, Rayleigh-Jean's law should be taken far more seriously than Wien's. It was derived rigorously on the basis of classical physics. It involves no arbitrary constants and where it does fit the experimental curve, it fits exactly. If the failure of the Wien's law was "too bad", then the failure of Rayleigh-Jean's law presented a crisis. It was a clear indication that classical theory was unable to account for an important experimental observation. This, then, was the situation to which Max Planck addressed himself.

### 19.12 Planck's law of radiation

Planck's first approach was essentially empirical. He found that by slightly changing Wien's formula to  $E_\lambda d\lambda = \frac{c_1 \lambda^{-5}}{e^{c_2/RT} - 1} d\lambda$  and by adjusting Wien's constants a formula at long wavelengths can be obtained that fitted the experimental curve everywhere. Planck knew that he had the correct formula and that it ought to be derivable. He tried every conceivable method to derive this correct formula from classical physics. Finally he was forced to conclude that there was no flaw in Rayleigh's derivation and that the flaw must lie in classical theory itself. Planck had to eliminate the 'ultraviolet catastrophe, which came into Rayleigh's derivation because of the assumption that the radiation standing waves can vibrate in an infinite number of modes. An average energy  $kT$  is associated with each one of these modes and

therefore the total energy content of the radiation field would be infinite. This, however, is clearly unreasonable. Perhaps the high-frequency modes are not excited or they are hard to excite. Let us again consider the principle of equipartition of energy.

Rayleigh, in effect, assumed that the energy content in one mode of oscillation could have any value from zero up and that the probability of finding an oscillator with energy is given by Boltzmann's distribution law. This gives an average energy

$$\bar{E} = \frac{\int E e^{-E/kT} dE}{\int e^{-E/kT} dE} = kT.$$

Another way of expressing this is to say that the modes can be in different energy states instead of assuming a continuum of states with a population proportional to the probability given by Boltzmann's law (Fig. 19.13a).

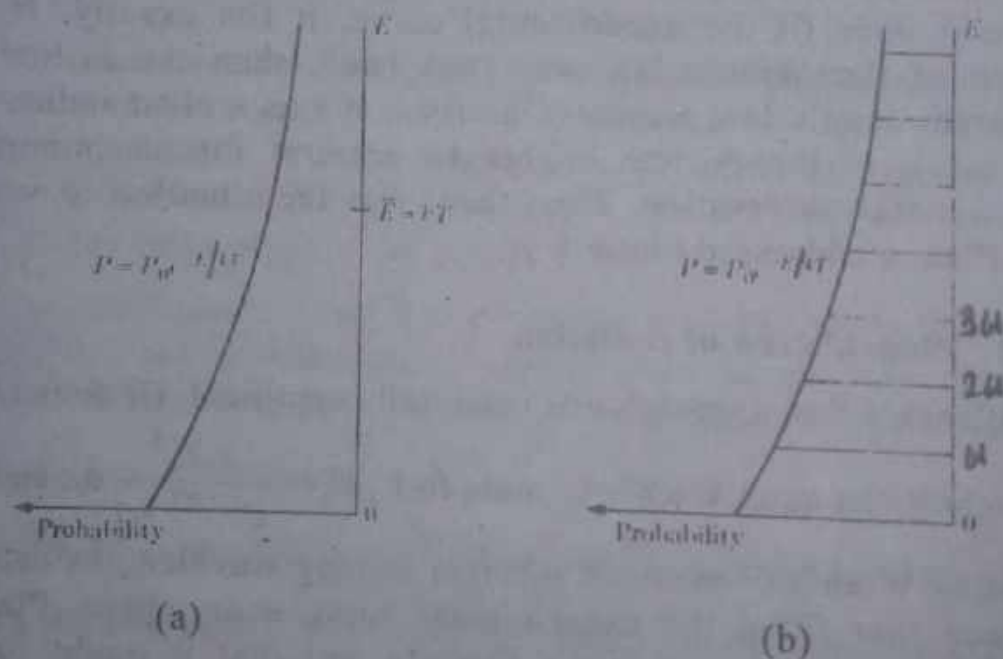


Fig. 19.13

Planck gave up this classical hypothesis of continuous emission of radiation and postulated, instead, that each mode could only receive energy from the walls in discrete steps. The energy content of one mode would then be  $0, \epsilon, 2\epsilon, 3\epsilon$ , etc. or in general,  $m\epsilon$  where  $m$  is an integer (Fig. 19.13b). Fig. 19.13(a) indicates the

probability  $P$  of finding an energy content  $\epsilon$  in a given mode of oscillation according to classical theory (Rayleigh) while (b) indicates the same as (a), but energy content can only take on discrete values,  $E = m\epsilon$  (Planck). As drawn, this is an *energy-level diagram* with the horizontal coordinate indicating *relative population* in each level. According to the procedure used by Boltzmann in 1877 to determine the distribution of kinetic energy among the molecules in a gas, the number of oscillators having the energy  $m\epsilon$  is

$$n_m = n_0 e^{-m\epsilon/kT}$$

The energy contributed by the  $n_m$  oscillators is obviously

$$m\epsilon n_m = m\epsilon n_0 e^{-m\epsilon/kT}$$

Therefore, the average energy  $\bar{E}$  of an oscillator is

$$\bar{E} = \frac{\sum_{m=0}^{\infty} m\epsilon n_0 e^{-m\epsilon/kT}}{\sum_{m=0}^{\infty} n_0 e^{-m\epsilon/kT}} \quad (ii)$$

Since  $m$  is an integer, eqn. (i) becomes

$$\bar{E} = \frac{0 + \epsilon e^{-\epsilon/kT} + 2\epsilon e^{-2\epsilon/kT} + 3\epsilon e^{-3\epsilon/kT} + \dots}{1 + e^{-\epsilon/kT} + e^{-2\epsilon/kT} + e^{-3\epsilon/kT} + \dots} \quad (iii)$$

Let  $x = e^{-\epsilon/kT}$ . Eqn. (ii) then becomes

$$\bar{E} = \epsilon x \frac{1 + 2x + 3x^2 + \dots}{1 + x + x^2 + \dots}$$

Taking note that  $x < 1$ , the limits of these convergent series can be found by the usual methods. The convergent limit of the series in the numerator is  $1/(1-x)^2$  while the denominator is a simple geometric progression converging to  $1/(1-x)$ . Substituting these limits in eqn. (iii), we get

$$\bar{E} = \epsilon x \frac{1/(1-x)^2}{1/(1-x)} = \frac{\epsilon x}{1-x} = \frac{\epsilon}{(1/x - 1)}$$



When  $x$  is replaced by its equivalent, the result is

$$\bar{E} = \frac{\epsilon}{e^{\epsilon/kT} - 1}$$

and not  $kT$  as given by the equipartition law.

The number of modes of vibration per unit volume between frequency range  $\nu$  and  $\nu + d\nu$  is, as already shown,

$$\frac{8\pi\nu^2 d\nu}{c^3}$$

So, the energy density of radiation in the frequency interval  $\nu$  and  $\nu + d\nu$  is

$$E_\nu d\nu = \frac{8\pi\nu^2 d\nu}{c^3} \times \frac{\epsilon}{e^{\epsilon/kT} - 1} \quad (\text{iv})$$

Expressed in terms of wavelength, the expression becomes

$$\begin{aligned} E_\lambda d\lambda &= \frac{8\pi c^2}{c^3 \lambda^2} \cdot \frac{c}{\lambda^2} d\lambda \times \frac{\epsilon}{e^{\epsilon/kT} - 1} \\ &= \frac{8\pi}{\lambda^4} \times \frac{\epsilon}{e^{\epsilon/kT} - 1} \times d\lambda \end{aligned} \quad (\text{v})$$

$$\text{since } \nu = \frac{c}{\lambda} \quad \text{so that } d\nu = \left| \frac{c}{\lambda^2} d\lambda \right| \quad (\text{vi})$$

Expression (v) gives the energy density in the wavelength range  $\lambda$  and  $d\lambda$  and begins to look like Planck's empirical law if  $\epsilon$  is not zero. Indeed, the denominators of these two equations become identical (except for the minus one) if a value of  $\epsilon$  is so chosen that the powers of the exponential terms are the same. To obtain this value of  $\epsilon$ , we let

$$\frac{c_2}{\lambda T} = \frac{\epsilon}{kT}$$

$$\text{or, } \epsilon = \frac{c_2 k}{\lambda} = \frac{c_2 k}{c} \cdot \nu \quad (\text{vii})$$

In eqn. (vii),  $c$  is the free-space velocity of light and  $\nu$  is the frequency of the oscillator and therefore also the frequency of the radiation it emits. If we replace the constants  $(c_2k/c)$  by another constant  $h$ , we have

$$\epsilon = \frac{hc}{\lambda} = h\nu$$

When this value of  $\epsilon$  is substituted in eqn. (iv) and (v), they become

$$E_\nu d\nu = \frac{8\pi h}{c^3} \frac{\nu^3}{(e^{h\nu/kT} - 1)} d\nu \quad (19.15)$$

and

$$E_\lambda d\lambda = \frac{8\pi hc}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1} d\lambda \quad (19.16)$$

Eqn. (19.15) is the Planck's law for the energy density of black body or cavity radiation in the frequency interval  $\nu$  and  $\nu + d\nu$  while eqn. (19.16) gives the Planck's law for energy density in the wavelength range  $\lambda$  and  $\lambda + d\lambda$  in the spectrum of a black body.

Eqn. (19.16) is plotted in Fig. 19.10 (solid curve) for a temperature  $T = 1646K$ . The equation does agree with the experimental results.

When the concept of discrete energy steps  $\epsilon$  was introduced, it was not indicated that  $\epsilon$  would be different for the different modes. However, this is what the analysis has shown to be the case. Going back to Fig. 19.13(b), it can now be explained qualitatively how Planck avoids the ultraviolet catastrophe. For small wavelengths, high frequencies, the distance from  $E = 0$  to the first discrete energy level  $\epsilon$  is so large that the probability for a mode to exist in that state of excitation becomes diminishingly small. All modes of very high frequency therefore have  $E = 0$ . This is also immediately evident from eqn.

$$\bar{E} = \frac{\epsilon}{e^{\epsilon/kT} - 1}$$

Thus Planck was led to his startling hypothesis that exchange of energy does not take place continuously as assumed in classical physics, but takes place in a discontinuous manner, in discrete bundles of energy which must be an integral multiple of the product of the constant  $h$  and the frequency  $\nu$  of the electromagnetic radiation it emits. If  $E$  represents the smallest permissible energy change, then according to Planck's famous quantum equation,

$$E = h\nu$$

The new constant  $h$  is called the Planck's constant and its value is  $(6.62620 \pm 0.00005) \times 10^{-34}$  J.s.

The quantum (quantum is the Latin word for *how much* or *how great*) concept, introduced by Planck in 1900, eventually led to the conclusion that radiation is not emitted in continuous amounts but in discrete bundles of energy each equal to  $h\nu$ . These bundles or packets of radiative energy are now called *quanta* or *photons*. This was the beginning of the atomic theory of radiation, which has grown to become the quantum theory. It is obvious, however, that quanta of radiation of different frequencies have different *sizes* (energies), and that they are atomic only in the sense that they are discrete. Planck at first thought that is *ad hoc* (*ad hoc* means literally *to this*, and is used to describe a hypothesis which is applicable to but one (this) situation) hypothesis applied only to the oscillators and, possibly, to the emitted radiation in their immediate neighbourhood and that, at most, it was a slight modification of Maxwell's theory of radiation. However, as will be seen later, it initiated a series of events which changed our whole concept of the interaction of electro-magnetic radiation with matter.

### 19.13 Deduction from Planck's law:

The classical laws of radiation such as Wien's law, Rayleigh-Jean's law etc can be deduced rather easily from the Planck's law of radiation.

#### Wien's law

In terms of wavelength  $\lambda$ , the Planck's law is given by

$$E_\lambda d\lambda = \frac{8\pi hc}{\lambda^5} = \frac{d\lambda}{e^{hc/\lambda kT} - 1} \quad (i)$$



For short wavelength (high frequency) and low temperature,  $\lambda T$  is so small that  $e^{hc/\lambda kT} \gg 1$ . Thus from eqn. (i) above, we have

$$E_{\lambda} d\lambda = \frac{8\pi hc}{\lambda^5} \frac{d\lambda}{e^{hc/\lambda kT}}$$

$$= c_1 \lambda^{-5} e^{-c_2/\lambda T} d\lambda \quad (ii)$$

where the constants  $c_1$  and  $c_2$  are given by  $c_1 = 8\pi hc$  and  $c_2 = hc / k$ .

Eqn. (ii) is *Wien's radiation law* and is an essentially empirical formula.  $c_1$  and  $c_2$  are two adjustable constant whose values were adjusted by Wien to obtain good fit with experimental results at long wavelengths.

### Rayleigh-Jeans law

For long wavelength (low frequency) and high temperature  $hc / \lambda kT \ll 1$ . Then from (i), we have

$$E_{\lambda} d\lambda = \frac{8\pi hc}{\lambda^5} \frac{d\lambda}{(1 + \frac{hc}{\lambda kT} + \dots) - 1}$$

$$= \frac{8\pi kT}{\lambda^4} d\lambda \quad (iii)$$

which is Rayleigh-Jeans law. It shows that the energy density of radiation from a black body is inversely proportional to the fourth power of  $\lambda$ .

### Wien's displacement law

Differentiating the right hand side of eqn. (i) with respect to  $\lambda$  and equating the result to zero, the value of  $\lambda$  (at temperature  $T$ ) can be obtained for which the energy density is maximum. If  $\lambda_m$  symbolizes this particular wavelength, we have

$$-\frac{5 \times 8\pi hc}{\lambda_m^6 (e^{hc/\lambda_m kT} - 1)} + \frac{8\pi hc}{\lambda_m^5} \cdot \frac{e^{hc/\lambda_m kT}}{(e^{hc/\lambda_m kT} - 1)^2} \cdot \frac{hc}{kT \lambda_m^2} = 0 \quad (iv)$$

Introducing  $x = hc / \lambda_m kT$  and eliminating the common factors, we get

$$xe^x / (e^x - 1) = 5$$

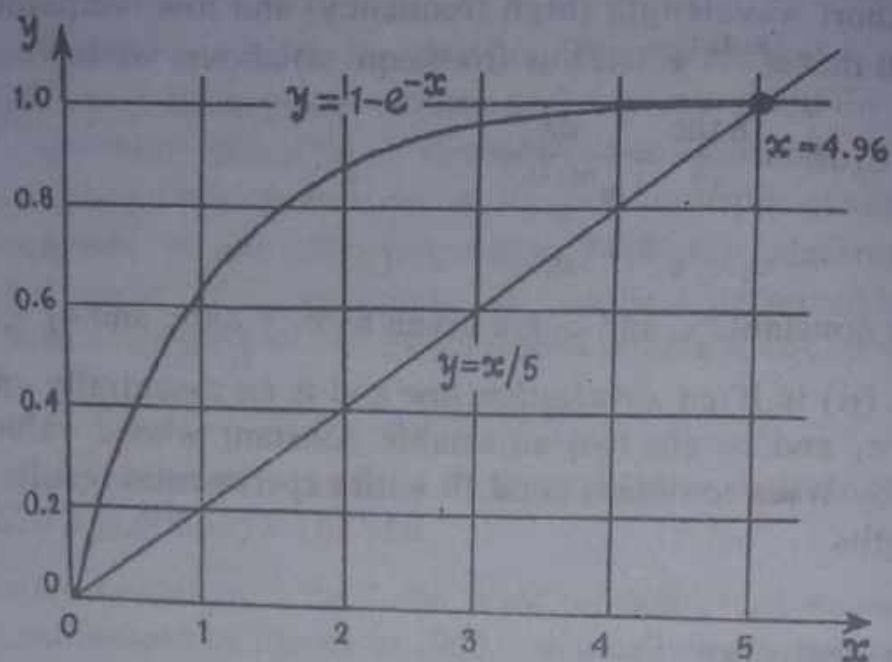


Fig. 19.14

This transcendental equation that can be solved graphically (Fig. 19.14) or numerically shows at once that there must be a root in the neighborhood of 5. Applying the usual method, the exact value turns out to be  $x = 4.965$ .

$$\therefore \frac{hc}{\lambda_m kT} = 4.965$$

$$\text{or, } \lambda_m T = \frac{hc}{4.965 k} = 0.2896 \text{ cm.K.}$$

= constant.

(v)

This is *Wien's displacement law* and states that as the temperature of a black body is increased, the position of maximum emission moves in the direction of shorter waves in such a way that the product  $\lambda_m T = \text{constant}$ . The temperature of the outer surface of all radiating bodies, including the heavenly ones, can be obtained with the help of this law.

### Stefan- Boltzmann's law

The  $T^4$ - law of Stefan and Boltzmann also follows easily from the Planck's radiation law.

The total radiation  $Q$ , emitted by a black body, is obtained by integrating  $E\nu d\nu$  for waves of all frequencies from 0 to  $\infty$ . Thus the total radiation emitted by a black body is

$$\begin{aligned} Q &= \int_0^{\infty} E\nu d\nu \\ &= \frac{8\pi h}{c^3} \int_0^{\infty} \frac{\nu^3}{e^{h\nu/kT} - 1} d\nu \\ &= \frac{8\pi k^4 T^4}{c^3 h^3} \int_0^{\infty} \frac{x^3}{e^x - 1} dx \end{aligned}$$

(substituting  $h\nu / kT = x$ , so that  $d\nu = \frac{kT}{h} \cdot dx$ )

$$\begin{aligned} \therefore Q &= \frac{8\pi k^4 T^4}{c^3 h^3} \times \frac{\pi^4}{15} \quad \left[ \because \int_0^{\infty} \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15} \right] \\ &= \alpha T^4 \quad \text{(vi)} \end{aligned}$$

$$\text{where } \alpha = \frac{8\pi^5 k^4}{15c^3 h^3} = \text{constant} \quad \text{(vii)}$$

$$\therefore \text{ Stefan constant, } \sigma = \frac{\alpha c}{4}$$

$$= \frac{2\pi^5 k^4}{15c^2 h^3}$$

Eqn. (vi) is the *Stefan - Boltzmann law* which states that the total energy emitted by a black body is proportional to the fourth power of the absolute temperature.

#### 19.14 Comparison of Wien's, Rayleigh- Jeans' and Planck's radiation laws.

The points in Fig. 19.10 represent the distribution of energy in the spectrum of the radiation from a black body as obtained by Lummer and Pringsheim at a temperature of  $1646^\circ \text{K}$ . The graph of Wien's law is shown by the dashed line while the Rayleigh - Jean's



law for the same temperature is shown on the right hand side of the figure. It is clear from the curves that Wien's law is closer to that of Planck at shorter wavelengths while Rayleigh- Jean's law approaches Planck's in the region of longer wavelengths.

Introducing the substitution  $ch / \lambda kT = x$ , we get

$$\text{Planck's law: } E_{\lambda} d\lambda = \frac{8\pi kT}{\lambda^4} \cdot \frac{x}{e^x - 1} d\lambda$$

$$\text{Rayleigh- Jean's law: } E_{\lambda} d\lambda = \frac{8\pi kT}{\lambda^4} d\lambda$$

$$\text{Wien's law: } E_{\lambda} d\lambda = \frac{8\pi kT}{\lambda^4} \cdot \frac{x}{e^x} d\lambda$$

which clearly depicts the difference in the three laws of radiation.

Planck's law differs from that of Rayleigh- Jean's in having the factor  $x / (e^x - 1)$  which for  $x > 2$  is not very different and for large  $x$  the difference is imperceptible. For small  $x$ , however, the difference is marked.

As shown earlier, when  $\lambda T$  is small,  $e^x$  becomes large compared to 1 and Planck's law becomes identical with that of Wien's law. Similarly for large  $\lambda T$ ,  $e^x - 1 \approx e^x$  and we obtain Rayleigh - Jean's law.

The three laws may be compared from another aspect, namely the maximum values of  $E_{\lambda}$  obtained from the three laws.

#### Planck's law

It has already been shown that from Planck's law

$$\lambda_m T = \frac{ch}{4.965 k} = 0.289 \text{ cm K.}$$

#### Wien's law

For maximum  $E_{\lambda}$ ,  $dE_{\lambda} / d\lambda = 0$ . If the corresponding value of  $\lambda$  is indicated by  $\lambda_m$ , then

$$-\frac{5}{\lambda_m} + \frac{x}{\lambda_m} = 0; \quad \text{or, } x = 5$$

$$\text{or, } \frac{ch}{\lambda_m kT} = 5$$

$$\therefore \lambda_m T = \frac{ch}{5k} = 0.287 \text{ cm K.}$$

### Rayleigh- Jean's law

Proceeding similarly as in Wien's law, *i.e.*, imposing  $dE_\lambda / d\lambda = 0$  at  $\lambda = \lambda_m$ , we get

$$-3(8\pi kT) \lambda_m^{-5} = 0$$

$\therefore$  Either  $T = 0$  or  $\lambda = \infty$ . But neither of these is possible. So of the three formulae, Rayleigh- Jean's law suffers the most serious drawback. But the agreement between that of Planck and Wien, in this respect, is excellent as both yield almost identical values for  $\lambda_m T$ .

### 19.15 Solar constant

The sun is the source of heat radiations and it emits radiations in all directions. The earth receives only a fraction of the energy emitted by the sun. A part of the heat radiations is also absorbed by the atmosphere. Air, clouds, dust particles, etc. present in the atmosphere scatter the radiations (heat and light) falling on them. From the quantity of heat radiations received by the earth, it is possible to estimate the temperature of the sun. A term called *solar constant* is defined in this connection.

The solar constant is defined as the amount of solar energy received per minute per unit area of a perfectly black body held at right angles to the direction of the sun's rays and placed at the mean distance of the earth from the sun in the absence of earth's atmosphere.

Values of solar constant ( $S$ )

$$S = 0.32 \text{ k cal / s / m}^2$$

$$= 1.34 \times 10^3 \text{ W / m}^2$$

### determination of solar constant

The apparatus used for determining the solar constant is called *pyroheliometer* (*pyro* means fire and *helios* means the sun).

The simplest and best apparatus available for this purpose is Angstrom's compensation pyroheliometer. It consists essentially of two identical thin stripes A and B of platinum or manganin. The strips are blackened and mounted side by side inside a tube such that when A is exposed to the solar radiation the other strip is effectively shielded from radiation by means of a screen S (Fig. 19.15). The junctions of a copper-constantan thermo-couple are kept very close to (but electrically insulated from) the backs of the strips A and B. There is an electrical circuit by which current can be passed through B to heat it electrically.

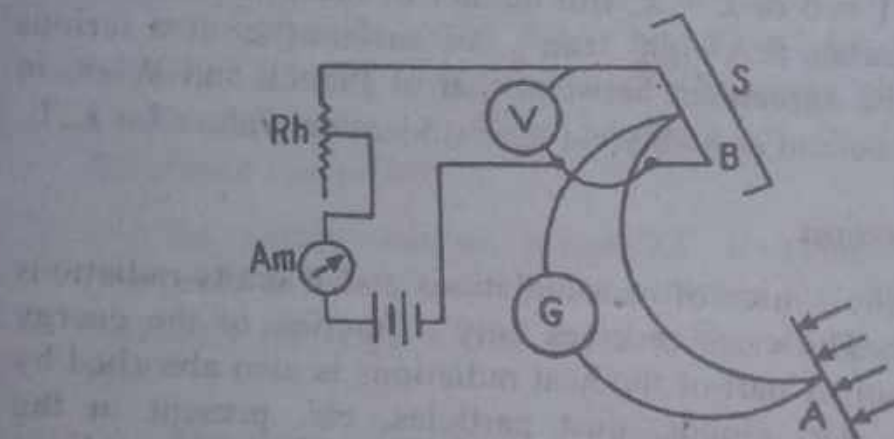


Fig. 19.15

As the parallel beam of radiant energy falls normally on the strip A, its temperature and hence that of the thermocouple junction rises. A temperature difference, therefore, develops between the junctions of the thermocouple, resulting in a deflection of the galvanometer G in the thermocouple circuit. When the deflection becomes steady, an electric current is passed through B to heat it up. The current is then adjusted till the galvanometer deflection is annulled.

Obviously, the temperature of both the strips at this stage is same and the radiant energy absorbed by A is equal to the electrical energy supplied to B. The supplied electrical energy is known from the readings of the ammeter Am and the voltmeter V.



If  $V$  is the potential difference across  $B$  and  $I$  the current through it, then heat supplied to it per minute is

$$= \frac{VI}{J} \times 60 \quad (i)$$

where  $J$  is the mechanical equivalent of heat.

If  $\alpha$  is the area of the exposed strip  $A$  and  $a$  its absorptive power, then energy absorbed by  $A$  is

$$= a S \alpha \quad (ii)$$

where  $S$  is the solar constant i.e., incident solar heat per minute per unit area.

Equating (i) and (ii), we have

$$a S \alpha = \frac{VI \times 60}{J}$$

$$\text{or, } S = \frac{VI}{a J \alpha} = \frac{VI \times 60}{4.2 \times a \times \alpha}$$

A correction for atmospheric absorption, however, is to be made to get the true value for the solar constant. Let  $S$  and  $S_0$  be the absorbed and true values of the solar constant. If  $S$  is determined for different elevations of the sun on the same day under similar sky conditions, then, as a first approximation,

$$S = S_0 a^{\sec Z}$$

where  $z$  is the sun's zenith distance and  $a$  the absorption coefficient having values 0.55 to 0.85.

Taking logarithms,

$$\log S = \log S_0 + \sec z \log a.$$

If  $\log S$  is plotted as ordinate and the corresponding  $\sec Z$  as abscissa, a straight line is obtained. Its intercept on the  $Y$ -axis gives  $\log S_0$  and hence  $S_0$ .

The mean value of solar constant, as evaluated by accurate experiments, is 1.937 caloric per minute per sq. cm.

### 19.16. Temperature of the Sun

The sun consists of a central core, called the *photosphere*, surrounded by a relatively cooler atmosphere, of burning gases known as the *chromosphere*. The central portion has a temperature of the order of  $10^7$  K. The chromosphere has a temperature of about 6000 K, called the effective temperature of the sun. Assuming the sun to be a perfectly black body, its temperature can be calculated either from Stefan's law or from Wien's law.

Let  $R$  be the mean distance of the sun from the earth and  $S$  be the solar constant. Then the total amount of heat energy received by the sphere of radius  $R$  in one minute  $= 4\pi R^2 S$ .

If  $r$  is the radius of the sun, then the amount of heat energy radiated by 1 sq. cm. of the surface in one minute

$$E = \frac{4\pi R^2 S}{4\pi r^2} = \left(\frac{R}{r}\right)^2$$

Taking  $R = 148.48 \times 10^7$  cm

$r = 6.928 \times 10^5$  cm

and  $S = 1.94$  cal per  $\text{cm}^2$  per minute

$$\therefore E = \left(\frac{148.48 \times 10^7}{6.928 \times 10^5}\right)^2 \times \frac{1.94}{60} \text{ cal per second (i)}$$

Also, according to Stefan's law

$$E = \sigma T^4$$

But  $\sigma = 5.75 \times 10^{-5}$  ergs per  $\text{cm}^2$  per second

$$= \frac{5.75 \times 10^{-5}}{4.2 \times 10^7} \text{ cal per cm}^2 \text{ per second}$$

$$\therefore E = \frac{5.75 \times 10^{-5}}{4.2 \times 10^7} \cdot T^4 \quad \text{(ii)}$$

Equating (i) and (ii)

$$\left( \frac{5.75 \times 10^{-5}}{4.2 \times 10^7} \right) T^4 = \left( \frac{148.48 \times 10^7}{6.928 \times 10^5} \right) \times \frac{1.94}{60}$$

$$\text{or, } T = 5730 \text{ K}$$

This temperature gives the effective temperature of the sun acting as a perfectly black body radiator. However, the actual temperature of the sun is somewhat higher than this value. The temperature of the sun is usually taken as 6000K.

#### From Wien's law:

From Wien's law of radiation, we have

$$\lambda_m T = \text{constant} = 0.287$$

Also, from the study of solar spectrum, Abbot gave

$$\lambda_m = 4753 \text{ \AA}$$

$$\therefore 4753 \times 10^{-8} T = 0.287$$

$$\text{or, } T = \frac{0.287}{4753 \times 10^{-8}} = 6038 \text{ K}$$

#### Newton's law of cooling from Stefan's law

Stefan's law is applicable for all temperature of a hot body. But Newton's law of cooling which states that the rate at which a body loses heat due to radiation is directly proportional to the excess temperature *i.e.*, the temperature difference between the body and the surroundings provided this difference is small. It can be easily derived from Stefan's law.

Consider a hot body at a temperature  $T_1$  placed in a uniform temperature enclosure at  $T_2$ . According to Stefan's law

$$E = e\sigma (T_1^4 - T_2^4) \quad (i)$$

where  $e$  is the emissivity of the surface of the hot body

$$\begin{aligned} E &= e\sigma (T_1^4 - T_2^4) (T_1^2 + T_2^2) \\ &= e\sigma (T_1 - T_2) (T_1 + T_2) + (T_1^2 + T_2^2) \\ &= e\sigma (T_1 - T_2) (T_1^3 + T_1^2 T_2 + T_2^2 T_1 + T_2^3) \quad (ii) \end{aligned}$$



If  $(T_1 - T_2)$  is small,  $T_1 \approx T_2$ .

Eqn. (ii) may then be written as

$$E = \sigma (T_1 - T_2) \times 4T_2^3 \quad (\text{app.})$$

$$\text{or, } E = c. (T_1 - T_2)$$

where  $c = 4T_2^3 \cdot \sigma$  = a constant, since  $T_2$  is constant.

$$\therefore E \propto (T_1 - T_2).$$

which is the Newton's law of cooling.

**Example 19.1.** Each square meter of the sun's surface radiates energy at the rate of  $6.3 \times 10^7 \text{ J / m}^2 / \text{sec}$ . Assuming that Stefan's law applies to the radiation; find the temperature of the sun's surface. Stefan's constant is  $5.669 \times 10^{-8} \text{ W / m}^2 / \text{K}^4$ .

**Soln.**

By Stefan's law, we have  $Q = \sigma T^4$ .

Here,  $Q = 6.3 \times 10^7 \text{ J / m}^2 / \text{sec}$ .

and  $\sigma = 5.669 \times 10^{-8} \text{ W / m}^2 / \text{K}^4$

$$\therefore T^4 = \frac{6.3 \times 10^7 \text{ J / m}^2 / \text{sec}}{5.669 \times 10^{-8} \text{ W / m}^2 / \text{K}^4}$$

$$\approx 1.11 \times 10^{15} \text{ K}^4$$

$$\text{or, } T \approx 5773 \text{ K}$$

**Example 19.2.** Two large closely spaced concentric spheres (both black body radiators) are maintained at temperatures of 200 K and 300 K respectively. The space in between the two spheres is evacuated. Calculate the net rate of energy transfer between the two spheres. [ $\sigma = 5.672 \times 10^{-8} \text{ M.K.S. units}$ ].

**Soln.**

$$E = \sigma (T_1^4 - T_2^4)$$

Here  $T_1 = 300 \text{ K}$   $T_2 = 200 \text{ K}$  and  $\sigma = 5.672 \times 10^{-8} \text{ M.K.S. units}$

$$\therefore E = 5.672 \times 10^{-8} [(300)^4 - (200)^4]$$

$$= 368.68 \text{ Watts / m}^2.$$

**Example 19.3.** The relative emittance of tungsten is approximately 0.35. A tungsten sphere of surface area  $10^{-3}$  sq. meters is suspended inside a large evacuated enclosure whose walls are 300 K. What power input is required to maintain the sphere at a temperature of 3000 K? The conduction of heat along the supports can be neglected? [ $\sigma = 5.672 \times 10^{-8}$  M.K.S units].

**Soln.**

$$E = \sigma A e [T^4 - T_0^4]$$

Here  $A = 10^{-3}$  sq. m.  $e = 0.35$

$$\sigma = 5.672 \times 10^{-8} \text{ M.K.S units.}$$

$$\therefore E = 5.672 \times 10^{-8} \times 10^{-3} \times 0.35 [(3000)^4 - (300)^4]$$

$$= 1609 \text{ watts.}$$

$$\therefore \text{the power input} = 1609 \text{ watts.}$$

**Example 19.4.** Calculate the energy radiated per minute from the filament of an incandescent lamp at 2000 K, if the surface area is  $5.0 \times 10^{-5}$  sq. meters and its relative emittance is 0.85.

**Soln.**

$$E = A e \sigma t (T^4)$$

Here  $A = 5.0 \times 10^{-5}$   $e = 0.85$

$$\sigma = 5.672 \times 10^{-8} \text{ M.K.S. units} \quad T = 2000\text{K} \quad t = 60 \text{ secs.}$$

$$\therefore E = 5.0 \times 10^{-5} \times 0.85 \times 5.672 \times 10^{-8} \times 60 \times (2000)^4$$

$$= 2315 \text{ Joules.}$$

**Example 19.5.** An iron furnace radiates  $1.53 \times 10^5$  calories per hour through an opening of cross-section  $10^{-4}$  sq. meter. If the relative emittance of the furnace is 0.80, calculate the temperature of the furnace. Given  $\sigma = 1.36 \times 10^{-8} \text{ cal / m}^2\text{-s-K}^4$ .

**Soln.**

$$E = A e \sigma t (T^4) ; \text{ or, } T^4 = \frac{E}{A e \sigma t}$$

Here  $E = 1.53 \times 10^5 \text{ cal}$ ,  $A = 10^{-4} \text{ m}^2$

$$e = 0.80 \quad t = 60 \times 60 = 3600 \text{ secs.}$$

$$\sigma = 1.36 \times 10^{-8} \quad T = ?$$

$$\therefore T = \left[ \frac{E}{Ae\sigma t} \right]^{\frac{1}{4}}$$

$$= \left[ \frac{1.53 \times 10^5}{10^{-4} \times 0.80 \times 1.36 \times 10^{-8} \times 3600} \right]^{\frac{1}{4}}$$

$$= 2500 \text{ K.}$$

**Example 19.6.** An aluminium foil of relative emittance 0.1 is placed in between two concentric spheres at temperatures 300 K and 200 K respectively. Calculate the temperature of the foil after the steady state is reached. Also calculate the rate of energy transfer between one of the spheres and the foil. Assume the spheres to be perfect black body radiators.  $\sigma = 5.672 \times 10^{-8} \text{ M. K. S. Units}$ .

**Soln.**

(i) Let  $x$  be the temperature of the foil after the steady state is reached. Then

$$e\sigma(T_1^4 - x^4) = e\sigma(x^4 - T_2^4)$$

$$\text{or, } T_1^4 - x^4 = x^4 - T_2^4$$

Here  $T_1 = 300 \text{ K}$  and  $T_2 = 200 \text{ K}$

$$\therefore 300^4 - x^4 = x^4 - 200^4$$

$$\text{or, } x^4 = 48.5 \times 10^8$$

$$\text{or, } x = 263.8 \text{ K.}$$

$$(ii) E = e\sigma(T_1^4 - x^4)$$

Here  $e = 0.1$

$$\sigma = 5.672 \times 10^{-8}$$

$$T_1 = 300 \text{ K}$$

$$x = 263.8 \text{ K}$$

$$\therefore E = 0.1 \times 5.672 \times 10^{-8} [(300)^4 - (263.8)^4]$$

$$= 18.5 \text{ watts / m}^2.$$



**Example 19.7.** A solid copper sphere cools at the rate of  $2.8^{\circ}\text{C}$  per minute when its temperature is  $127^{\circ}\text{C}$ . At what rate will a solid copper sphere of twice the radius cool when its temperature is  $227^{\circ}\text{C}$ , if in both cases the surroundings are maintained at  $27^{\circ}\text{C}$ . Assume Stefan's law to be valid.

**Soln.**

Let  $V$  be the volume,  $\rho$  the density and  $s$  the specific heat of the first copper sphere. Heat lost by it per second is

mass  $\times$  sp. ht.  $\times$  fall in temp. / sec.

$$= V \cdot \rho \times s \times \frac{2.8}{60} \text{ cal / sec.} \quad (\text{i})$$

According to Stefan's law, net loss of energy by the body is

$$E = \sigma A (T_1^4 - T_2^4) \text{ cal / sec} \quad (\text{ii})$$

where  $A$  is the surface area of the body.

Equating (i) and (ii) we have

$$V \rho s \frac{2.8}{60} = \sigma A (400^4 - 300^4)$$

It  $r$  is the radius of the sphere, then

$$V = \frac{4}{3} \pi r^3$$

$$\text{and } A = 4\pi r^2$$

$$\therefore \frac{4}{3} \pi r^3 \rho s \times \frac{2.8}{60} = \sigma 4\pi r^2 \times 175 \times 10^8 \quad (\text{iii})$$

In the case of the second sphere, let  $\frac{d\theta}{dt}$  is the rate of cooling.

Obviously its radius is  $2r$ .

So, proceeding as above, we have

$$\frac{4}{3} \pi (2r)^3 \rho s \times \frac{d\theta}{dt} = \sigma \times 4\pi (2r)^2 \times [(600)^4 - (300)^4]$$

$$\text{or, } 8 \times \frac{4}{3} \times \pi r^3 \rho s \frac{d\theta}{dt} = \sigma \times 16\pi r^2 \times 1215 \times 10^8 \quad (\text{iv})$$

Dividing (iv) by (iii), we have

$$\frac{d\theta}{dt} = 9.72^\circ\text{C} / \text{minute.}$$

**Example 19.8.** Radiations coming from a perfectly black body at  $127^\circ\text{C}$  were allowed to fall on one face of the thermo-couple connected with a mirror galvanometer when the deflection produced was  $2^\circ$ . The thermo-couple was next turned towards a furnace and deflection of  $32^\circ$  was observed. Calculate the temperature of the furnace, assuming the other junction to be at zero degree celsius and the behavior of the thermo-couple is linear.

**Soln.**

Let  $E_1$  and  $E_2$  be the radiations emitted by the black body and the furnace respectively. As the deflection produced in the galvanometer is proportional to the intensity of the heat energy incident on the thermo-couple,

$$E_1 \propto 2 \quad \text{and} \quad E_2 \propto 32$$

$$\text{or, } E_2 / E_1 = 16$$

According to Stefan's law,

$$E_1 = \sigma T_1^4 \quad \text{and} \quad E_2 = \sigma T_2^4$$

$$\therefore \frac{E_2}{E_1} = \frac{\sigma T_2^4}{\sigma T_1^4} = 16$$

$$\begin{aligned} \text{or, } T_2^4 &= T_1^4 \times 16 \\ &= (400)^4 \times 16 \\ &= (800)^4 \end{aligned}$$

$$\therefore T_2 = 800 \text{ K} = 527^\circ\text{C.}$$

**Example 19.9.** Calculate the pressure exerted by sunlight on the earth's surface, given the solar constant is  $8.148 \text{ J} / \text{cm}^2 / \text{min}$ .

**Soln.**

Here,  $I$  = intensity of radiation

$$= \frac{8.148 \times 10^4}{60} \text{ J / m}^2 \text{ / s}$$

The solar rays incident on the earth's surface is parallel. So the pressure of radiation

$$p = \frac{I}{c} = \frac{8.148 \times 10^4}{60 \times 3 \times 10^8} \\ = 0.0452 \times 10^{-4} \text{ N / m}^2.$$

**Example 19.10.** Assuming the earth to be a spherical black body moving in a circular orbit of radius  $1.5 \times 10^8$  km round the sun, find its equilibrium temperature if the sun is considered to be a sphere of radius  $7 \times 10^5$  km radiating as a black body at a temperature  $6 \times 10^3$  K. Assume that the radius of the earth is small compared to the radius of its orbit.

**Soln.**

Energy radiated per second from the sun,

$$Q = \sigma \times 4\pi R_s^2 \times (6 \times 10^3)^4 \quad (i)$$

where  $R_s$  is the radius of the sun.

This energy is contained within a sphere of radius  $R$  where  $R$  is the radius of the earth's orbit. So the energy received by the earth per second is

$$\frac{Q}{4\pi R^2} \times \pi R_e^2 = \frac{QR_e^2}{4R^2} \\ = 4\pi\sigma R_s^2 \times (6)^4 \times 10^{12} \cdot \frac{R_e^2}{4R^2} \quad (\text{using (i)})$$

where  $R_e$  is the radius of the earth ( $R_e \ll R_s$ )

For thermal equilibrium, the rate at which energy is received must be equal to the rate at which energy is emitted.



$$\therefore 4\pi R_e^2 \times \sigma \cdot T_e^4 = 4\pi\sigma R_s^2 \times (6)^4 \times 10^{12} \frac{R_e^2}{4R^2}$$

$$\therefore T_e^4 = \left(\frac{R_s}{2R}\right)^2 \times (6)^4 \times 10^{12}$$

$$\text{or, } T_e = \left(\frac{R_s}{2R}\right)^{\frac{1}{2}} \times 6 \times 10^3$$

$$\text{Here } R_s = 7 \times 10^5 \text{ km}$$

$$R = 1.5 \times 10^8 \text{ km}$$

$$\therefore T_e = 6 \times 10^3 \times \left(\frac{7 \times 10^5}{2 \times 1.5 \times 10^8}\right)^{\frac{1}{2}}$$

$$= 290 \text{ K.}$$

**Example 19.11.** Calculate the black body temperature of sun from the following data:

Stefan's constant,  $\sigma = 1.37 \times 10^{-12} \text{ cal/cm}^2/\text{s}$

Solar constant,  $S = 2.3 \text{ cal/cm}^2/\text{min}$

Radius of the sun,  $r = 7 \times 10^{10} \text{ cm}$

Distance between the sun and the earth  $R = 1.5 \times 10^{13} \text{ cm}$ .

**Soln.**

Total amount of heat energy received by a sphere of radius  $r$  one minute  $= 4\pi R^2 \cdot S$ .

Then the amount of heat energy radiated by 1 sq. cm. surface of the sun is one minute

$$E = \frac{4\pi R^2 S}{4\pi r^2} = \left(\frac{R}{r}\right)^2 \cdot S$$

But by Stefan's law  $E = \sigma T^4$  where  $T$  is the temperature of the sun

$$\therefore \sigma T^4 = \left(\frac{R}{r}\right)^2 \cdot S$$

$$T^4 = \left(\frac{R}{r}\right)^2 \cdot \frac{S}{\sigma} = \left(\frac{1.5 \times 10^{13}}{7 \times 10^{10}}\right)^2 \frac{2.3}{60 \times 1.37 \times 10^{-12}}$$

$$= (5987)^4 \text{ K}^4$$

$$\therefore T = 5987 \text{ K.}$$

**Example 19.12.** A black body at a temperature of 1646 K has the wavelength corresponding to the maximum emission ( $\lambda_m$ ) equal to 1.78 micron. Find the temperature of the moon (assumed to be a black body) if  $\lambda_m$  for the moon is 14 micron.

**Soln.**

From Wien's displacement law,  $= \lambda_m T$  const.

$$\therefore \lambda_m T = \lambda'_m T'$$

where  $\lambda_m = 1.78$  micron,

$\lambda'_m = 14$  micron,

$T = 1646 \text{ K}$

$T' = ?$

$$\therefore 1.78 \times 1646 = 14 \times T'$$

$$\text{or, } T' = \frac{1.78 \times 1646}{14} = 209.27 \text{ K.}$$

**Temperature of the moon = 209.27 K.**

**Example 19.13.** Use Stefan's law to show that the total radiant energy emitted by sun per second is  $3.95 \times 10^{26} \text{ J}$ . Also show that the rate at which energy is reaching the top of the earth's atmosphere is  $1.4 \text{ KW.m}^{-2}$ . Assume the sun to be a black body at temperature 5800 K.

**Soln.** Given:

radius of the sun,  $r = 7 \times 10^8 \text{ m}$

distance of the earth's atmosphere from the sun,  $R = 1.5 \times 10^{11}$  m

Stefan's constant,  $\sigma = 5.672 \times 10^{-8}$  SI unit.

Using Stefan's law, the total energy  $Q$  emitted by the sun is given by

$$\begin{aligned} Q &= \sigma A T^4 \\ &= \sigma \cdot 4\pi r^2 \cdot T^4 \\ &= 5.672 \times 10^{-8} \times 4 \times 3.142 \times (7 \times 10^8)^2 \times (5800)^4 \\ &= 3.95 \times 10^{26} \text{ J.} \end{aligned}$$

Solar energy reaching unit area of the earth's atmosphere per second is

$$\begin{aligned} E &= \frac{Q}{4\pi R^2} = \frac{3.95 \times 10^{26}}{4 \times 3.142 \times (1.5 \times 10^{11})^2} \\ &= 1.4 \times 10^3 \text{ W/m}^2. \\ &= 1.4 \text{ KW/m}^2. \end{aligned}$$

**Example 19.14.** The temperature of a body falls from  $30^\circ\text{C}$  to  $20^\circ\text{C}$  in 5 minutes. The air temperature is  $13^\circ\text{C}$ . Find the temperature of the body after another 5 minutes.

**Soln.** From Newton's law of cooling, we have

$$-\frac{d\theta}{dt} = C(\theta - \theta_s) \quad \text{(i)}$$

$$\text{or, } \frac{d\theta}{\theta - \theta_s} = -C dt \quad \text{(ii)}$$

Integrating  $\ln(\theta - \theta_s) = -Ct + A$  ( $A = \text{a const.}$ )

At  $t = 0$ ,  $\theta = 30^\circ\text{C}$  and  $\theta_s = \text{surrounding temperature} = 13^\circ\text{C}$ .

$$\therefore \ln(30 - 13) = A = \ln 17.$$

when  $t = 5$  min,  $\theta = 20^\circ\text{C}$ ,  $\theta_s = 13^\circ\text{C}$ .

Therefore from (ii)

$$\ln(20 - 13) = -C \times 5 + \ln 17$$



$$\therefore -C = \frac{1}{5} \ln \frac{7}{17}$$

After another 5 minutes, i.e., at  $t = 10$  minutes from start,  $\theta_s = 13^\circ\text{C}$ ,  $\theta = ?$

$$\therefore \ln(\theta - \theta_s) = -C \times 10 + \ln 17 = 2 \ln \frac{7}{17} + \ln 17$$

$$\therefore \theta - 13 = \left(\frac{7}{17}\right)^2 \times 17 = \frac{49}{17}$$

$$\therefore \theta = 13 + \frac{49}{17} = 15.88^\circ\text{C}.$$

**Example 19.15.** Show that the average energy of the photons at temperature  $T$  of a cavity radiation is given by

$$\bar{\epsilon} = 2.70 kT$$

and the total number of photons emitted per second per unit area by a black body at temperature  $T$  is

$$\frac{2\pi k^3 T^3}{c^2 h^3} \times 2.404$$

**Soln.** From Planck's law of radiation, the spectral energy density  $E_\nu$  in frequency range  $\nu$  and  $\nu + d\nu$  is

$$E_\nu = \frac{8\pi\nu^2 d\nu}{c^3 (e^{h\nu/kT} - 1)} \cdot h\nu$$

$$= N_\nu d\nu h\nu$$

The number of photons in a cavity radiation at temperature  $T$  in the frequency range  $\nu$  and  $\nu + d\nu$  is, therefore,

$$N_\nu d\nu = \frac{8\pi\nu^2 d\nu}{c^3 (e^{h\nu/kT} - 1)}$$

Total number of photons per cc at  $T$  is given by

$$N = \int_0^{\infty} N_v dv = \int_0^{\infty} \frac{8\pi v^2 dv}{c^3 (e^{hv/kT} - 1)}$$

$$= \int_0^{\infty} \frac{8\pi k^3 T^3}{c^3 h^3} \cdot \frac{x^2 dx}{e^x - 1} \quad (\text{substituting } hv/kT \text{ by } x)$$

$$\text{or, } N = \frac{8\pi k^3 T^3}{c^3 h^3} \int_0^{\infty} \frac{x^2}{e^x - 1} dx$$

$$\text{But } \int_0^{\infty} \frac{x^2}{e^x - 1} dx = 2 \xi(3) = \left[ 1 + \frac{1}{2^3} + \frac{1}{3^3} + \dots \right]$$

$$= 2 \times 1.202 = 2.404$$

where  $\xi(3)$  is Riemann zeta function of order 3.

$$\therefore N = \frac{8\pi k^3 T^3}{c^3 h^3} \times 2.404$$

The average energy  $\bar{\epsilon}$  of the photons at temperature  $T$  of the cavity is obtained by dividing total energy per c.c. by the number of photons per c.c.

$$\bar{\epsilon} = \frac{\int_0^{\infty} E v dv}{\int_0^{\infty} N_v dv} = \frac{\alpha T^4}{\frac{8\pi k^3 T^3}{c^3 h^3} \times 2.404}$$

$$= \frac{kT\pi^4}{15 \times 2.404} = kT \times 2.70$$

Since  $\alpha = 8\pi^5 k^4 / 15 c^3 h^3$  vide eqn. (viii) Art. 19.13

$\therefore$  Average energy of the photons at temperature  $T$  of the cavity radiation is given by

$$\bar{\epsilon} = 2.70 kT$$

The total number of photons emitted per sec. per unit area by a black body at temperature  $T$  is given by

$$\frac{1}{4} Nc = \frac{1}{4} C \cdot \frac{8\pi k^3 T^3}{c^3 h^3} \times 2.404$$

$$= \frac{2\pi k^3 T^3}{c^3 h^3} \times 2.404$$

### EXERCISE

- [1] Define emissive power and absorptive power for a substance in connection with thermal radiation. What is a black body and what do you mean by black body radiation?
- [2] Define the term 'perfectly black body'. Discuss the possibility of realizing the same in practice. Discuss the analogy between black body radiation and a perfect gas. Derive an expression for the pressure of radiation inside the black body chamber.
- [3] What is a black body? State and prove Kirchhoff's law for thermal radiation.
- [4] Show that the pressure due to diffuse black body radiation is one-third of the energy density of radiation.
- [5] Show that the energy density of black body radiation is proportional to the fourth power of its absolute temperature.
- [6] Show that the energy density of radiation inside a uniformly heated enclosure is given by

$$E = \frac{4\pi K}{c}$$

where  $K$  is the specific intensity of radiation and  $c$  be the velocity of light.

- [7] A material at an initial temperature  $T_1$  is placed within a hollow enclosure maintained at a constant temperature  $T$ . Show that the equilibrium condition will be attained when

(i) the temperature of the material will be  $T$



and (ii)  $e_\lambda/a_\lambda = E_\lambda$  where  $e_\lambda$  and  $a_\lambda$  are the emissive and absorptive powers respectively of the material and  $E_\lambda$  the emissive power of a perfect black body for the wavelength  $\lambda$  at temperature  $T$ .

- [8] Applying the principles of thermodynamics prove that the energy density  $E$  of black body radiation in a cavity with perfectly reflecting walls maintained at a constant temperature  $T$  is given by  $E = aT^4$ , where  $a$  is a constant.
- [9] State and prove Stefan-Boltzmann's law of radiation. Show that Newton's law of cooling follows from Stefan-Boltzmann's law.
- [10] Derive Wien's law for the energy distribution in the spectrum of black body radiation. Hence deduce Wien's displacement law.
- [11] Derive and discuss Rayleigh-Jeans's formula for the distribution of energy in the spectrum of a black body.
- [12] What is quantum theory of radiation? Derive Planck's formula for energy distribution in the spectrum of a black body at an absolute temperature  $T$ . Show that Wien's and Rayleigh-Jeans's formulae are particular cases of Planck's formula.
- [13] Write down Stefan-Boltzmann's law of radiation and derive it from Planck's formula.
- [14] Define solar constant. Describe a method for experimental determination of solar constant.
- [15] Write notes on:
- (i) Kirchhoff law and its application in spectroscopy and astrophysics.
  - (ii) Stefan-Boltzmann's law
  - (iii) Black body
  - (iv) Wien's displacement law
  - (v) Rayleigh-Jeans's law
  - (vi) Planck's law of radiation
  - (vii) Solar constant.
- [16] Calculate the radiant emittance of a black body at a temperature of (i) 200 K and (ii) 2000 K.  $\sigma = 5.672 \times 10^{-8} \text{ SI units}$ .
- (Ans. (i) 90.75 watts/m<sup>2</sup> (ii) 907.5 kilowatts/m<sup>2</sup>).

- [17] Two ideal black bodies A and B at temperatures  $127^{\circ}\text{C}$  and  $227^{\circ}\text{C}$  are placed in an evacuated enclosure whose walls are blackened and kept at  $27^{\circ}\text{C}$ . Compare their rates of cooling.

[Ans. Ratio of cooling of A: B is 175: 544]

- [18] Calculate the energy radiated per minute from the filament of an incandescent lamp at  $3000\text{ K}$ , if the surface area is  $10^{-4}\text{ sq. metres}$  and its relative emittance is 0.425.

$$\sigma = 5.67 \times 10^{-8} \text{ joules/m}^2/\text{s/k}^4.$$

(Ans: 11710 joules)

- [19] A small blackened sphere of copper of diameter  $2\text{ cm}$  is maintained at a temperature of  $127^{\circ}\text{C}$  in surroundings at a temperature of  $27^{\circ}\text{C}$ . What is the net amount of heat in calories radiated by the sphere in one minute.

$$\text{Stefan's constant} = 5.67 \times 10^{-8} \text{ joules/ m}^2/\text{s/ k}^4$$

$$J = 4.2 \text{ Joules/cal}$$

[Ans : 18 cal]

- [20] The relative emittance of tungsten is approximately 0.70. A tungsten sphere area  $5 \times 10^{-4}\text{ sq. meters}$  are suspended inside a large evacuated enclosure whose walls are at  $300\text{ K}$ . What power input is required to maintain the sphere at a temperature of  $3000\text{ K}$ . The conduction of heat along the supports can be neglected.

$$\sigma = 5.672 \times 10^{-8} \text{ SI unit.}$$

[Ans. 1609 watts]

- [21] Calculate the maximum net rate of loss of heat by radiation from a sphere of radius  $10\text{ cm}$  at a temperature of  $2000^{\circ}\text{C}$  when the surroundings is at a temperature of  $20^{\circ}\text{C}$ .

[Ans. 72.7 cal/s]

- [22] Deduce the temperature at which a black body loses thermal energy at the rate of  $1\text{ w / cm}^2$ . Given  $\sigma = 5.672 \times 10^{-8} \text{ SI units}$ .

[Ans. 648 K]

- [23] An aluminum foil of relative emittance 0.2 is placed in between two concentric spheres at temperatures  $300\text{ K}$  and  $200\text{ K}$  respectively. Calculate the temperature of the foil after the steady state is reached. Assume that the spheres are perfect black body radiators. Also

calculate the rate of energy transfer between one of the spheres and the foil.

$$\sigma = 5.672 \times 10^{-8} \text{ S.I. units.}$$

[Ans. 263.8 K, 37 watts/m<sup>2</sup>]

- [24] Calculate the wavelength corresponding to the maximum in the spectral distribution of the sun assuming that it radiates with the properties of a black body radiator at 6000 K.

[Ans.  $4.83 \times 10^{-10}$  m]

- [25] An iron furnace radiates  $7.65 \times 10^4$  calories per hour through an opening of cross-section  $10^{-4}$  Sq. metres. If the relative emittance of the furnace is 0.40 calculate the temperature of the furnace.

[Ans. 2500 K]

- [26] The back side of a perfectly black disc is thermally insulated. If the sun rays are incident normally on the disc, what will be the equilibrium temperature? Assume the sun to be a black radiator at 6200 K. Sun's radius =  $6.92 \times 10^5$  km; sun-earth distance =  $1.5 \times 10^8$  km.

[Ans. 422 K]

- [27] Calculate the effective temperature of the sun from the following data:

Solar constant = 2.279 cal / sq. cm / min;

Radius of the sun =  $4.3 \times 10^5$  miles

Earth-sun distance =  $9.3 \times 10^7$  miles

and Stefan's constant =  $1.37 \times 10^{-12}$  cal / sq. cm / deg<sup>4</sup>

- [28] It is found that the wavelengths corresponding to maximum emission are as under:

Temperature in degree Kelvin	723	903	1094	1460	1646
Wavelength (maximum emission) in micron	4.08	3.28	2.71	2.04	1.78

Show that the results are in agreement with Wien's displacement law.



## CHAPTER XX

## THERMODYNAMICS

*Introduction – the Zeroth law of thermodynamics – First law of thermodynamics – Thermodynamic equilibrium – Reversible and irreversible processes – Work and the indicator diagram of a gas – Carnot cycle – Efficiency of Carnot engine – Refrigerators – Second law of thermodynamics – Carnot's theorem – Absolute scale of temperature – Solved problems – Exercises.*

**20.1 Introduction**

Thermodynamics, as the name implies, is a branch of physics which deals with heat in motion. The branch had its start in the early part of the nineteenth century, primarily as a result of attempts to improve the efficiency of heat engines. These were devices, where there was an input in the form of heat, and whose output was mechanical work. Thus thermodynamics was concerned with both thermal and mechanical, or dynamical concepts. As the subject developed and its basic laws were more fully understood, its scope became broader. Today scientists and engineers use the principle of thermodynamics in the design of internal combusting engines, refrigeration and air-conditioning systems, conventional and nuclear power stations, and propulsion systems for aircrafts, rockets, missiles, ship and land vehicles. Broadly speaking, thermodynamics deals with the transformation of heat energy to other forms of energy, such as mechanical, electrical, magnetic and radiant, etc.

**20.2 The zeroth law of thermodynamics**

In every measurement of temperature, the correctness of a law is tacitly assumed. This law, known as the zeroth law of thermodynamics, states that *when any two bodies are each separately in thermal equilibrium with a third, then they are also in thermal equilibrium with each other.*

**20.3 First law of thermodynamics**

There are two slightly different forms in which the first law of thermodynamics may be stated.

(i) The first form of the law establishes equivalence between mechanical work and the heat energy. According to this law, a definite amount of mechanical work is needed to produce a definite amount of heat and *vice versa*. For example, to produce one calorie of heat, 4.186 joules of work must always be done somehow. In general, if  $W$  is the amount of work done, then the heat produced  $H$  is given by

$$W = JH$$

where  $J$  is a constant called the *mechanical equivalent of heat*.

$$\begin{aligned} J &= 4.186 \text{ joules per calorie} \\ &= 4.186 \times 10^7 \text{ ergs per calorie} \\ &= 4186 \text{ joules per kilocalorie} \end{aligned}$$

(ii) The second form of the law is an extension of the *law of conservation of energy*. According to conservation law, energy of a system cannot be created or destroyed but can be transformed from one form into other. The total energy is constant. Extending this general law, the first law of thermodynamics may be stated as :

*In all transformations the amount of heat supplied to a system must be balanced by the sum of the gain in internal energy of the system due to the rise in temperature and the external work done.* Mathematically,

$$dQ = dU + \frac{dW}{J}$$

where

$dQ$  = heat supplied

$dU$  = change internal energy

$\frac{dW}{J}$  = heat equivalent of the work done

## 20.4 Thermodynamic equilibrium

When an arbitrary body or system is isolated and left to itself, its properties will in general change with time. If initially there are temperature differences between different parts of the body or

system, then after a sufficiently long time the temperature of all parts of the body or system will become same. The body or the system is then said to be in *thermal equilibrium*.

If there are variations in pressure or elastic stress within the body or the system, parts of the system may move, expand or contract. Eventually these motions, expansions or contractions will cease. When this happens, *i.e.*, there is no unbalanced forces in the interior of the body or the system, the body or the system is said to be in *mechanical equilibrium*.

Suppose the system contains substances that can react chemically. After a sufficiently long time has elapsed, all possible chemical reactions will have taken place; there will be no chemical reaction within the system and also no movement of any chemical constituent from one part of the system to the other. The system is then said to be in *chemical equilibrium*.

A system which is in thermal, mechanical and chemical equilibrium is said to be in *thermodynamic equilibrium*. In our discussion we shall consider system that are in thermodynamic equilibrium, or those in which the departure from thermodynamic equilibrium is negligibly small. Unless otherwise specified, the *state* of a system implies an equilibrium state.

## 20.5 Reversible and irreversible processes

A process which can be retraced in the opposite direction so that the working substance passes through exactly the same states in all respects as in the direct process is called a *reversible process*. Thus, if heat is absorbed by the working substance in a step in the direct process, it is given out in the reverse process and *vice versa*. If work is done *by* the working substance in the direct process, then work is done *on* the working substance in the reverse process.

As an example of reversible process, let us consider the alternate heating and cooling of a mixture of water and ice at  $0^{\circ}\text{C}$  in a vessel in which the temperature can be closely controlled and with which heat exchanges can be made as required. If the mixture is cooled so that its temperature falls by the slightest amount, then water will freeze and its volume will increase. If the



mixture is now heated so that its temperature rises *by the same slightest amount*, then the ice will melt and so decrease in volume. Hence, we find that changes in volume reverse as the substance freezes or melts. Since these changes take place at practically constant temperature, they can be considered as isothermal changes as well. Alternately, *isothermal changes when carried out very slowly are regarded as reversible changes*. From this it can be seen that a reversible process must be in thermodynamic equilibrium at all instants.

**Conditions of reversibility :** (i) The changes in pressure and volume of the working substance must take place at an extremely slow rate, so that the substance undergoing a reversible change is, at all instants, in thermodynamic equilibrium with its surroundings. (ii) There should be no friction and no loss of heat by conduction, convection or radiation.

A process which cannot be retraced in the opposite direction is called an *irreversible process*. In an irreversible process the system with the surroundings can never be completely restored to its initial condition by reversing the controlling factors. All changes which occur suddenly like explosions, etc. may be considered as irreversible. Heat produced by friction, heat generated when a current flows through an electrical resistance, sudden unbalanced expansion are some examples of irreversible process. *It is to be noted, however, that all natural processes are irreversible.*

## 20.6 Work and the PV (indicator) diagram for a gas

Let us suppose that a mass of a gas is confined in a cylinder provided with a piston. When heat is supplied to this mass of a gas, the gas expands, the piston moves and the gas does work against the pressure of the atmosphere.

Let A denote the initial state of the gas before the expansion takes place and B the final state of the gas after expansion has taken place. A change from one state to another is called a *thermodynamic process or transformation*. There are different ways of representing such a change of which the P-V plot is the most common. Fig. 20.1 represents a process where a gas has been taken from a state A, whose variables are  $P_1$  and  $V_1$ .

to a state B with variables  $P_2$  and  $V_2$ . The temperature  $T$  will, in general, have different values at different points of the path. The total work done by the gas during its passage from A to B.

$$W_{AB} = \int_{V_1}^{V_2} P \cdot dV = P (V_2 - V_1)$$

and is represented by the area between the  $P$ - $V$  curve and the  $V$ -axis bounded by ordinates  $P_1$  and  $P_2$ . This  $P$ - $V$  diagram which represents the work done by the gas due to supply of heat is called a work diagram or an *indicator diagram*.

Let us consider that the two states of the gas, mentioned above, are at the same temperature. If the gas is ideal,  $P_1 V_1 = P_2 V_2$ . Curves in Fig. 20.1 illustrates several possible kinds of expansion connecting these states. Along path A (Fig. 20.1) the pressure is kept constant until the gas reaches the volume  $V_2$ , and then the volume is kept constant while pressure is reduced to its final pressure  $P_2$ . The work done along this path is  $P_1 (V_2 - V_1)$  for the horizontal (constant pressure) part of the path and zero along the constant volume. This work is indicated by the shaded area under the curve. Along the path B (Fig. 20.2) the gas is first cooled at constant volume to pressure  $P_2$  and then allowed to expand at constant pressure until it reaches volume  $V_2$ . The work done along this path is  $P_2 (V_2 - V_1)$  and is clearly less than that along A, as indicated by the area under

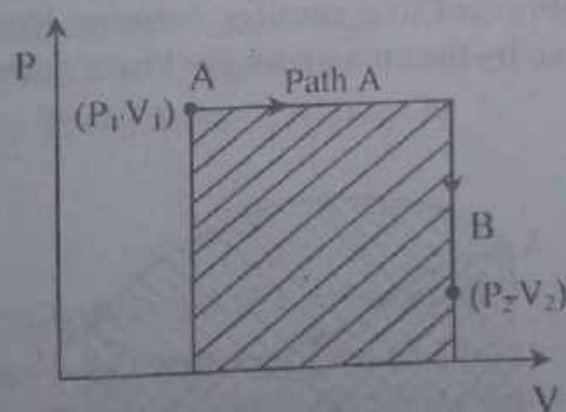


Fig. 20.1

the curve for this path. Along path C (Fig. 20.3) both the pressure and volume change for each part of the path. The work done is again the area under this curve connecting the initial and final states.

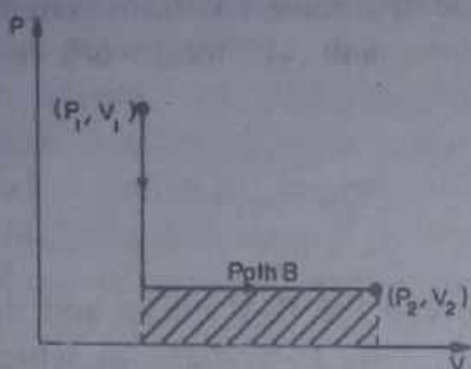


Fig. 20.2

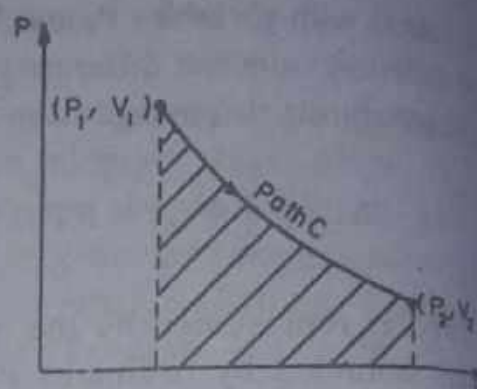


Fig. 20.3

Thus it is clear that the work done in going from state 1 to state 2 depends on how the gas goes from one state to the other. In other words, *work cannot be a function of the state; it depends on the nature of the path on the P-V diagram connecting the initial and final states.*

*Any transformation that returns the gas to its original state is called a cycle.* Fig. 20.4 represents a cycle, the gas being taken from A to B and back to A. According to first law of thermodynamics, the internal energy of a gas is a function of state. Thus the net change for the internal energy for the cyclic path indicated in the figure must be zero since the system is in its original state at the end. During the passage from A to B the gas expands and does an amount of work equal to the area under the upper curve. In passing from B to A the gas is compressed to a smaller volume. Hence work is done on the gas and is given by the area under the lower curve. Since the work

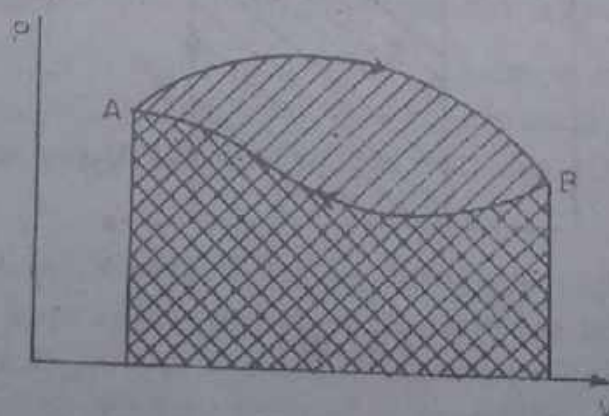


Fig. 20.4



done *does* depend on the nature of the path while internal energy does not, the net work done by the gas during the cycle is the area on the P-V plot that is enclosed by the cycle. Thus, although the substance returns to its initial state in a cycle, the net work done is not zero. This is because the gas receives heat during expansion and gives out heat during compression. Like work, heat added also depends on the nature of the path. Thus the heat received during the performance of the cycle is not zero and this heat has been transformed into work.

If the cycle in Fig. 20.4 had been traversed in the opposite sense, so that what was previously a compression has now become an expansion and *vice versa*, the work done by the gas is under the lower curve, and the work done on the gas is the area under the upper curve. Thus the work enclosed by the cycle is no longer the work done by the gas but that done on the gas during the cycle. In general, then, if a cycle on the P-V diagram is traversed in a *clockwise* manner, its area represents the work done by the system; if it is traversed in *counter clockwise* manner, it represents negative work done by the system *i.e.*, work done on the system by external agencies.

## 20.7 Efficiency of heat engines

A heat engine is a device for converting heat into mechanical energy. The efficiency of a heat engine is defined as the ratio of the output to the input – the output being the mechanical work done during a cycle while the input is the heat absorbed during the cycle. Both the work and heat should be expressed in same units. Thus if,  $W$  be the amount of work obtainable from a heat engine in one cycle at the expense of  $Q$  amount of heat, then its efficiency  $\eta$  is given by

$$\eta = \frac{W}{Q}$$

both  $W$  and  $Q$  being expressed in work units.

Of two engines which take in heat during the performance of a cycle of operations, the more efficient is the one which converts the greater amount of heat absorbed into work.

## 20.8 The Carnot cycle

A cycle in which the working substance starting from a given condition of temperature, pressure and volume is made to undergo

two successive expansions (one isothermal and another adiabatic) and then two successive compressions (one isothermal and another adiabatic) at the end of which the working substance is brought back to its initial condition, is called Carnot's cycle.

This important but simple reversible cycle was introduced by the French engineer Sadi Carnot in 1824. Sadi Carnot was the first to approach the problem of the efficiency of a heat engine from a truly fundamental standpoint. He argued that the essentials of a heat engine are:

(i) *a cylinder S* : the walls of the cylinder are perfectly non-conducting but its bottom is perfectly conducting. The cylinder is provided with a non-conducting piston P which moves horizontally inside the cylinder without friction (Fig. 20.5). The working substance inside the cylinder is a perfect gas.

(ii) *a source* : it is a hot body or heat reservoir of unlimited heat capacity, kept always at a constant high temperature  $T_1$  K. The temperature of the body is supposed to remain constant even if an amount of heat is taken from it.

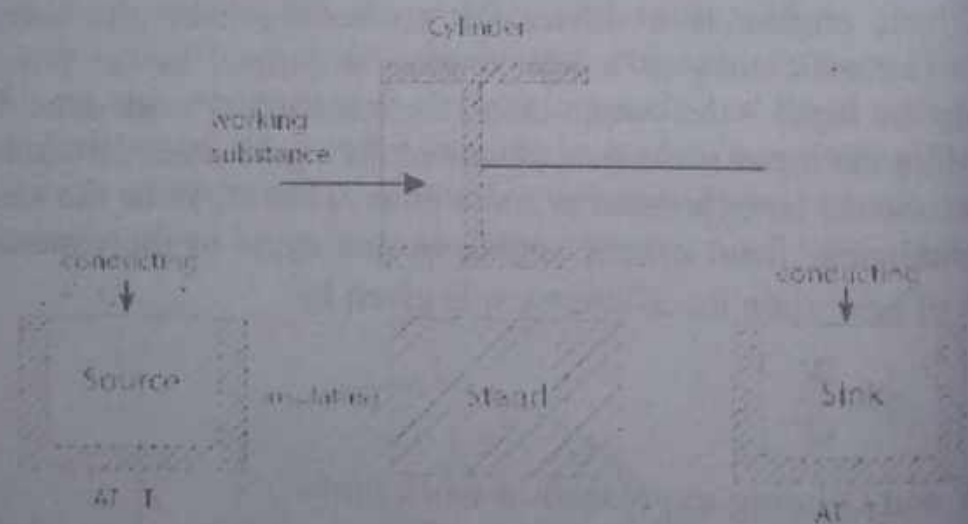


Fig. 20.5

(iii) *a sink* : it is a cold body of unlimited heat-receiving capacity, kept at a constant temperature  $T_2$  K (lower than  $T_1$ ). The temperature is supposed to remain constant even if an amount of heat is given to it.

(iv) *a non-conducting stand D* : it is a block of non-conducting material.

All the imperfections of an actual heat engine have been overlooked in a Carnot cycle. Hence the Carnot cycle is a very ideal heat engine – one never realized in practice.

Let us now consider the four stages of a Carnot cycle. Let the engine cylinder contain  $m$  gms of air as the working substance and let its original condition be represented by the point  $a$  on the  $P$ - $V$  diagram (Fig. 20.6). At point  $a$ , let the air have pressure  $P_1$ , volume  $V_1$  and the same temperature  $T_1$  K as that of the source.

#### operation I (isothermal expansion)

Let the bottom of the cylinder be placed in contact with the source and the gas is allowed to expand slowly. The fall in temperature during expansion is compensated by absorption of heat from the source. So the gas expands isothermally to the point  $b$  represented by  $P_2$ ,  $V_2$  and  $T_1$ . During the process an amount of heat

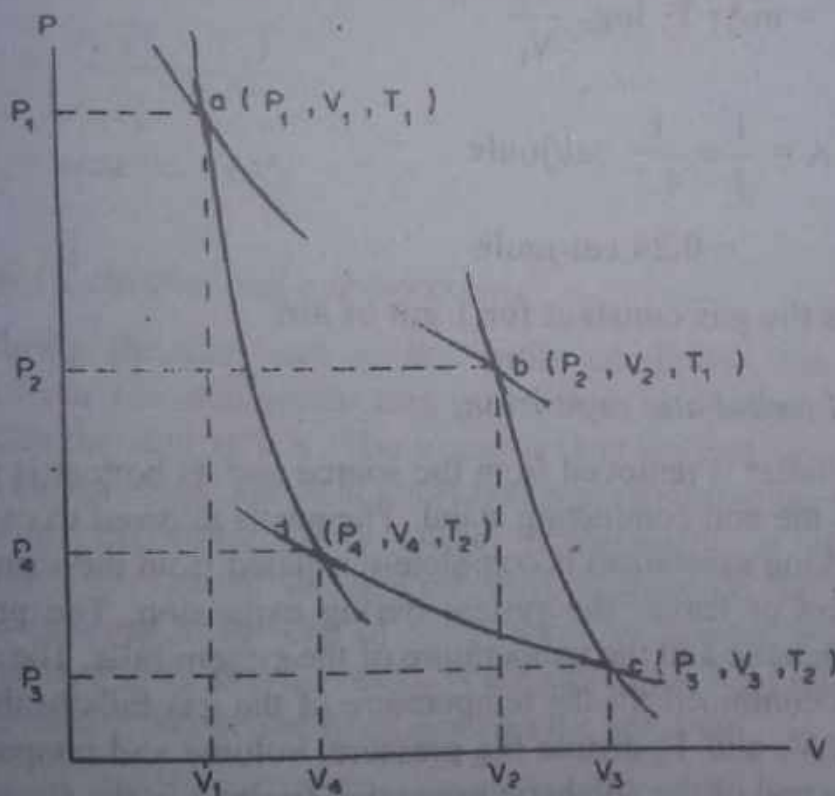


Fig. 20.6

energy  $Q_1$  is absorbed from the source at constant temperature. This energy is equivalent to the amount of external work  $W_1$ , done by the gas during the expansion. The amount of work done by air



$$\begin{aligned}
 W_1 &= \int_{V_1}^{V_2} P.dV \\
 &= mrT_1 \int_{V_1}^{V_2} \frac{dV}{V} \quad [\because PV = mrT_1] \\
 &= mrT_1 \log_e \frac{V_2}{V_1} \quad \text{or, } P = \frac{mrT_1}{V} \\
 &= \text{area } a b V_2 V_1
 \end{aligned}$$

Heat received during the process

$$\begin{aligned}
 Q_1 &= \frac{W_1}{J} = \frac{mrT_1 \log_e \frac{V_2}{V_1}}{J} \\
 &= mA_r T_1 \log_e \frac{V_2}{V_1}
 \end{aligned}$$

$$\begin{aligned}
 \text{where } A &= \frac{1}{J} = \frac{1}{4.2} \text{ cal/joule} \\
 &= 0.24 \text{ cal/joule}
 \end{aligned}$$

and  $r$  is the gas constant for 1 gm of air.

#### *operation II (adiabatic expansion)*

The cylinder is removed from the source and its bottom is placed in contact with the non-conducting stand. The gas is allowed to expand. The gas (working substance) is completely isolated from the surroundings; no heat enters or leaves the system during expansion. The process is therefore, adiabatic and the temperature of the system falls. The adiabatic expansion is continued till the temperature of the gas falls to that of the sink. Let  $P_3$ ,  $V_3$  and  $T_2$  denote the pressure, volume and temperature of the gas at the end of the adiabatic expansion (point  $c$  in the figure). If  $W_2$  denote the work done by the gas during adiabatic expansion represented by  $bc$ , we have

$$W_2 = \int_{V_2}^{V_3} P dV$$

$$= K \int_{V_2}^{V_3} \frac{dV}{V^\gamma}$$

(for adiabatic change,  $PV^\gamma = K$ .)

$$= K \left[ \frac{V^{-\gamma+1}}{-\gamma+1} \right]_{V_2}^{V_3}$$

$$= \frac{K}{\gamma-1} [V_2^{-\gamma+1} - V_3^{-\gamma+1}]$$

$$= \frac{P_2 V_2 - P_3 V_3}{\gamma-1} \quad (K = P_2 V_2^\gamma = P_3 V_3^\gamma)$$

$$= \frac{m r T_1 - m r T_2}{\gamma-1}$$

$$= \frac{m r (T_1 - T_2)}{\gamma-1}$$

$$= \text{area } bc \, V_3 \, V_2.$$

### operation III (isothermal compression)

To bring the gas back to the initial conditions, the cylinder is removed from the non-conducting stand and its bottom is placed in contact with the sink at  $T_2 K$ . The piston is then pressed very slowly and the gas is compressed. The heat generated is given out to the sink and the compression, therefore, takes place isothermally at the constant temperature  $T_2 K$ . During the process work is done on the gas. Let the position of the gas at the end of the process be denoted by the point d represented by  $P_4$ ,  $V_4$  and  $T_2$ . Let  $Q_2$  is the amount of heat rejected (given out) to the sink. If  $W_3$  be the work done on the gas, then

$$W_3 = \int_{V_3}^{V_4} P \cdot dV$$

$$= m r T_2 \log_e \frac{V_4}{V_3}$$

$$\begin{aligned}
 &= mrT_2 \log_e \frac{V_3}{V_4} \\
 &= \text{area } cV_3V_4d \\
 &\quad mrT_2 \log_e \frac{V_3}{V_4} \\
 \text{and } Q_2 &= \frac{J}{J} \\
 &= mrT_2 \log_e \frac{V_3}{V_4}
 \end{aligned}$$

*operation IV (adiabatic compression)*

The cylinder is then removed from the sink and its bottom is placed in contact with the insulating stand. The gas is compressed adiabatically until its temperature rises to  $T_1$  K and its pressure and volume become  $P_1$  and  $V_1$ , represented by the point a. If  $W_4$  is the amount of work done on the gas during the adiabatic compression, then

$$\begin{aligned}
 W_4 &= \int_{V_4}^{V_1} P.dV \\
 &= - \frac{P_4 V_4 - P_1 V_1}{\gamma - 1} \quad \left[ = - \frac{mr(T_2 - T_1)}{\gamma - 1} \right] \\
 &= \frac{P_1 V_1 - P_4 V_4}{\gamma - 1} \quad \left[ = \frac{mr(T_1 - T_2)}{\gamma - 1} \right] \\
 &= \text{area } dV_4V_1a
 \end{aligned}$$

The net work  $W$  done by the system during the cycle is given

$$\begin{aligned}
 W &= W_1 + W_2 - W_3 - W_4 \\
 &= W_1 - W_3 \quad (\text{since } W_2 = W_4).
 \end{aligned}$$

This is represented by the area  $abcd$  in Fig. 20.4 ( $= \text{area } abV_2V_1 + \text{area } bcV_3V_4 - \text{area } cV_3V_4d - \text{area } dV_4V_1a$ ).

The net work done can be written down directly from the first law of thermodynamics as follows :

The net amount of heat energy received by the system in the cycle is  $Q_1 - Q_2$  where  $Q_1$  is the amount of heat received in operation I and  $Q_2$



that given up in operation III. Since the initial and final states are the same there is no net change in the internal energy  $dU$  of the system. Hence from the first law of thermodynamics ( $dQ = dU + dW$ ) we have

$$W = Q_1 - Q_2 \quad (dU = 0).$$

for the cycle.  $Q_1$  and  $Q_2$  are both taken as positive quantities.

The result of the cycle is that heat has been converted into work by the system. Any required amount of work can be obtained by simply repeating the cycle. Thus the system acts like a heat engine.

An ideal gas has been taken as the working substance in the cycle. But the working substance can be anything at all, it may be a solid, liquid or a gas which is not ideal. It may even change from one phase to another during the cycle. But the  $P$ - $V$  diagrams would be different for different substances. Common heat engines use steam or a mixture of fuel and air, or fuel and oxygen as their working substance. Heat may be obtained from the combustion of a fuel such as gasoline or coal, or from the annihilation of mass in nuclear fission process in nuclear reactors. Heat may be discharged at the exhaust or to a condenser. *Although real heat engines do not operate on a reversible cycle*, the Carnot cycle, which is reversible, gives useful information about the behaviour of any heat engine.

### Flow diagram

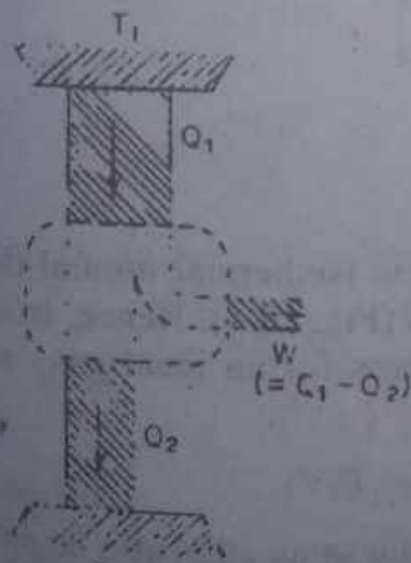


Fig. 20.7

It is useful to represent the operation of any heat engine by a schematic flow diagram like that in Fig. 20.7. The engine itself is represented by the circle. The cross-section of the *incoming pipe line* at the top of the diagram is taken to be proportional to the heat  $Q_1$  supplied to the engine by the hot reservoir (source). The cross-section of the *outgoing pipe line* at the bottom of the circle is proportional to the heat  $Q_2$  which is rejected to the cold reservoir (sink). The branch line to the right represents that portion of the heat supplied which has been converted by the engine into mechanical work.

### 20.9 Efficiency of Carnot engine

The efficiency of a Carnot engine (heat engine) is defined as the ratio of the mechanical work done during a cycle to the heat supplied during the cycle, both the work and the heat being expressed in the same unit. It is denoted by the symbol  $\eta$ .

If  $Q_1$  and  $Q_2$  be the respective amounts of heat taken in at temperature  $T_1$ K and given out at  $T_2$ K by the engine then  $Q_1 - Q_2$  represents the amount of heat converted into work, i.e., the mechanical work done. Then

$$\begin{aligned}\eta &= \frac{\text{work done}}{\text{heat drawn from the source}} \\ &= \frac{\text{heat converted into work}}{\text{heat drawn from the source}} \\ &= \frac{Q_1 - Q_2}{Q_1} = 1 - \frac{Q_2}{Q_1} \quad (20.1)\end{aligned}$$

#### Efficiency in terms of the temperatures of the source and the sink

From the expressions for the work done during each operation of a Carnot cycle (Art. 20.8) we get

$$\begin{aligned}\frac{Q_1}{Q_2} &= \frac{W_1}{W_3} = \frac{m A r T_1 \log_e (V_2/V_1)}{m A r T_2 \log_e (V_3/V_4)} \\ &= \frac{T_1 \log_e (V_2/V_1)}{T_2 \log_e (V_3/V_4)}\end{aligned}$$

Now the points  $a$  and  $b$  lie on the same isothermal  $ab$  and the points  $c$  and  $d$  lie on the same isothermal  $cd$  (Fig. 19.4). Hence, from the equation describing an isothermal process for an ideal gas, we obtain for the paths  $ab$  and  $cd$ ,

$$P_1 V_1 = P_2 V_2 \quad \text{and} \quad P_3 V_3 = P_4 V_4$$

Again, since the points  $b$  and  $c$  lie on the same adiabatic  $bc$  and the points  $d$  and  $a$  lie on the same adiabatic  $da$ , we have, from the equation describing an adiabatic process for an ideal gas,

$$P_2 V_2^\gamma = P_3 V_3^\gamma \quad \text{and} \quad P_4 V_4^\gamma = P_1 V_1^\gamma$$

Multiplying these four equations together we have

$$P_1 V_1 P_3 V_3 P_2 V_2^\gamma P_4 V_4^\gamma = P_2 V_2 P_4 V_4 P_3 V_3^\gamma P_1 V_1^\gamma$$

Cancelling  $P_1 P_2 P_3 P_4$  on both sides

$$V_1 V_2^\gamma V_3 V_4^\gamma = V_2 V_3^\gamma V_4 V_1^\gamma$$

$$\text{or, } \frac{V_2^\gamma V_4^\gamma}{V_2 V_4} = \frac{V_3^\gamma V_1^\gamma}{V_3 V_1}$$

$$\text{or, } (V_2 V_4)^{\gamma-1} = (V_3 V_1)^{\gamma-1}$$

$$\text{or, } V_2 V_4 = V_3 V_1$$

$$\text{or, } \frac{V_2}{V_1} = \frac{V_3}{V_4}$$

Hence,

$$\frac{Q_1}{Q_2} = \frac{T_1 \log_e \frac{V_2}{V_1}}{T_2 \log_e \frac{V_3}{V_4}} = \frac{T_1}{T_2}$$

$$\text{or, } \frac{Q_2}{Q_1} = \frac{T_2}{T_1}$$

Substituting this in the expression for efficiency

$$\eta = 1 - \frac{Q_2}{Q_1} = 1 - \frac{T_2}{T_1} = \frac{T_1 - T_2}{T_1} \quad (20.2)$$

### Remarks

(i) The expression for efficiency shows that the *efficiency depends only upon the temperatures of the source and the sink* and is always less than unity, since  $T_1 > (T_1 - T_2)$ . The greater the difference between  $T_1$  and  $T_2$  the greater will be the efficiency. In



other words, *heat should be taken in at as high a temperature as possible and rejected at as low a temperature as practicable.* Hundred per cent efficiency can be achieved only when  $T_2$  reaches absolute zero which is impossible to achieve in practice.

(ii) Eqn. (20.2) also shows that the efficiency of a Carnot engine is independent of the nature of the working substance; it depends only on the temperature of the source and the sink. It also follows from this that *two reversible engines working between the same two temperature limits have the same efficiency.*

(iii) Another important conclusion from the consideration of Carnot's cycle follows from the relation

$$\frac{Q_1}{Q_2} = \frac{T_1}{T_2}; \quad \text{or,} \quad \frac{Q_1}{T_1} = \frac{Q_2}{T_2} \quad (20.3)$$

*It means that the amount of heat taken in or rejected out are proportional to the absolute temperatures at which they are taken in or rejected out.*

(iv) It should be noted that, in practice, it is impossible to construct an engine based on Carnot's cycle which consists of perfectly isothermal and adiabatic operations. This is because, the isothermal process will take place only when the piston will move very slowly so as to allow sufficient time for the transfer of heat to take place. On the other hand, the adiabatic process will take place only when the piston moves very fast so that no heat transfer can take place. It is obvious that such sudden changes in the speed of an engine are not possible in practice.

Beside, in any actual engine, it is impossible to eliminate completely the friction between its various moving parts and loss of heat due to radiation and convection.

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**Example 20.1.** Find the efficiency of a Carnot engine working between the steam point and the ice point.

**Soln.**

$$\eta = 1 - \frac{T_2}{T_1}$$

$$T_1 = 273 + 100 = 373 \text{ K}$$

$$T_2 = 273 + 0 = 273 \text{ K}$$

$$\therefore \eta = 1 - \frac{273}{373}$$

$$= \frac{100}{373} = 0.2681$$

$$\% \text{ of efficiency} = 0.2681 \times 100 = 26.81\%.$$

**Example 20.2.** Find the efficiency of a Carnot engine working between  $127^\circ\text{C}$  and  $27^\circ\text{C}$ .

**Soln.**

$$\eta = 1 - \frac{T_2}{T_1}$$

$$T_1 = 273 + 127 = 400 \text{ K}$$

$$T_2 = 273 + 27 = 300 \text{ K}$$

$$\therefore \eta = 1 - \frac{300}{400} = 0.25$$

$$\% \text{ of efficiency} = 0.25 \times 100 = 25\%.$$

**Example 20.3.** A Carnot engine whose low temperature reservoir is at  $7^\circ\text{C}$  has an efficiency of 50%. It is desired to increase the efficiency to 70%. By how many degrees should the temperature of the high temperature reservoir be increased?

**Soln.**

In the first case,

$$\eta = 50\% = 0.5$$

$$T_2 = 273 + 7 = 280 \text{ K}$$

$$\eta = 1 - \frac{T_2}{T_1}$$

$$\text{or, } 0.5 = 1 - \frac{T_2}{T_1} = 1 - \frac{280}{T_1}$$

$$\text{or, } 0.5 = \frac{280}{T_1}$$

$$\text{or, } T_1 = \frac{280}{0.5} = 560 \text{ K.}$$

In the second case,

$$\eta = 0.7$$

$$T_2 = 280 \text{ K.}$$

$$\eta = 1 - \frac{T_2}{T_1}$$

$$\text{or, } 0.7 = 1 - \frac{280}{T_1} \quad \text{or, } 0.3 = \frac{280}{T_1}$$

$$\text{or, } T_1 = \frac{280}{0.3} = 933.33 \text{ K.}$$

Hence the temperature of the high temperature reservoir should be raised by  $(933.33 - 560) = 373.33 \text{ K.}$

**Example 20.4.** The efficiency of a Carnot engine is  $\frac{1}{6}$ . If on reducing the temperature of the sink by  $65^\circ\text{C}$ , the efficiency becomes  $\frac{1}{3}$ , find the temperatures of the source and the sink between which the engine is working.

**Soln.**

Let  $T_1$  and  $T_2$  be the temperatures of the source and sink respectively.

In the first case,

$$\eta = 1 - \frac{T_2}{T_1}; \quad \text{or, } \frac{1}{6} = 1 - \frac{T_2}{T_1} \quad (i)$$



In the second case, temperature of the sink =  $T_2 - 65$

$$\eta = 1 - \frac{T_2 - 65}{T_1}$$

$$\text{or, } \frac{1}{3} = 1 - \frac{T_2 - 65}{T_1} \quad (\text{ii})$$

From (i) and (ii) we get

$$\frac{T_2}{T_1} - \frac{T_2 - 65}{T_1} = \frac{1}{3} - \frac{1}{6}$$

$$\text{or, } \frac{65}{T_1} = \frac{1}{6}$$

$$\text{or, } T_1 = 65 \times 6 = 390 \text{ K} = 390 - 273 = 117^\circ\text{C}.$$

From relation (i)

$$1 - \frac{T_2}{390} = \frac{1}{6}$$

$$\text{or, } \frac{T_2}{390} = \frac{5}{6}$$

$$\text{or, } T_2 = \frac{390 \times 5}{6} = 325 \text{ K} = 325 - 273 = 52^\circ\text{C}.$$

**Example 20.5.** A Carnot engine whose temperature of the source is 400 K takes 200 calories of heat at this temperature and rejects 150 calories of heat in the sink. What is the temperature of the sink? Calculate the efficiency of the engine.

**Soln.**

From the relation  $\frac{Q_1}{T_1} = \frac{Q_2}{T_2}$  we have

$$T_2 = \frac{Q_2}{Q_1} \times T_1$$

$$Q_2 = 200 \text{ cal}$$

$$Q_2 = 150 \text{ cal}$$

$$T_1 = 400 \text{ K.}$$

$$\therefore T_2 = \frac{150}{200} \times 400 = 300 \text{ K.}$$

$$\text{Efficiency } \eta = 1 - \frac{T_2}{T_1}$$

$$= 1 - \frac{300}{400} = 0.25$$

$$\% \text{ of efficiency} = 0.25 \times 100 = 25\%.$$

**Example 20.6.** A Carnot engine is operated between two reservoirs at temperatures of 450 K and 350 K. If the engine receives 1 kcal of heat from the source in each cycle, calculate (i) amount of heat rejected to the sink in each cycle, (ii) efficiency of the engine and (iii) work done by the engine in each cycle.

**Soln.**

$$(i) \quad \frac{Q_1}{T_1} = \frac{Q_2}{T_2}$$

$$\text{or, } Q_2 = \frac{T_2}{T_1} \times Q_1$$

$$T_1 = 450 \text{ K}$$

$$T_2 = 350 \text{ K}$$

$$Q_1 = 1 \text{ kcal} = 1000 \text{ cal.}$$

$$\therefore Q_2 = \frac{350}{450} \times 1000$$

$$= 777.77 \text{ cal.} = 0.778 \text{ kcal.}$$

$$(ii) \quad \eta = 1 - \frac{T_2}{T_1}$$

$$= 1 - \frac{350}{450} = 0.222\%.$$

$$\% \text{ of efficiency} = 22.2\%.$$

$$\begin{aligned} \text{(iii) Work done} &= Q_1 - Q_2 = 1 - 0.778 \\ &= 0.222 \text{ kcal.} \end{aligned}$$

$$J = 4186 \text{ joules/kcal.}$$

$$\therefore W = 0.222 \times J \text{ joules.}$$

$$= 0.222 \times 4186 \text{ joules}$$

$$= 930 \text{ joules.}$$

**Example 20.7.** An engine works in a Carnot cycle between the temperatures  $100^\circ\text{C}$  and  $0^\circ\text{C}$ . If the work done in the cycle is 1200 kilogram-metres, find how much heat, measured in calories, is taken in at the higher temperature.

**Soln.**

If  $Q_1$  is the heat taken in at  $(273 + 100) = 373 \text{ K}$ , then

$$\frac{Q_1}{Q_2} = \frac{373}{273}; \quad \text{or, } Q_2 = \frac{273}{373} \times Q_1$$

If  $W$  is the work done in ergs, then

$W = J \cdot (Q_1 - Q_2)$  where  $J$  is the mechanical equivalent of heat.

$$\text{or, } 1200 \times 1000 \times 100 \times 981 = 4.18 \times 10^7 \left( Q_1 - \frac{273}{373} Q_1 \right).$$

$$\begin{aligned} [1200 \text{ kilogram-metres} &= 1200 \times 1000 \times 100 \text{ gm-cm}] \\ &= 1200 \times 1000 \times 100 \times 981 \text{ dynes}] \end{aligned}$$

$$\text{or, } 1.2 \times 10^8 \times 981 = 4.18 \times 10^7 \times \frac{100}{373} Q_1$$

$$\begin{aligned} \therefore Q_1 &= \frac{1.2 \times 10^8 \times 981 \times 373}{4.18 \times 10^7 \times 100} \\ &= 1.08 \times 10^4 \text{ calories} \end{aligned}$$

**Example 20.8.** Find the efficiency of an engine requiring  $3 \times 10^6$  cal. of heat per horse-power hour and compare it with that of a



perfect reversible engine, assuming that the source is at  $1000^{\circ}\text{C}$  and the sink at  $0^{\circ}\text{C}$ .

**Soln.**

Amount of work done by the engine in an hour

$$= 746.4 \times 60 \times 60 \text{ joules}$$

$$\text{Efficiency, } \eta = \frac{\text{work done}}{\text{heat supplied}}$$

$$= \frac{W \text{ joules}}{Q \text{ cal}} = \frac{W \text{ joules}}{J.Q \text{ joules}}$$

$$= \frac{746.4 \times 60 \times 60}{4.2 \times 3 \times 10^6} = 0.213$$

$$\% \text{ efficiency} = 0.213 \times 100 = 21.3\%$$

Efficiency of the perfect reversible engine

$$= 1 - \frac{T_2}{T_1} = 1 - \frac{(273 + 0)}{(273 + 1000)}$$

$$= 1 - \frac{273}{1273} = 0.785$$

$$\% \text{ efficiency} = 0.785 \times 100 = 78.5\%$$

The ratio of the efficiencies = 21 : 78 (approx.)

## 20.10 Refrigerator

A refrigeration may be regarded as a heat engine operating in reverse. Carnot's cycle is perfectly reversible; therefore, it can work as a heat engine and also as a refrigerator. When it works as a heat engine it absorbs an amount of heat  $Q_1$  from the source at a higher temperature  $T_1$ , does an amount of work  $W$  and rejects an amount of heat  $Q_2$  to a sink at a lower temperature  $T_2$ . When it works as a refrigerator, it absorbs an amount of heat  $Q_2$  from the sink at the lower temperature  $T_2$ , an amount of work  $W$  is done on it by some

external agency and rejects an amount of heat  $Q_1$  to the source at a higher temperature  $T_1$ . Thus in the case of refrigerator, *heat flows from a body at a lower temperature to a body at a higher temperature with the help of an external agency*. In the case of an ordinary household refrigerator, the food and ice cubes constitute the sink, working substance is freon and work is done by the electric motor and the hot reservoir is the air surrounding the refrigerator.

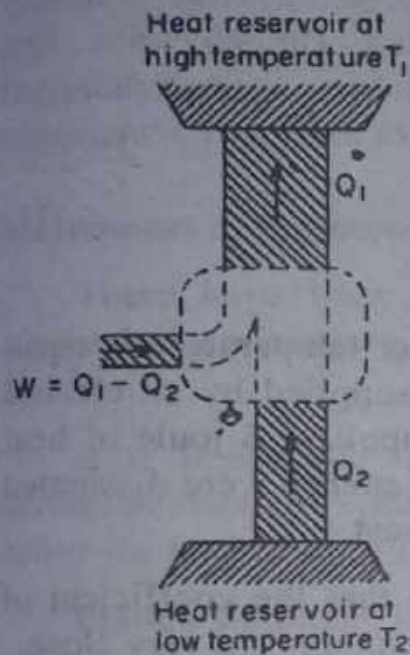


Fig. 20.8

The flow diagram of a refrigerator is shown in Fig. 20.8. The P-V diagram of the refrigerator cycle will be similar to Fig. 20.7 with the arrows reversed.

The purpose of any refrigerator is to extract as much heat as possible from a cold reservoir with the expenditure of as little work as possible. The *output*, so to speak, is the heat extracted from the cold reservoir, and the *input* is the work supplied by the external agency to operate the machine. How to define the efficiency of a refrigerator then?

A convenient measure of the efficiency or, better still, the performance of a refrigerator is expressed by the *coefficient of performance*,  $E$  which is the ratio of the heat extracted from the cold reservoir to the work done.

$$E = \frac{\text{heat extracted from the cold reservoir}}{\text{work done to operate the refrigerator}}$$

$$= \frac{Q_2}{W} = \frac{Q_2}{Q_1 - Q_2} \quad (20.4)$$

From the Carnot's principle we know

$$\frac{Q_1}{T_1} = \frac{Q_2}{T_2}$$

$$\therefore E = \frac{Q_2}{Q_1 - Q_2} = \frac{T_2}{T_1 - T_2} \quad (20.5)$$



In the case of a heat engine, the efficiency cannot be more than unity, but in the case of a refrigerator, the coefficient of performance may be considerably larger than unity. If, for the sake of argument, one assumes the value 5,

$$E = \frac{Q_2}{W} = 5$$

$$\text{But, } Q_2 = Q_1 - W$$

$$\text{hence, } \frac{Q_1 - W}{W} = 5$$

$$\text{or, } \frac{Q_1}{W} = 6$$

Therefore, the heat liberated at the higher temperature is equal to six times the work done. If the work is supplied by an electric motor, for every joule of electrical energy supplied, 6 joule of heat will be liberated; whereas if 1 J of electrical energy were dissipated in a resistor, one could obtain at most 1 J of heat.

From the relation (20.5) it can be seen that the coefficient of performance is very high when the two temperatures are very close.

### 20.11 Background of the second law of thermodynamics

Even with the improved engineering design, the heat engines constructed were inefficient devices. Only a fraction of the heat absorbed at the high-temperature source could be converted into useful mechanical work, discharging a sizable fraction of this heat at the low-temperature exhaust of the engine. This discharged heat represented a waste as it remained unconverted into mechanical energy. It remained a hope to design an engine that could convert all the heat absorbed into mechanical energy, and also could take heat from an abundant reservoir, like the *ocean*, thus dispensing with the necessity of providing a source of heat at a higher temperature than the surroundings by burning fuel. Likewise, it also remained a hope to design an engine that could simply transfer heat from a cold body to a hot body without the aid of an external agency. Neither of these expectations was against the first law of thermodynamics. The first law of thermodynamics simply sets out the equivalence between the



quantity of heat used and the mechanical work done. But the first law of thermodynamics is a qualitative statement; it does not specify the condition under which conversion of heat into work is possible neither the direction in which heat transfer can take place. It does not preclude the possibility of the existence of a 100 per cent efficient engine or a self-acting reservoir. Nevertheless, neither of these expectations has ever been realized in practice and there is reason to believe they never will be. This gap between expectations and achievements has been bridged by the second law of thermodynamics, which asserts, as a result of generalization of experience, that such expectations can never be realized in practice.

### Statements of the second law of thermodynamics

There have been many statements of the second law, each emphasizing a particular facet of the law. The original statement of Kelvin is

*It is impossible by means of inanimate material agency to derive mechanical effect from any portion of matter by cooling it below the temperature of the coldest of the surrounding objects.*

Planck stated the law in the form

*It is impossible to construct an engine which, working in a complete cycle, will produce no effect other than the raising of a weight and the cooling of a heat reservoir.*

A new term reservoir is used to explain the second law. A reservoir is a device having infinite thermal capacity and which can absorb, retain or reject unlimited quantity of heat without any change in its temperature. The statements of Kelvin and Planck may be combined into one equivalent statement, hereafter referred to as **Kelvin-Planck** statement of the second law, as

*It is impossible to construct an engine which, operating in a cycle, will produce no other effect than the extraction of heat from a reservoir and the performance of an equivalent amount of work.*

or,

*No cyclic process exist that produces no other effect than the extraction of heat from a body and its conversion into an equivalent amount of mechanical work.*

There is another alternative statement for the second law by Clausius who formulated his law from a *negative* standpoint. He considered the working of not a heat engine, but of a *refrigerator*. We have seen that refrigerator is a device that performs a cycle in the direction opposite to that of a heat engine. Its net result is thus to abstract some heat at a lower temperature and reject a larger amount at a higher temperature. Our experience shows that to transfer heat from a cold body to a hot reservoir, work must always be *done* on the working substance (*refrigerant*). Without the expenditure of work such a transfer is *not* possible.

**Clausius - statement** – *It is impossible for a self-acting machine working in a cyclical process, unaided by any external agency, to convey heat from a body at a lower temperature to a body at a higher temperature.*

It may be noted that the above statements do not preclude the possibility of converting heat into work or transferring heat from a colder to a hotter body, but they repudiate that these effects can be produced by cyclic processes. The two alternative statements which underscore these possibilities of occurrence are :

(i) It is impossible to extract heat continuously from a single reservoir at a single temperature and convert it into work unless a positive amount of heat is delivered at a lower temperature. It implies that there should be two reservoirs for any heat engine. One reservoir, called the *source*, is taken at a higher temperature and the other reservoir, called the *sink* is taken at a lower temperature.

(ii) It is impossible to transfer heat from a colder body to a hotter one without the expenditure of mechanical work.

The statements of Kelvin-Planck and Clausius appear, at first sight, unrelated. But a closer scrutiny would reveal that they are, in fact, equivalent in all aspects.

It is not possible to prove the second law of thermodynamics by direct experiment but its validity is fully substantiated by the experimental verification of a great number of predictions based upon it. The second law is not a deduction from the first but stands by itself as a separate law of nature, referring to an aspect of nature different from that contemplated by the first law. The first law denies the possibility of creating or destroying energy; the second denies the possibility of



utilizing energy in a particular way. The continual operation of a machine that creates its own energy and thus violates the first law is called the *perpetual motion of the first kind*.

If the second law as stated above were not true, it would be possible to drive a steamship across the ocean by extracting heat from the ocean or to run a power plant by extracting heat from the surrounding air. A cyclic device that would *continuously* abstract heat from a single reservoir and convert the same *completely* to mechanical work is called a *perpetual machine of the second kind*. It would not violate the first law (conservation of energy principle) since it would not create energy. The second law is thus sometimes stated as : "*A perpetual machine of the second kind is impossible*".

## 20.12 Carnot's theorem

The second law of thermodynamics leads to two important deductions. These were first stated in a theorem of great practical importance by Carnot and has come to be known as Carnot's theorem.

(i) All reversible engines working between the same temperature limits have the same efficiency.

(ii) Working between the same initial and final temperatures, no engine can be more efficient than a reversible one.

### Proof

(i) Let us start with two reversible engines A and B operating between the temperatures  $T_1$  and  $T_2$  where  $T_1 > T_2$ . The initial lengths and pressures of the stroke and the working substance of the engines may be different. We shall see if it is possible for one of the engines to be more efficient than the other. To start with, let us assume that the efficiency  $\eta$  of A is greater than the efficiency  $\eta'$  of B.

Let A work as a heat engine and B as a refrigerator, conveying heat from the sink to the source (Fig. 20.9). A takes an amount of heat  $Q_1$  from the source at a temperature  $T_1$ , does an external work  $W$  and gives out heat  $Q_2$  at  $T_2$ . The engine B extracts an amount of heat  $Q'_2$  from the sink at  $T_2$ , does an amount of work  $W'$  on the working substance and gives out heat  $Q'_1$  to the source at  $T_1$ . The



engines are then coupled mechanically and their stroke length adjusted so that the work done per cycle by A is just sufficient to operate B.

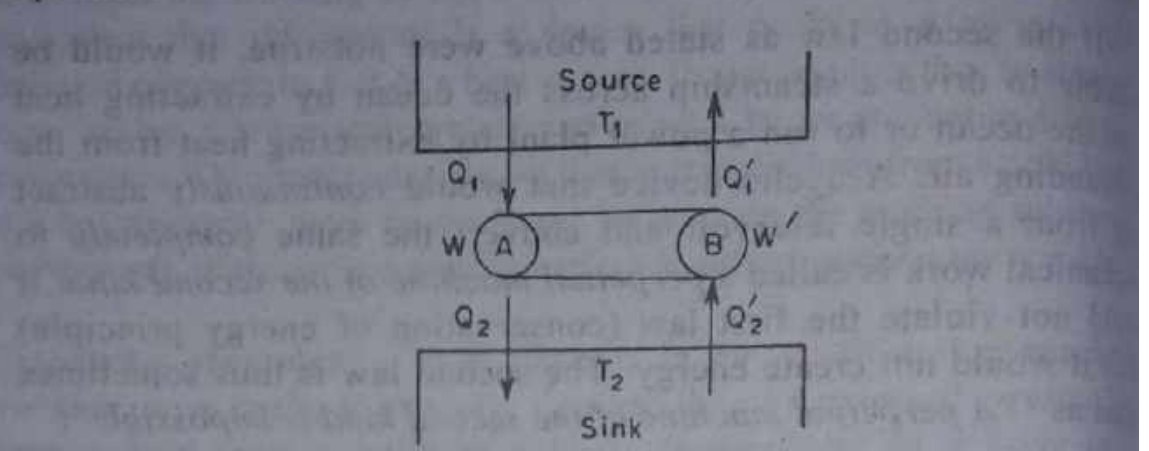


Fig. 20.9

Then,

since  $\eta > \eta'$  (assumption),

$$\frac{Q_1 - Q_2}{Q_1} > \frac{Q'_1 - Q'_2}{Q'_1}$$

Since the work done per cycle by A equals the work done per cycle on B,

$$W = W'$$

$$\text{or, } Q_1 - Q_2 = Q'_1 - Q'_2$$

$$\text{or, } \frac{1}{Q_1} > \frac{1}{Q'_1}$$

$$\text{or, } Q_1 < Q'_1$$

Hence, from the equality of works

$$Q_2 < Q'_2$$

Thus both  $(Q'_1 - Q_1)$  and  $(Q'_2 - Q_2)$  are positive quantities. This means the hot source gains heat  $(Q'_1 - Q_1)$  and the cold source loses heat  $(Q'_2 - Q_2)$ . But no work is done in the process by the combined system A+B. Thus heat is being transferred from a body at a lower temperature to a body at a higher temperature without any external

work being done on the system. This is a direct contradiction of the Clausius' statement of the second law of thermodynamics. Thus  $\eta$  cannot be greater than  $\eta'$ . Hence two reversible engines working between the same two temperature limits have the same efficiency.

(ii) Let us now consider two engines S and R of which S is irreversible and R reversible (Fig. 20.10). Both the engines are working between the same two temperature limits  $T_1$  and  $T_2$  where  $T_1 > T_2$ . To start with, we shall assume that the efficiency  $\eta$  of the irreversible engine S is greater than the efficiency  $\eta'$  of the reversible engine R.

Let the two engines be coupled together in such a way that the irreversible engine S drives the reversible engine R backwards. This means that R is made to act as a refrigerator, conveying heat from the sink to the source.

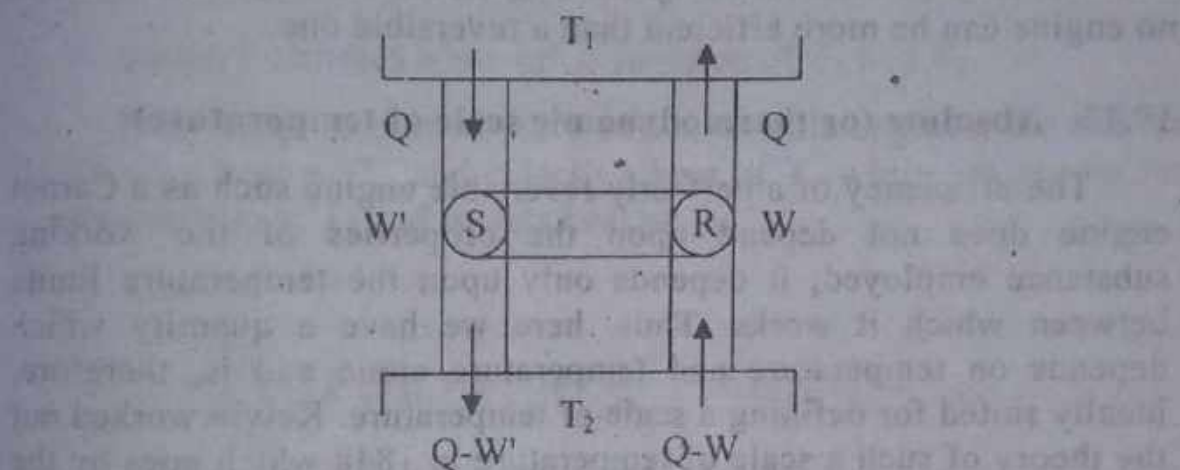


Fig. 20.10

Let S absorb an amount of heat  $Q$  from the source at the temperature  $T_1$ , perform an amount of work  $W'$  and give out the rest  $Q - W'$  to the sink at temperature  $T_2$ . R in its turn is made to extract a certain amount of heat from the sink and return the same amount of heat  $Q$  to the source after a work  $W$  has been performed on it. This is achieved by adjusting the quantity of working substance. The amount of heat that R abstracts from the sink must, therefore, be  $Q - W$ .

Now since  $\eta > \eta'$  (assumption)

$$\frac{W'}{Q} > \frac{W}{Q}$$



$$\text{or, } W' > W$$

$$\text{Hence } Q - W > Q - W'$$

This means R extracts more heat from the sink than S delivers to it. Thus the net result is that the combined system extracts a certain quantity of heat  $(Q - W) - (Q - W') = W' - W$  per cycle from the sink and converts the whole of it into work, *while the source remains unaffected. This means that the source is unnecessary.* A situation, therefore, arises in which, the system of combined machines, acting as a self-acting machine are able to perform external work, unaided by any external agency, without the necessity of a separately maintained source of heat. This is contrary to the second law of thermodynamics. Hence S cannot be more efficient than R. Thus working between the same temperature limits no engine can be more efficient than a reversible one.

### 19.13 Absolute (or thermodynamic scale of temperature)

The efficiency of a perfectly reversible engine such as a Carnot engine does not depend upon the properties of the working substance employed; it depends only upon the temperature limits between which it works. Thus, here we have a quantity which depends on temperature and temperature alone and is, therefore, ideally suited for defining a scale of temperature. Kelvin worked out the theory of such a scale of temperature in 1848 which goes by the name of *Kelvin's absolute or thermodynamic scale of temperature*. The advantage of this scale over all others is that it does not depend on the particular property of any substance and is in fact based on the fundamental principles of thermodynamics only. It was later shown that it agrees with the perfect scale.

Let  $T_1$  and  $T_2$  be the temperatures on any arbitrary scale of temperature of the source and sink respectively between which a Carnot engine works. If  $Q_1$  is the amount of heat received at  $T_1$  and  $Q_2$  is that rejected at  $T_2$ , then the efficiency of the engine is given by

$$\begin{aligned} \eta &= \frac{\text{output energy}}{\text{input energy}} \\ &= \frac{\text{work done}}{\text{heat taken in}} \end{aligned}$$



$$= \frac{Q_1 - Q_2}{Q_1}$$

$$= 1 - \frac{Q_2}{Q_1} = f(T_1, T_2)$$

where  $f$  is an unknown function of  $T_1$  and  $T_2$ . Rearranging the above equation we get

$$\frac{Q_2}{Q_1} = 1 - f(T_1, T_2)$$

$$\text{or, } \frac{Q_1}{Q_2} = \frac{1}{1 - f(T_1, T_2)} = F(T_1, T_2) \quad (20.6)$$

where  $F$  denotes some other function of  $T_1$  and  $T_2$ .

Now suppose that we have two reversible engines, the first one receiving heat at  $T_1$  and rejecting heat at  $T_2$  while the second one receives heat at  $T_2$  and rejects heat at  $T_3$ .

Then,

$$\frac{Q_1}{Q_2} = F(T_1, T_2) \quad \text{and} \quad \frac{Q_2}{Q_3} = F(T_2, T_3)$$

Since the heat rejected by the first engine,  $Q_2$ , is the heat received by the second engine, the two engines can be combined to constitute a third engine, receiving heat  $Q_1$  at  $T_1$  and rejecting heat  $Q_3$  at  $T_3$ .

Then

$$\frac{Q_1}{Q_3} = F(T_1, T_3)$$

Since

$$\frac{Q_1}{Q_3} = \frac{Q_1}{Q_2} \times \frac{Q_2}{Q_3}$$

we, therefore, have

$$F(T_1, T_3) = F(T_1, T_2) \times F(T_2, T_3)$$

The above relation indicates that the functions  $F(T_1, T_2)$  and  $F(T_2, T_3)$  must be of the forms  $\frac{\psi(T_1)}{\psi(T_2)}$  and  $\frac{\psi(T_2)}{\psi(T_3)}$  respectively where  $\psi$  denotes yet another function of  $T$ ; for in that case

$$\begin{aligned} F(T_1, T_2) \times F(T_2, T_3) &= \frac{\psi(T_1)}{\psi(T_2)} \times \frac{\psi(T_2)}{\psi(T_3)} \\ &= \frac{\psi(T_1)}{\psi(T_3)} = F(T_1, T_3) \end{aligned}$$

Therefore, accepting the form of the function indicated above, we have

$$\frac{Q_1}{Q_2} = F(T_1, T_2) = \frac{\psi(T_1)}{\psi(T_2)} \quad (20.7)$$

It will be seen that if  $T_1 > T_2$ , then  $Q_1 > Q_2$  and therefore  $\psi(T_1) > \psi(T_2)$  or the function  $\psi(T)$  is such that its magnitude increases with the increase of  $T$ . This means that  $\psi(T)$  is a linear function of  $T$  and may, therefore, be used to measure the temperature.

Let us denote  $\psi(T)$  by  $\theta$  where  $\theta$  is a temperature on the new scale.

Then we have

$$\frac{Q_1}{Q_2} = \frac{\psi(T_1)}{\psi(T_2)} = \frac{\theta_1}{\theta_2} \quad (20.8)$$

Eqn. (20.8) is used to define the absolute or thermodynamic scale or Kelvin's absolute scale of temperature. Following eqn. (20.8) Kelvin stated that *the absolute values of two temperatures are to one another in the proportion of heat taken in and the heat rejected in a perfectly reversible engine working between these temperatures as source and sink respectively*. This new scale of temperature depends only upon the characteristics of a reversible heat engine and does not depend upon the nature of the working substance. *Hence the scale so defined is absolute in the sense that it does not depend upon the properties of any particular substance.*

Thus the efficiency will be given by

$$\eta = \frac{Q_1 - Q_2}{Q_1} = \frac{\theta_1 - \theta_2}{\theta_1} = 1 - \frac{\theta_2}{\theta_1}$$

### ***Absolute zero on Kelvin scale***

Consider an engine working between the steam point and zero degree on the absolute scale. Let the temperatures be  $\theta_{\text{steam}}$  and  $\theta_0$  ( $= 0$ ). Then the efficiency of the engine

$$\eta = 1 - \frac{\theta_0}{\theta_{\text{steam}}} = 1$$

i.e., the engine has an efficiency equal to 1.

Similarly an engine working between ice point and absolute zero will have an efficiency

$$\eta = 1 - \frac{\theta_0}{\theta_{\text{ice}}} = 1$$

But efficiency

$$\eta = \frac{Q_1 - Q_2}{Q_1} = 1 - \frac{Q_2}{Q_1}$$

$$\therefore \text{ if } 1 - \frac{Q_2}{Q_1} = 1, \text{ then } \frac{Q_2}{Q_1} = 0;$$

$$\text{or, } Q_2 = 0.$$

It means that if the sink is at the temperature of zero degree absolute, no heat is rejected and the whole of the available energy has been used up in doing useful work and in this case the engine will have 100 per cent efficiency.

Thus the absolute zero temperature on the Kelvin scale is defined as the temperature at which a system undergoes a reversible isothermal process (between two reversible adiabatic surfaces) without any transfer of heat.

No temperature on the absolute scale can be less than zero i.e., negative; for if it would be so,  $Q_2$  would be negative which implies that the engine would convert more heat to work than it takes from



the source, i.e., it should be able to create energy out of nothing. This is impossible and hence  $\theta = 0$  i.e., the absolute zero is the lowest temperature conceivable.

Having found the zero from which temperatures are now to be measured, the only thing which remains arbitrary is the size of the degree. This may be fixed in the same manner as for an ordinary temperature scale by choosing a certain number of degrees between two fixed temperatures. As in all scientific work, the temperature interval between the steam and the ice points is divided into 100 equal divisions on this scale. Each such division may be called 1 Kelvin degree. Hence if  $\theta_{\text{ice}}$  is the ice point, then  $\theta_{\text{ice}} + 100$  is the steam point. A Carnot engine working between the steam point and the ice point will satisfy the condition

$$\begin{aligned} \frac{Q_{\text{steam}}}{Q_{\text{ice}}} &= \frac{\theta_{\text{steam}}}{\theta_{\text{ice}}} \\ &= \frac{\theta_{\text{ice}} + 100}{\theta_{\text{ice}}} = 1 + \frac{100}{\theta_{\text{ice}}} \end{aligned}$$

where  $Q_{\text{steam}}$  is the amount of heat extracted from the source at the steam point, and  $Q_{\text{ice}}$  is the quantity delivered to the sink at the ice point. By measuring  $Q_{\text{steam}}$  and  $Q_{\text{ice}}$ , the value of  $\theta_{\text{ice}}$  can be found from the above relation. This enables us to set up the new scale.

### Equality of ideal-gas temperature and Kelvin temperature

A practical realization of the Kelvin scale may be effected by considering a Carnot engine which uses perfect or ideal gas as the working substance. If  $T_1$  and  $T_2$  are the temperatures of the source and sink respectively on the gas scale, then

$$\frac{Q_1}{Q_2} = \frac{T_1}{T_2}$$

where  $Q_1$  and  $Q_2$  are the amounts of heat extracted and rejected respectively. If the temperatures of the source and the sink are  $\theta_1$  and  $\theta_2$  respectively on the absolute scale, then, again

$$\frac{Q_1}{Q_2} = \frac{\theta_1}{\theta_2}$$

Thus we have,

$$\frac{Q_1}{Q_2} = \frac{\theta_1}{\theta_2} = \frac{T_1}{T_2}$$

If  $\theta_2 = 0$ , then  $Q_2 = 0$ . Therefore,  $T_2$  will also be zero. Thus the absolute zero of the thermodynamic scale coincides with the zero of the perfect gas scale. Again the number of degrees between the ice and the steam point is 100 in both the absolute and the gas scale. Therefore, we have

$$\frac{Q_{\text{steam}}}{Q_{\text{ice}}} = \frac{T_{\text{ice}} + 100}{T_{\text{ice}}} = \frac{\theta_{\text{ice}} + 100}{\theta_{\text{ice}}}$$

$$\text{or, } \theta_{\text{ice}} = T_{\text{ice}}$$

i.e., the ice points on the two scales are the same. Thus Kelvin's thermodynamic or absolute scale of temperature is numerically equal to the ideal gas scale temperature and can be measured with a gas thermometer whose readings have been corrected to the ideal gas scale.

Again, if  $Q_1$  and  $Q_2$  are the heat extracted and rejected at temperatures  $T_1$  and  $T_2$  on the gas scale, then we have

$$\frac{Q_1}{Q_2} = \frac{T_1}{T_2}$$

But, as proved earlier

$$\frac{Q_1}{Q_2} = \frac{\psi(T_1)}{\psi(T_2)} = \frac{\theta_1}{\theta_2}$$

$\theta_1$  and  $\theta_2$  being the corresponding temperatures on the Kelvin scale.

$$\therefore \frac{T_1}{T_2} = \frac{\theta_1}{\theta_2}$$

Therefore, the ratio of the temperatures on the perfect gas scale is the same as the ratio of the temperatures on the Kelvin's scale.

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**Example 20.9.** A Carnot engine working as a refrigerator between 260 K and 300 K receives 500 calories of heat from the reservoir at the lower temperature. Calculate (i) the amount of heat rejected to the reservoir at the higher temperature and (ii) the amount of work done in each cycle to operate the reservoir.

**Soln.**

$$(i) \quad \frac{Q_1}{T_1} = \frac{Q_2}{T_2}; \quad \text{or,} \quad Q_1 = \frac{T_1}{T_2} \times Q_2$$

$$T_1 = 300 \text{ K} \quad T_2 = 260 \text{ K} \quad Q_2 = 500 \text{ cal.}$$

$$\therefore Q_1 = \frac{300}{260} \times 500 = 576.92 \text{ cal.}$$

$$(ii) \quad W = Q_1 - Q_2 = 576.92 - 500$$

$$= 76.92 \text{ cal}$$

$$= 76.92 \times 4.2 \text{ joules}$$

$$= 323 \text{ joules.}$$

**Example 20.10.** A Carnot refrigerator takes heat from water at 0°C and discards it to a room at 27°C. 1 Kg of water at 0°C is to be changed into ice at 0°C. (i) How many calories of heat are discarded to the room? (ii) What is the work done by the refrigerator in this process? (iii) What is the co-efficient of performance of the machine? (latent heat of fusion of ice = 80 cal/gm)

**Soln.**

$$T_1 = 273 + 27 = 300 \text{ K}$$

$$T_2 = 273 + 0 = 273 \text{ K} \quad Q_1 = ?$$

$$Q_2 = m.L = 1000 \times 80$$

$$= 80,000 \text{ cal}$$

$$(i) \quad \frac{Q_1}{Q_2} = \frac{T_1}{T_2}; \quad \text{or,} \quad Q_1 = \frac{T_1}{T_2} \times Q_2$$

$$= \frac{300}{273} \times 8 \times 10^4 = 87,912 \text{ cal.}$$



$$\begin{aligned}
 \text{(ii) work done, } W &= Q_1 - Q_2 \\
 &= 87912 - 80000 = 7912 \text{ cal.} \\
 &= 7912 \times 4.2 \text{ joules} \\
 &= 33230 \text{ joules.}
 \end{aligned}$$

(iii) coefficient of performance, E

$$= \frac{Q_2}{W} = \frac{Q_2}{Q_1 - Q_2} = \frac{80,000}{7912} = 10.11.$$

**Example 20.11.** Assuming that a domestic refrigerator can be regarded as a reversible engine working between the temperature of melting ice and the room temperature  $17^\circ\text{C}$ , calculate the work that must be done in terms of K.W.H. to freeze 1kg of water already at  $0^\circ\text{C}$  ( $J = 4.2 \text{ Joules/cal.}$ , latent heat of fusion of ice =  $80 \text{ cal/gm}$ ).

**Soln.**

Heat that must be extracted to convert 1kg of water at  $0^\circ\text{C}$  into ice at  $0^\circ\text{C}$  is

$$Q_2 = mL = 1000 \times 80 = 8 \times 10^4 \text{ cal.}$$

$$\text{From } \frac{Q_1}{T_1} = \frac{Q_2}{T_2}$$

Heat rejected at  $(273 + 17) = 290 \text{ K}$  is

$$Q_1 = \frac{Q_2}{T_2} \cdot T_1 = \frac{8 \times 10^4}{273} \times 290$$

$$[T_2 = (273 + 0) = 273 \text{ K.}]$$

$$\therefore Q_1 = 8.5 \times 10^4 \text{ cal.}$$

Hence work done,

$$\begin{aligned}
 W &= Q_1 - Q_2 = (8.5 - 8.0) \times 10^4 \text{ cal} \\
 &= 5 \times 10^3 \text{ cal} \\
 &= 5 \times 10^3 \times 4.2 \text{ joules} \\
 &= 21 \times 10^3 \text{ joules}
 \end{aligned}$$

$$1 \text{ K.W.H.} = 1000 \text{ joules / sec} \times 3600 \text{ sec} \\ = 36 \times 10^5 \text{ joules}$$

$$\therefore 21 \times 10^3 \text{ joules} = \frac{21 \times 10^3}{36 \times 10^5} \\ = 0.58 \times 10^{-3} \text{ K.W.H.}$$

**Example 20.12.** A refrigerator is driven by a 1000 watt electric motor, which is operating at an efficiency of 60 per cent. If the refrigerator can be treated as a reversible heat engine operating between  $0^\circ\text{C}$  and room temperature which is  $20^\circ\text{C}$ , calculate the time required by it to freeze 100 kg of water, which is at  $0^\circ\text{C}$ . Heat loss may be neglected. (Latent heat of fusion of ice = 80 cal/gm,  $J = 4.2 \text{ joule/cal}$ ).

**Soln.**

Quantity of heat,  $Q_2$ , that must be extracted to convert 100 kg of water at  $0^\circ\text{C}$  into ice at  $0^\circ\text{C}$  is

$$= 100 \times 1000 \times 80 \text{ cal} \\ = 8 \times 10^6 \text{ cal.}$$

Quantity of heat,  $Q_1$ , that must be delivered to the source, i.e., the atmosphere at  $20^\circ\text{C}$  is obtained from the relation

$$\frac{Q_1}{T_1} = \frac{Q_2}{T_2}; \text{ or, } Q_1 = \frac{Q_2}{T_2} \times T_1$$

$$T_1 = (273 + 20) = 293 \text{ K}$$

$$T_2 = (273 + 0) = 273 \text{ K}$$

$$\therefore Q_1 = \frac{8 \times 10^6}{273} \times 293 \\ = 8.58 \times 10^6 \text{ cal.}$$

$$\text{Work done, } W = Q_1 - Q_2 \\ = (8.58 - 8.0) \times 10^6 \text{ cal} \\ = 0.58 \times 10^6 \times 4.2 \text{ joules} \\ = 2.436 \times 10^6 \text{ joules}$$

Efficiency of the motor is 60%. So the motor works at 600 watt.  
or,  $E.i. = 600 \text{ watt}$

$\therefore$  Energy input = E.i.t. joules  
= work done.

$\therefore$  E.i.t. =  $2.436 \times 10^6$  joules

or,  $600t = 2.436 \times 10^6$  joules

or,  $t = \frac{2.436 \times 10^6}{600}$  secs

=  $0.00406 \times 10^6$  sec.

= 4060 sec. = **67.66 minutes.**

**Example 20.13.** A Carnot engine is operated between two reservoirs at temperatures 400 K and 300 K.

(a) If the engine receives 1200 cal from the reservoir at 400 K in each cycle, how many calories does it reject to the reservoir at 300 K?

(b) If the engine is operated in reverse as a refrigerator and receives 1200 cal from the reservoir at 300 K, how many calories does it deliver to the reservoir at 400 K?

**Soln.**

(a) From  $\frac{Q_1}{T_1} = \frac{Q_2}{T_2}$

we have  $Q_2 = \frac{Q_1}{T_1} \times T_2$

$Q_1 = 1200$  cal       $T_1 = 400$  K       $T_2 = 300$  K

$\therefore Q_2 = \frac{1200}{400} \times 300 = \mathbf{900}$  cal.

(b) Again from  $\frac{Q_1}{T_1} = \frac{Q_2}{T_2}$

we have  $Q_1 = \frac{Q_2}{T_2} \times T_1$

$T_1 = 400$  K       $T_2 = 300$  K       $Q_2 = 1200$  cal.

$\therefore Q_1 = \frac{1200}{300} \times 400 = \mathbf{1600}$  cal.



## EXERCISES

- [1] State and fully explain the significance of the first law of thermodynamics.
- [2] Distinguish between reversible and irreversible processes. To which category the following processes belong and why?
  - (a) isothermal expansion of a gas;
  - (b) adiabatic compression of a gas;
  - (c) transfer of heat from one body to another body at lower temperature;
  - (d) production of heat by friction;
  - (e) operation of an engine on Carnot's cycle.
- [3] Explain what is meant by thermal equilibrium. Distinguish between a thermodynamic process and a thermodynamic cycle.
- [4] Describe a Carnot cycle. Obtain expressions for the work done in each operation of the cycle and the net work done in the cycle.
- [5] Describe a Carnot cycle and show how the work done in one operation is represented on a P-V diagram.  
Calculate the work done in each operation of the cycle when the working substance is a perfect gas.
- [6] What is efficiency of an engine? Derive an expression for the efficiency of a Carnot engine in terms of the temperatures of the source and the sink.
- [7] What do you mean by efficiency of a heat engine. Show that the efficiency of a Carnot engine using an ideal gas as the working substance is  $\eta = \frac{T_1 - T_2}{T_1}$  where  $T_1$  and  $T_2$  are the temperatures of the source and the sink respectively.
- [8] "A refrigerator is a heat engine operating in reverse". Explain. Obtain an expression for the coefficient of performance of a refrigerator.
- [9] What is the difference between a heat engine and a refrigerator? Can a Carnot cycle work both as a heat engine and a refrigerator? Why is it not possible to construct an engine based on Carnot's cycle?

- [10] State second law of thermodynamics. State and prove Carnot's theorem.
- [11] What is a reversible cycle and what do you mean by a reversible engine?
- Prove that the efficiency of all reversible engines working between the same two temperatures must be the same and show also that the reversible form of engine is the most efficient form.
- [12] State the second law of thermodynamics in as many ways as you can. Show that no engine can be more efficient than a Carnot engine working between the same two temperatures.
- [13] Derive an expression for the efficiency of a Carnot engine in terms of the temperatures of the source and the sink. Show how an absolute scale of temperature, independent of the nature of the working substance, can be defined with the help of this expression.
- [14] Show how Carnot's principle leads to an absolute scale of temperature. Show that the ideal gas scale and the absolute (thermodynamic) scale are identical. How is the absolute scale realised in practice?
- [15] Define a temperature scale without making use of the peculiarities of any selected thermometric substance. Show that the ratio of the two temperatures as measured on the Kelvin scale is identical with the ratio of the same two temperatures on a perfect gas scale.
- [16] Define Kelvin's absolute or thermodynamic scale of temperature. What is the meaning of absolute zero on this scale? Is a negative temperature possible on this scale? Show how this scale agrees with that of a perfect gas scale.
- [17] A Carnot engine works between  $450^{\circ}\text{C}$  and  $20^{\circ}\text{C}$ . What is its efficiency? [58.4%].
- [18] An ideal heat engine has an efficiency of 25 per cent when its low-temperature reservoir is  $25^{\circ}\text{C}$ . It is desired to increase the efficiency to 35 per cent. What change should be made in either (a) the temperature of the low-temperature reservoir, or (b) temperature of the high-temperature reservoir? [ $-39^{\circ}\text{C}$ ,  $61^{\circ}\text{C}$ ].
- [19] Calculate the maximum efficiency of a reversible heat engine working between the temperatures  $167^{\circ}\text{C}$  and  $57^{\circ}\text{C}$ . [0.25].
- [20] An engine works in a Carnot cycle between the temperatures  $100^{\circ}\text{C}$  and  $15^{\circ}\text{C}$ . If the work done in each cycle is 400 kilogram-metres,



how much heat, measured in calories, is taken in at the upper temperature? [ $4.12 \times 10^3$  cal]

- [21] A Carnot engine whose temperature of the source is 400 K takes 500 calories of the heat at this temperature and rejects 400 calories of heat to the sink. What is the temperature of the sink? Calculate the efficiency of the engine. [320 K, 20%].
- [22] A Carnot engine, with the cold body temperature of  $17^\circ\text{C}$  has 50% efficiency. By how much should the temperature of its hot body be changed to increase the efficiency to 60%? When will the efficiency be 100%? [ $145^\circ\text{C}$ , when  $T_2 = 0$  K].
- [23] A Carnot engine is operated between two reservoirs at temperatures 500 K and 400 K. If the engine receives 2000 calories of heat from the source in each cycle, calculate (i) the amount of heat rejected to the sink in each cycle, (ii) the efficiency of the engine and (iii) the work done by the engine in each cycle in joules and kilo-watt hours. [(i) 1600 cal (ii) 20% (iii) 1780 joules,  $4.944 \times 10^{-4}$  K.W.hr].
- [24] A Carnot engine working as a refrigerator between 250 K and 300 K receives 1000 calories of heat from the reservoir at the lower temperature. (i) Calculate the amount of heat rejected to the reservoir at the higher temperature. (ii) Calculate the amount of work done in each cycle to operate the refrigerator.
- [25] A Carnot engine, having a perfect gas as working substance, is driven backwards and is used for freezing water already at  $0^\circ\text{C}$ . The engine is driven by a 500-W electric motor, having an efficiency of 60%, how long will it take to freeze 9 kg of water. Take  $15^\circ\text{C}$  and  $0^\circ\text{C}$  as the working temperatures of the engine and assume that there are no heat losses in the refrigerating system. Latent heat of fusion of ice = 80 cal/gm;  $J = 4.2$  joules/cal. [9.23 min].
- [26] An inventor claims to have developed an engine working between 600 K and 300 K capable of having an efficiency of 52%. Comment on his claim. [The claim is invalid].
- [27] A Carnot engine is operated between the ice point and the steam point. If the engine receives 746 calories from the source in each cycle, how many calories does it reject to the sink? If the engine is now used as a refrigerator and receives the same heat from the sink, how many calories it delivers to the source? Find the work done by the engine in each case. [546 cal; 1019 cal; 200 cal; 273 cal].



## CHAPTER XXI

## ENTROPY

*Clausius theorem of inequality – Entropy – Change in entropy for a reversible process – Change in entropy for an irreversible process – Entropy and second law of thermodynamics – Temperature and entropy diagram – Calculation of entropy changes in reversible process – Entropy of a perfect gas: general case – Approximate value of heat absorbed – Concept of entropy – Solved problems – Exercises.*

## 21.1. Clausius theorem of inequality

Before considering entropy, we shall consider a system K (Fig. 21.1) performing a cyclic process, consisting of a sequence of states through which the system passes. During the process, the system interacts with a series of heat reservoirs  $R_1, R_2, \dots, R_n$  at temperatures  $T_1, T_2, \dots, T_n$  respectively. It is obvious that the initial state  $i$  and the final state  $f$  of the process coincide. During the process, the system exchanges heat energies  $Q_1, Q_2, \dots, Q_n$  with the heat reservoirs. If, during interaction with the reservoirs, the system absorbs heat,  $Q$ 's are positive and  $Q$ 's are negative when heat is rejected by it.

We are to prove that

$$\sum_{i=1}^n \frac{Q_i}{T_i} \leq 0 \quad (i)$$

according as the cycle is irreversible or reversible.

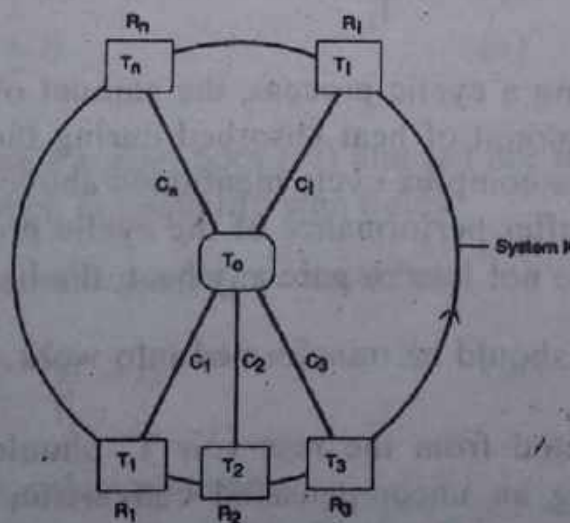


Fig. 21.1.

**Proof**

To prove the inequality, let us introduce a heat reservoir at any arbitrary temperature  $T_0$  and consider  $n$  – Carnot engines (Fig. 21.1)  $C_1, C_2, \dots, C_n$  operating between  $(T_0, T_1), (T_0, T_2), \dots, (T_0, T_n)$ . The sizes of the cycles are so adjusted that the amount of heat rejected by a typical Carnot cycle  $C_i (T_0, T_i)$  at  $T_i$  is exactly equal to the amount of heat absorbed by  $K$  during the cyclic process.

Let us now consider a complex cycle consisting of the cycle performed by  $K$  and the  $n$ -Carnot cycles ( $C_i$ 's), each performing once.

$$\therefore \text{Complex cycle} = K_{\text{cycle}} + C_1 + C_2 + \dots + C_n$$

As a consequence of the complex cycle,

- (i) the system  $K$  returns to its initial state,
  - (ii) none of the heat reservoirs  $R_1, R_2, \dots, R_n$  loses or gains heat during the complex cycle,
- and (iii) the arbitrary reservoir at  $T_0$  loses an amount of heat given by

$$\sum_{i=1}^n Q_{i,0} = Q_0 = T_0 \sum \frac{Q_i}{T_i} \quad (\text{ii})$$

$$[\text{For a typical cycle, } C_i (T_0, T_i) : \frac{Q_{i,0}}{Q_i} = \frac{T_0}{T_i};$$

$$\text{or, } Q_{i,0} = T_0 \frac{Q_i}{T_i} = \text{heat lost by } T_0]$$

Now during a cyclic process, the amount of work performed is equal to the amount of heat absorbed during the process. Since the system  $K$  in the complex cycle mentioned above returns to its initial configuration after performance of the cyclic process, reservoirs  $R_1, R_2, \dots, R_n$  do not lose or gain any heat; the heat absorbed from  $T_0$  ( $= T_0 \sum \frac{Q_i}{T_i}$ ) should be transformed into work. This means that the heat  $Q_0$  extracted from the reservoir  $T_0$  should be converted into work, implying an uncompensated conversion of heat into work. This does not contradict the first law of thermodynamics. According



to this law, it is a matter of indifference whether the reservoir gains or loses heat as long as the mechanical system loses or gains the same quantity of work. The situation is different when the second law of thermodynamics is considered.

If the heat reservoir loses heat and the mechanical system gains an equal amount of work, it means that the whole of the heat absorbed from the reservoir is converted into work. But this violates the Kelvin-Planck statement of the second law of thermodynamics.

$$\therefore W = Q_0 = T_0 \sum \frac{Q_i}{T_i} \text{ must be either negative or zero.}$$

$$\text{or, } T_0 \sum \frac{Q_i}{T_i} \leq 0 \quad (\text{iii})$$

$$\text{or, } \sum \frac{Q_i}{T_i} \leq 0 \quad (\text{iv})$$

which proves the proposition as given in eqn. (i).

If the cycle described by the system K is thermodynamically *reversible*, then  $Q_i$ 's can be replaced by  $-Q_i$ 's when the cycle is described in the opposite direction. Then eqn. (iii) can be written as

$$T_0 \sum -\frac{Q_i}{T_i} \leq 0$$

$$\text{or, } \sum \frac{Q_i}{T_i} \geq 0 \quad (\text{v})$$

For a reversible cycle, both (iv) and (v) are to be satisfied and this is possible when the equality sign holds.

$\therefore$  For *reversible cycles*, we must have

$$\sum \frac{Q_i}{T_i} = 0 \quad (\text{vi})$$

And, for any *arbitrary cycle*,



$$\sum \frac{Q_i}{T_i} \leq 0 \quad (\text{vii})$$

the equality holding for reversible cycles only.

When the number of reservoirs is infinite and the heat exchanges take place by infinitesimal amounts, eqn. (vii) can be written as

$$\oint \frac{dQ}{T} \leq 0 \quad (21.1)$$

$\oint$  means sum over the cycle and  $dQ$  an infinitesimal amount of heat absorbed by the system (K) from a reservoir at temperature  $T$ . Eqn. (vii) or (viii) is known as *Clausius theorem*.

The relation holds good for any cyclic process, reversible or irreversible. The following example will help to illustrate this theorem.

**Example 21.1** (i) Consider two blocks at temperatures 300K and 200K that are in contact. The block at higher temperature conducts heat to the block at the lower temperature. Suppose, 200 joules of heat is transferred.

**Soln.**

Here,

$$Q_1 = 200 \text{ joules} \quad T_1 = 300 \text{ K}$$

$$Q_2 = -200 \text{ " } \quad T_2 = 200 \text{ K}$$

$$\therefore \sum \frac{dQ}{T} = \frac{Q_1}{T_1} + \frac{Q_2}{T_2}$$

$$= \frac{200}{300} + \frac{(-200)}{200}$$

$$= -\frac{1}{3}$$

$$= -0.33 \text{ joules/degree.}$$

(ii) Consider an actual heat engine working between the temperatures 500 K and 300 K. Suppose the efficiency of the engine is 20% and that it takes 1000 joules of heat from the reservoir at the higher temperature.

Since the efficiency of the engine is 20%,

$$\eta = \frac{20}{100} = 1 - \frac{Q_2}{Q_1} = 1 - \frac{Q_2}{1000}$$

$$\text{or, } Q_2 = 800 \text{ joules}$$

$$\begin{aligned} \therefore \sum \frac{Q}{T} &= \frac{Q_1}{T_1} + \frac{Q_2}{T_2} & T_1 &= 500\text{K} & Q_1 &= +1000 \text{ joules} \\ & & T_2 &= 300\text{K} & Q_2 &= -800 \text{ joules} \\ &= \frac{1000}{500} + \frac{(-800)}{300} & & & & \text{(heat rejected is considered negative)} \\ &= -\frac{2}{3} \text{ joules/degree} \end{aligned}$$

(iii) Now consider a Carnot reversible engine working between the temperatures 500K and 300K. Suppose the engine absorbs 1000 joules of heat energy from the high temperature reservoir.

$$\text{From } \frac{Q_1}{T_1} = \frac{Q_2}{T_2} \text{ we have}$$

$$\frac{1000}{500} = \frac{Q_2}{300} \quad \text{or, } Q_2 = 600 \text{ joules.}$$

$$\text{From } \sum \frac{Q}{T} = \frac{Q_1}{T_1} + \frac{Q_2}{T_2}, \text{ we have}$$

$$\begin{aligned} &= \frac{1000}{500} + \frac{(-600)}{300} & & \text{(heat rejected is considered negative)} \\ &= 0 \end{aligned}$$

The above example shows that  $\sum \frac{Q}{T} = 0$ , only in the limiting case of a reversible engine. In no case  $\sum \frac{Q}{T}$  is greater than zero.

## 21.2 Entropy

The property of the reversible cycle expressed by the equation

$$\oint \frac{dQ}{T} = 0$$

enables us to define a new *function of state* called *entropy* of the system.

If a system undergoes a reversible transformation from an initial state 1 to a final state 2, it can be depicted on an indicator diagram, say a P-V diagram. It is possible to take the system back reversibly from 2 to 1 through more than one path.

Let us consider a close system undergoing a reversible process from state 1 to state 2 along the path A and from state 2 to state 1 along the path B (Fig. 21.2). Since this is a reversible process

$$\oint \frac{dQ}{T} = 0$$

$$\therefore \int_{1A}^{2A} \frac{dQ}{T} + \int_{2B}^{1B} \frac{dQ}{T} = 0 \quad (i)$$

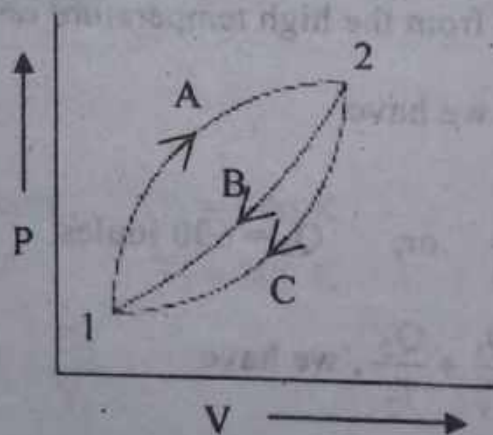


Fig. 21.2

Let us now consider the reversible cycle from state 1 to state 2 along the path C. For this reversible process, we have

$$\int_{1A}^{2A} \frac{dQ}{T} + \int_{2C}^{1C} \frac{dQ}{T} = 0 \quad (ii)$$



From eqn. (i) and (ii), we get

$$\int_{2B}^{1B} \frac{dQ}{T} = \int_{2C}^{1C} \frac{dQ}{T} \quad \text{(iii)}$$

Eqn. (iii) shows that  $\int \frac{dQ}{T}$  has the same value for all the reversible paths from state 2 to state 1. Thus the quantity  $\int \frac{dQ}{T}$  is independent of the path and is a function of the end states only, therefore it is a property. This property is called *entropy*.

Entropy is a thermodynamical property and is defined by the relation

$$dS = \frac{dQ}{T} \quad (21.2)$$

$$\text{or, } S_2 - S_1 = \int_1^2 \frac{dQ}{T} \quad (21.3)$$

The quantity  $S_2 - S_1$  represents the change in entropy of the system when it is changed from state 1 to state 2.

The literal meaning of the term entropy is transformation. *In the whole field of physics, there is hardly any concept which is more difficult to understand than the concept of entropy; not is there one which is, however, more fundamental.* It is a very important thermodynamic quantity which tells us about the direction of flow of heat. Although the physical concept of entropy is very difficult, yet once understood, it leads us very easily and quickly to important results which, by other means, can be obtained much more laboriously.

When a body or a system receives or gives out a quantity of heat  $Q$  at a constant temperature  $T_K$  by a reversible process, then the quantity  $\frac{Q}{T}$  is called the increase or decrease in entropy (change in entropy) of the body or the system. If during the process the temperature changes, then the heat may be supposed to be absorbed or given up in successive small amounts  $dQ$  so that the temperature

remains sensibly constant for each infinitesimal amount. Then the infinitesimal small change in entropy is given by  $\frac{dQ}{T}$ .

The total change in entropy is found by integrating the above relation and is denoted by S. Or

$$S = \int \frac{dQ}{T}$$

Thus, there are only two important expressions for entropy namely

$$\frac{Q}{T} \text{ and } \int \frac{dQ}{T}$$

### Graphical study

A set of isothermals a, b, c at temperatures  $T_1, T_2, T_3$  (in absolute scale) and two adiabatics d and e are shown in Fig. 21.3. In passing from one adiabatic to another along an isothermal, a certain amount of heat, depending upon the particular isothermal, is either absorbed or rejected by the body. Let  $Q_1, Q_2$  and  $Q_3$  be the amounts of heat absorbed or rejected by the body in passing along the isothermals at  $T_1, T_2$  and  $T_3$  respectively. According to the definition of Kelvin's absolute thermodynamic scale of temperature (Art. 20.13), we have

$$\frac{Q_1}{T_1} = \frac{Q_2}{T_2} = \frac{Q_3}{T_3} ; \quad \text{i.e., } \frac{Q}{T} = \text{constant,}$$

Hence, when a body passes from one adiabatic to another, the quantity  $\frac{Q}{T}$  i.e., the change in entropy is the same. The change in

entropy is, therefore, a property which is determined by the two adiabatics and which distinguishes one adiabatic from the other. Evidently, *all points on an adiabatic have equal entropy*. Thus, as already stated the entropy of a body may be defined as the quantity which remains constant in a reversible adiabatic process. If the adiabatics are very close to each other and  $dQ$  is the small amount of heat absorbed or rejected at  $TK$ , then the small change in entropy  $dS$  is given by

$$dS = \frac{dQ}{T}$$



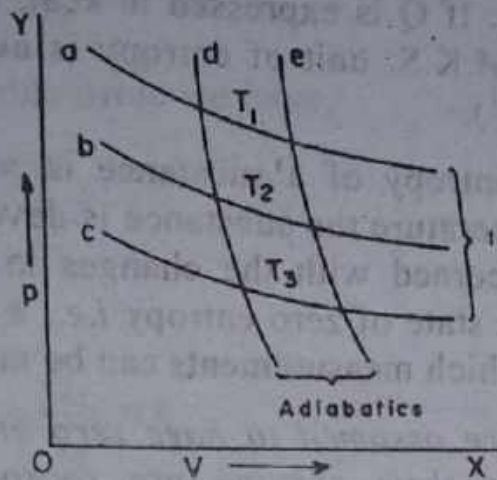


Fig. 21.3

Then the change in entropy in passing from adiabetic  $d$  to adiabetic  $e$  is given by

$$S_d - S_e = \int_d^e \frac{dQ}{T}$$

Since no heat enters or leaves during an adiabatic change,  $dQ$  is zero for an adiabatic change. Hence the change in entropy  $\frac{dQ}{T} = 0$

which implies that the *entropy is constant during an adiabatic change*. Clausius was the first to realize that *something remains constant in an adiabatic process* like the temperature in an isothermal process. He called this something *entropy*. Adiabatic curves on the  $P$ - $V$  diagram are, therefore, referred to as *isentropic curves*, i.e., curves of constant entropy.

*It is to be noted that the change in entropy of a body does not depend on the particular reversible process by which it passes from one state to the other – it depends wholly on the initial and final states of the body. Thus, entropy of a body is a definite function of its state just as its volume and pressure.*

### Units of entropy

$$\text{Change in entropy} = \frac{\text{heat added or subtracted}}{\text{absolute temperature}}$$



Therefore, the unit of entropy depends on the unit of heat and the absolute temperature. If  $Q$  is expressed in kcal, then the unit of entropy is  $\text{kcal/K}$ . The M.K.S. unit of entropy is evidently 1 joule per degree Kelvin ( $1 \text{ JK}^{-1}$ ).

Theoretically, the entropy of a substance is *zero at absolute zero* because at that temperature the substance is devoid of any heat. Since we are more concerned with the changes in entropy rather than its absolute value, a state of zero entropy *i.e.*, a reference point should be chosen from which measurements can be made.

*Fluids like water are assumed to have zero entropy when at  $0^\circ\text{C}$ . Hence changes in their entropy are reckoned from this temperature.*

If an addition of a small amount of heat  $dQ$  at a temperature  $T\text{K}$  increases the entropy by  $dS$ , then

$$dS = \frac{dQ}{T}; \text{ or, } dQ = T.dS \quad (21.4)$$

The total heat added can be found by integrating the above relation.

$$\text{Total heat supplied} = \int T.dS$$

### 21.3 Change in entropy for a reversible process

In reversible cycle, say in a Carnot cycle, there are two isothermal and two adiabatic processes. In the adiabatic processes there is neither absorption nor rejection of heat and as such no change in entropy. Of the two isothermal processes, one is an expansion and the other is a compression. During expansion an amount of heat  $Q_1$  is absorbed at a temperature  $T_1^\circ\text{K}$ . Hence there is an increase (or gain) in entropy which is equal to  $\frac{Q_1}{T_1}$ . In the compression process an amount of heat  $Q_2$  is rejected at a temperature  $T_2\text{K}$ . Hence there is a loss (or decrease) in entropy by an amount  $\frac{Q_2}{T_2}$ . Hence the total change in entropy for the whole reversible process.

$$\int dS = \frac{Q_1}{T_1} - \frac{Q_2}{T_2}$$

But for a reversible cycle we have,

$$\frac{Q_1}{T_1} = \frac{Q_2}{T_2}$$

Hence,

$$\int dS = \frac{Q_1}{T_1} - \frac{Q_2}{T_2} = 0$$

or,  $S = \text{constant}$ .

Hence the total change in entropy of the working substance in a complete reversible process is zero. Or in other words, *the entropy of a body in all reversible processes remains constant*.

#### 21.4 Change in entropy for an irreversible process

Let us consider a simple irreversible cycle in which the working substance absorbs a quantity of heat  $Q_1$  at  $T_1$  and rejects a quantity of heat  $Q_2$  at  $T_2$ . Then the efficiency of the engine performing such a cycle is given by

$$\eta' = \frac{Q_1 - Q_2}{Q_1}$$

Let a reversible engine work between the same two temperature limits. Then its efficiency

$$\eta = \frac{T_1 - T_2}{T_1}$$

According to Carnot's theorem

$$\eta' < \eta$$

$$\text{or, } \frac{Q_1 - Q_2}{Q_1} < \frac{T_1 - T_2}{T_1}$$

$$\text{or, } 1 - \frac{Q_2}{Q_1} < 1 - \frac{T_2}{T_1}$$

$$\text{or, } \frac{Q_2}{Q_1} > \frac{T_2}{T_1}$$

$$\text{or, } \frac{Q_2}{T_2} > \frac{Q_1}{T_1}$$

$$\text{or, } \frac{Q_2}{T_2} - \frac{Q_1}{T_1} > 0 \quad (21.5)$$

The heat transfer in an irreversible process may be assumed to have actually taken place in a reversible fashion, the irreversibility of the cycle as a whole arising from some other cause. At the end of the cycle the working substance of course returns to its initial state, and there is no question of any change in entropy. But in case of the working substance taken as a whole it is observed that the source loses entropy by an amount  $\frac{Q_1}{T_1}$  while the sink gains entropy by an amount  $\frac{Q_2}{T_2}$ . The change in entropy in the whole process is, therefore, given by

$$\frac{Q_2}{T_2} - \frac{Q_1}{T_1}$$

which according to eqn. (21.5) is a positive quantity. Thus, *the irreversible cycle there is a net increase (or gain) in entropy*. This conclusion is valid for any irreversible process. Two examples are given below:

- (i) *conduction or radiation of heat* : Let us consider the simple case of conduction and radiation of heat which are irreversible processes. Suppose a system consists of two bodies at temperatures  $T_1$  and  $T_2$  where  $T_1 > T_2$ . Let the body at temperature  $T_1$  lose an amount of heat  $Q$  - by way of conduction or radiation - to the body at lower temperature  $T_2$ .



Then the loss of entropy by the hot body =  $\frac{Q}{T_1}$  and gain of entropy by the cold body =  $\frac{Q}{T_2}$ . Hence considering the system as a whole, the net gain of entropy by the system

$$= \frac{Q}{T_2} - \frac{Q}{T_1} = Q \left( \frac{1}{T_2} - \frac{1}{T_1} \right)$$

which is a positive quantity since  $T_1 > T_2$ .

Thus the processes of conduction and radiation which are irreversible, result in an increase in entropy.

(ii) *free expansion of a gas in vacuum (or Joule expansion)* : Let us take a perfect gas in a vessel and allow it to rush into an evacuated vessel (vacuum) which is an irreversible process. The entire system is thermally insulated. Thus the temperature and hence the internal energy of the perfect gas does not change in the process.

Let the gas at temperature  $T$  and pressure  $P$  change in volume from  $V_1$  to  $V_2$  isothermally. The work done for a small change in volume  $dV$  is  $P.dV$ .

Then the change in entropy

$$dS = \frac{dQ}{T} = \frac{dU + PdV}{T}$$

$$= \frac{P.dV}{T} \quad (\text{since } dU = 0 \text{ for isothermal change})$$

$$= \frac{R.dV}{V}$$

$\therefore$  the total change in entropy

$$S = \int dS = \int_{V_1}^{V_2} \frac{RdV}{V}$$

$$= R \ln \frac{V_2}{V_1}$$

Since  $V_2 > V_1$  and  $R$  is a positive quantity,  $S$  is positive and hence entropy of the system increases.

It therefore follows that while entropy in a reversible process remains constant, entropy increases in all irreversible processes. Since most natural processes are irreversible, the entropy of the universe increases. This led Clausius to state that *the entropy of the universe tends towards a maximum*.

### 21.5 Entropy and second law of thermodynamics

Let the entropies of a body in an initial state 1 and a final state 2, as measured from some arbitrarily chosen zero entropy, be  $S_1$  and  $S_2$ . Then the change in entropy is given by

$$S_2 - S_1 = \int_1^2 \frac{dQ}{T} \quad (21.6)$$

When the two states 1 and 2 are infinitesimally close to each other, the relation (21.5) will reduce to the form

$$dS = \frac{dQ}{T} ; \quad \text{or, } dQ = T.dS \quad (21.7)$$

Equation (21.6) embodies the mathematical expression of the second law of thermodynamics.

### 21.6 Temperature – entropy diagram

Entropy is a property of a system and hence its value in an equilibrium state of the system (apart from an arbitrary constant) can be expressed in terms of the variables specifying the system. Thus for a PVT system, the entropy can be expressed as a function of  $P$  and  $V$ ,  $V$  and  $T$ , or  $T$  and  $P$ . Then just as with the internal energy, entropy can also be considered as one of the variables specifying the state of the system. Thus, the state of a system can be specified in terms of the entropy  $S$  and one other variable. If temperature  $T$  is selected as the other variable, every state of a system corresponds to a point on a  $T$ - $S$  diagram and every reversible process corresponds to a curve in this diagram.



Let us consider the case of an isothermal operation taking place at a constant temperature  $T$ . Since  $T$  remains constant during an isothermal operation. Say isothermal expansion, it is represented by a horizontal line  $AB$  in Fig. 21.4. The entropy of the gas increases from an initial value of  $S_1$  to a final value of  $S_2$ . The total heat added (or the heat flow into the system) is given by the area under the line  $AB$  and is

$$Q = T.dS = T (S_2 - S_1)$$

During an adiabatic expansion of the gas, no heat enters or leaves the gas, and hence  $dQ = 0$ . Since  $dS = \frac{dQ}{T}$ , it is obvious that *the change in entropy of the gas is also zero*. In other words, the entropy of the gas remains constant during an adiabatic operation (expansion or compression). The graph of an adiabatic operation (say an expansion) is, therefore, a vertical straight line – line  $CD$  in the Fig. 21.4. Point  $C$  represents the initial condition of the gas when

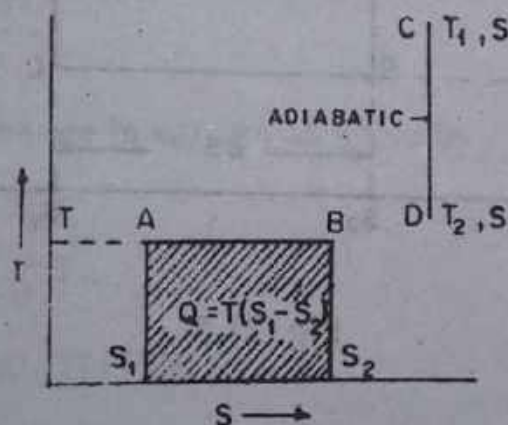


Fig. 21.4

its temperature is  $T_1$  and entropy, say  $S$ . It then expands to point  $D$  where its temperature has fallen to  $T_2$  but its entropy is found to be the same as before.

### Carnot cycle

A Carnot cycle has an especially simple form in a  $T$ - $S$  diagram, since it is bounded by two isothermals, along which  $S$  is constant. Thus a Carnot cycle is represented by a rectangle  $a-b-c-d-a$  in a  $T$ - $S$  diagram (Fig. 21.5) where the letters correspond to those used



in Fig. 20.6. The area under  $ab$  represents the heat absorbed in the isothermal expansion and that under  $cd$  is the heat expelled in the isothermal compression. The difference (or the algebraic sum) of the heat entries is equal to the work done and is represented by the area enclosed by the rectangle. The efficiency of a Carnot cycle can be deduced at once from Fig. 21.5. The heat absorbed at the higher temperature is  $T_1 \cdot dS$ . The work  $W$  is the area  $(T_1 - T_2) dS$ . Hence the efficiency

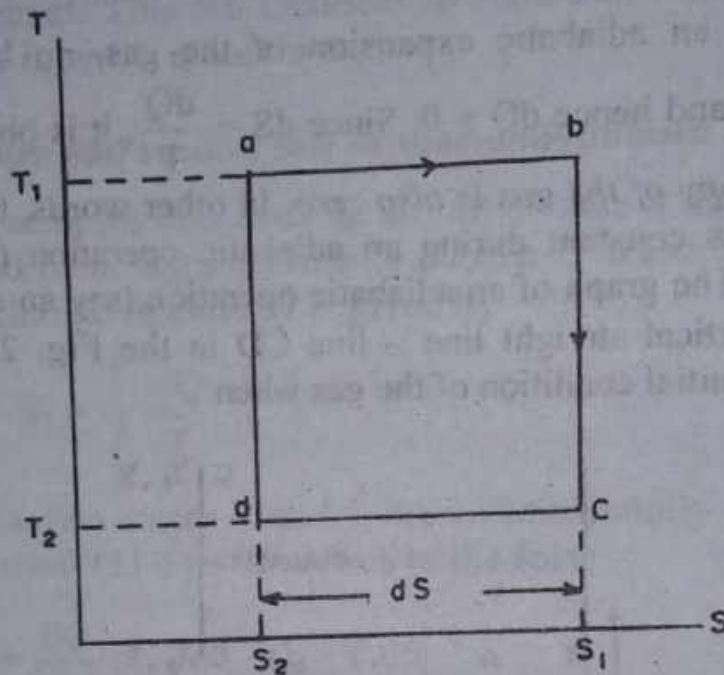


Fig. 21.5

$$\eta = \frac{\text{work done}}{\text{heat supplied}} = \frac{W}{Q_1} = \frac{(T_1 - T_2) \cdot dS}{T_1 \cdot dS} = \frac{T_1 - T_2}{T_1}$$

### 21.7 Calculation of entropy change in reversible processes

Changes in the entropy of a gas will now be considered

- (i) when it is heated at constant volume
- (ii) when it is heated at constant pressure

- (iii) during an isothermal expansion  
and (iv) when it is heated in any general manner.

The expression

$$\int dS = \int \frac{dQ}{T}$$

will be used in each case.

- (i) heating at constant volume

Let  $m$  kg of a gas be heated at constant volume from an initial temperature of  $T_1$  absolute to a final temperature of  $T_2$  absolute.

If a small amount of heat  $dQ$  increases the temperature by  $dT$ , then

$$dQ = m C_v dT$$

$$\text{But } dS = \frac{dQ}{T}$$

$$\therefore dS = \frac{m C_v dT}{T}$$

Then the total change in entropy is given by

$$\int_{S_1}^{S_2} dS = m C_v \int_{T_1}^{T_2} \frac{dT}{T}$$

$$\text{or, } S_2 - S_1 = m C_v [\ln T]_{T_1}^{T_2}$$

$$= m C_v \ln \frac{T_2}{T_1}$$

$$= m C_v \ln \frac{T_2}{T_1} \text{ kcal/K}$$

$$= m C_v \ln \frac{P_2}{P_1} \text{ kcal/K} \quad (21.8)$$

$$\text{since at constant volume, } \frac{P_2}{P_1} = \frac{T_2}{T_1}$$

The above change in entropy is shown by the logarithmic curve AB in Fig. 21.6.

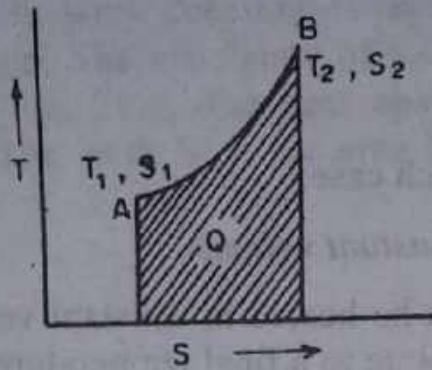


Fig. 21.6

(ii) *heating at constant pressure*

If  $dQ$  be the required amount of heat for a small increase  $dT$  in temperature, then

$$dQ = mC_p dT$$

$$\therefore dS = \frac{dQ}{T} = \frac{mC_p dT}{T}$$

$$\text{or, } \int_{S_1}^{S_2} dS = mC_p \int_{T_1}^{T_2} \frac{dT}{T}$$

$$\therefore S_2 - S_1 = mC_p \ln \frac{T_2}{T_1}$$

$$= mC_p \ln \frac{T_2}{T_1} \text{ kcal/K}$$

$$= mC_p \ln \frac{V_2}{V_1} \text{ kcal/K} \quad (21.9)$$

$$\text{since at constant pressure } \frac{V_2}{V_1} = \frac{T_2}{T_1}$$

The change in entropy as given by the above expression will be given by a logarithmic curve when represented on a T-S diagram.



It may be noted that the above calculations equally hold good for the heating of a liquid at constant pressure.

(iii) *isothermal expansion*

During an isothermal expansion, heat is taken in at constant temperature. If  $Q$  is the amount of heat taken in at a constant temperature  $T$ , then increase in entropy is

$$S_2 - S_1 = \frac{Q}{T}$$

Since heat taken in is equal to the work done by the gas during the isothermal expansion,

$$Q = \frac{mRT}{J} \ln r$$

where  $r = \frac{V_2}{V_1}$  is the isothermal expansion ratio.

$$\begin{aligned} \therefore S_2 - S_1 &= \frac{mRT}{JT} \ln r \\ &= \frac{mR}{J} \ln \frac{V_2}{V_1} \\ &= m (C_p - C_v) \ln \frac{V_2}{V_1} \quad [C_p - C_v = \frac{R}{J}] \\ &= -m (C_p - C_v) \ln \frac{P_2}{P_1} \end{aligned} \quad (21.10)$$

since at constant temperature  $\frac{V_2}{V_1} = \frac{P_1}{P_2}$

$$\text{and } \ln \frac{P_1}{P_2} = - \ln \frac{P_2}{P_1}.$$

## 21.8 Entropy change of a perfect gas : general case

The change of entropy of a gas when it expands in any general manner may also be found in the manner described above.

Let us suppose that the initial temperature, pressure and volume of  $m$  kg of a gas be  $T_1$ ,  $P_1$  and  $V_1$  respectively. Let this gas be heated in a general manner to a final temperature  $T_2$ , final pressure  $P_2$  and final volume  $V_2$ .

From the general heat energy equation, we have

$$\begin{aligned} dQ &= dU + \frac{dW}{J} \\ &= mC_v dT + \frac{PdV}{J} \end{aligned}$$

Dividing both sides by  $T$  we get

$$\frac{dQ}{T} = \frac{mC_v dT}{T} + \frac{PdV}{JT}$$

But  $\frac{dQ}{T}$  = change in entropy =  $dS$

$$\therefore dS = \frac{mC_v dT}{T} + \frac{PdV}{JT} \quad (21.11)$$

From the general gas equation  $PV = mRT$  we get

$$\frac{P}{T} = \frac{mR}{V}$$

Substituting this value of  $\frac{P}{T}$  in eqn. (21.10) we have

$$dS = mC_v \frac{dT}{T} + \frac{mR}{J} \cdot \frac{dV}{V}$$

Integrating both sides

$$\int_{S_1}^{S_2} dS = mC_v \int_{T_1}^{T_2} \frac{dT}{T} + \frac{mR}{J} \int_{V_1}^{V_2} \frac{dV}{V}$$

$$\text{or, } S_2 - S_1 = mC_v \ln \frac{T_2}{T_1} + \frac{mR}{J} \ln \frac{V_2}{V_1}$$

$$= m \left( C_v \ln \frac{T_2}{T_1} + \frac{R}{J} \ln \frac{V_2}{V_1} \right) \quad (21.12)$$

Eqn. (21.11) expresses the change in entropy in terms of the temperature and volume of the gas. With the help of the general gas equation either  $\frac{T_2}{T_1}$  or  $\frac{V_2}{V_1}$  can be eliminated from eqn. (21.12) as shown below.

$$P_1 V_1 = mRT_1 \text{ and } P_2 V_2 = mRT_2$$

$$\therefore \frac{P_2}{P_1} \cdot \frac{V_2}{V_1} = \frac{T_2}{T_1}$$

Substituting this value of  $\frac{T_2}{T_1}$  in eqn. (21.12), we get

$$\text{or, } S_2 - S_1 = mC_v \ln \frac{P_2}{P_1} \cdot \frac{V_2}{V_1} + \frac{mR}{J} \ln \frac{V_2}{V_1}$$

$$= m C_v \ln \frac{P_2}{P_1} + mC_v \ln \frac{V_2}{V_1} + \frac{mR}{J} \ln \frac{V_2}{V_1}$$

$$= m C_v \ln \frac{P_2}{P_1} + m \left( C_v + \frac{R}{J} \right) \ln \frac{V_2}{V_1}$$

$$\text{But } C_p - C_v = \frac{R}{J}; \text{ hence } C_v + \frac{R}{J} = C_p$$

$$\begin{aligned} \therefore S_2 - S_1 &= mC_v \ln \frac{P_2}{P_1} + mC_p \ln \frac{V_2}{V_1} \\ &= m \left( C_v \ln \frac{P_2}{P_1} + C_p \ln \frac{V_2}{V_1} \right) \end{aligned} \quad (21.13)$$

Again from  $P_1 V_1 = mRT_1$  and  $P_2 V_2 = mRT_2$ , we have

$$\frac{V_2}{V_1} = \frac{T_2}{T_1} \cdot \frac{P_1}{P_2}$$

Substituting this value of  $\frac{V_2}{V_1}$  in eqn. (21.12), we get



$$\begin{aligned}
 S_2 - S_1 &= mC_v \ln \frac{T_2}{T_1} + \frac{mR}{J} \ln \frac{T_2}{T_1} \cdot \frac{P_1}{P_2} \\
 &= mC_v \ln \frac{T_2}{T_1} + \frac{mR}{J} \ln \frac{T_2}{T_1} + \frac{mR}{J} \ln \frac{P_1}{P_2} \\
 &= m \left( C_v + \frac{R}{J} \right) \ln \frac{T_2}{T_1} - m \frac{R}{J} \ln \frac{P_2}{P_1} \\
 &= mC_p \ln \frac{T_2}{T_1} - \frac{mR}{J} \ln \frac{P_2}{P_1} \\
 &= m \left( C_p \ln \frac{T_2}{T_1} - \frac{R}{J} \ln \frac{P_2}{P_1} \right) \quad (21.14)
 \end{aligned}$$

Thus

$$\begin{aligned}
 S_2 - S_1 &= m \left( C_v \ln \frac{T_2}{T_1} + \frac{R}{J} \ln \frac{V_2}{V_1} \right) \text{ kcal/K} \\
 &= m \left( C_v \ln \frac{P_2}{P_1} + C_p \ln \frac{V_2}{V_1} \right) \text{ kcal/K} \\
 &= m \left( C_p \ln \frac{T_2}{T_1} - \frac{R}{J} \ln \frac{P_2}{P_1} \right) \text{ kcal/K.}
 \end{aligned}$$

## 21.9 Entropy of a Van der Waals gas

Let 1gm.mol. of a Van der Waals gas be changed from a pressure  $p_1$ , volume  $V_1$  and temperature  $T_1$  to pressure  $p_2$ , volume  $V_2$  and temperature  $T_2$ . The change in entropy is given by

$$S_2 - S_1 = \int_{T_1}^{T_2} \frac{dQ}{T} \quad \dots \dots \dots (i)$$

But from the first law of thermodynamics

$$\begin{aligned}
 \frac{dQ}{T} &= \frac{dU}{T} + \frac{pdV}{T} \\
 &= C_v \frac{dT}{T} + \frac{R}{V-b} dV \quad \dots \dots \dots (ii)
 \end{aligned}$$

(since, to a first approximation

$$p = \frac{RT}{V-b} \text{ for a Van der Waals gas)}$$

So from (i) and (ii), assuming  $C_v$  to remain constant,

$$\begin{aligned} S_2 - S_1 &= C_v \int_{T_1}^{T_2} \frac{dT}{T} + R \int_{V_1}^{V_2} \frac{dV}{V-b} \\ &= C_v \ln \frac{T_2}{T_1} + R \ln \frac{V_2 - b}{V_1 - b} \quad \dots \quad (\text{iii}) \end{aligned}$$

### 21.10 Approximate value of heat absorbed

The heating of a gas is represented on a  $T$ - $S$  diagram by a curve, say  $AB$ , in Fig. 21.7. However, the curve has only a slight curvature. In fact, it is so slightly curved that it can be assumed to be practically straight.

Let 1 kg of a gas at point  $A$  be heated to a point  $B$ . The heating takes place along the line  $AB$ . Let the initial temperature and entropy of the gas at the point  $A$  be  $T_1$  and  $S_1$  respectively; the corresponding final values at the point  $B$  being  $T_2$  and  $S_2$ . The heat absorbed by the gas is represented by the shaded area in Fig. 21.7. The approximate value of the heat can be found by assuming  $AB$  to be

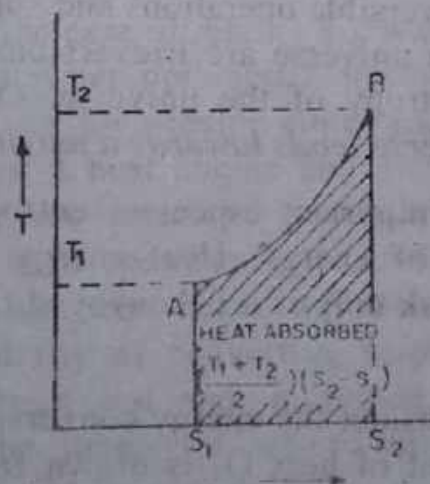


Fig. 21.7

a straight line.

$$\begin{aligned}
 \therefore \text{heat absorbed} &= \text{shaded area under AB} \\
 &= \text{base} \times \text{mean height} \\
 &= (S_2 - S_1) \left( \frac{T_1 + T_2}{2} \right) \\
 &= \text{change in entropy} \times \text{mean absolute temperature.}
 \end{aligned}$$

### 21.11 Concept of entropy

From the definition of the change of entropy of a substance ( $dS = \frac{dQ}{T}$ ), entropy was shown to be a *physical entity* – a definite single valued function of the thermodynamic variables. However, it is really difficult to form a tangible idea about the entropy since it cannot be physically felt like the many other physical quantities viz., temperature, heat, weight, pressure, etc. Although there is nothing strictly physical to represent it, still entropy is very much a physical entity and not a physical illusion. Some properties of entropy are discussed below.

(1) As discussed in Art. 21.3, the net change in entropy in any reversible cycle is zero; or in other words the entropy of a system remains constant in any reversible process.

(2) As proved in Art. 21.4, there is a net increase in the entropy during irreversible operations and since all actual operations taking place in the universe are irreversible, there is a continuous increase in the entropy of the universe. Or, in other words, *the entropy of the universe tends towards a maximum.*

(3) Another important aspect of entropy is that it represents the *unavailability* of energy. Heat energy becomes available for conversion into work only if it is conveyed from a higher to a lower temperature.

(i) Let us consider the energy conversion in a Carnot's engine in which an amount of heat  $Q_1$  is drawn from a hot body at  $T_1$ K, while an amount  $Q_2$  is rejected to a cold body at  $T_2$ K. The difference  $(Q_1 - Q_2)$  is converted into work and is said to be available energy.

Therefore, the available energy



$$\begin{aligned}
 &= Q_1 - Q_2 = Q_1 \left(1 - \frac{Q_2}{Q_1}\right) \\
 &= Q_1 \left(1 - \frac{T_2}{T_1}\right) \quad \left(\because \frac{Q_2}{Q_1} = \frac{T_2}{T_1}\right)
 \end{aligned}$$

Hence, heat unavailable for work

$$= Q_1 \times \frac{T_2}{T_1} = T_2 \left(\frac{Q_1}{T_1}\right)$$

Now  $\frac{Q_1}{T_1}$  represents the increase in entropy during the isothermal expansion.

$\therefore$  *unavailable energy*

$$= T_2 \times \text{increase in entropy}$$

$$= T_2 dS$$

If the temperature of the sink *i.e.*,  $T_2$  is constant, then amount of unavailable energy is proportional to the increase in entropy. It can be seen that the smaller the temperature  $T_2$  of the cold body, the smaller will be the unavailable energy. If  $T_0$  is the temperature of the coldest body available, the maximum unavailable energy will be  $Q_1 \frac{T_0}{T_1}$ .

(ii) Now consider the case of mixing hot water and cold water. Had the two amount of water not mixed, the hot and cold water could have been used as high-temperature and low-temperature reservoirs respectively of a heat engine and during the process of removing heat from the hot water and giving it to the cold water, some of the heat could have been converted into mechanical work. But once the waters have been mixed and brought to a uniform temperature this opportunity of converting some of the heat into mechanical work has been lost and, moreover, it has been lost irretrievably because the lukewarm water will never afterwards unmix itself into colder and hotter portions.

Before mixing takes place, suppose an amount of heat  $dQ$  was in the hot water at  $T_1$  K. If  $T_0$  is the lowest available temperature,

then the maximum available energy out of  $dQ$  was  $dQ (1 - \frac{T_0}{T_1})$ .

After mixing, this amount of heat is available in the cold body at temperature  $T_2$  K so that the maximum available energy then becomes  $dQ (1 - \frac{T_0}{T_2})$ .

Hence energy unavailable for work,

$$= dQ (1 - \frac{T_0}{T_1}) - dQ (1 - \frac{T_0}{T_2})$$

$$= T_0 (\frac{dQ}{T_2} - \frac{dQ}{T_1})$$

$$= T_0 \times \text{increase in entropy.}$$

It may be particularly noted that there is no decrease in heat energy when hot and cold waters are mixed together, but there has occurred an increase in the unavailability (or a decrease in the availability) of heat energy for conversion into mechanical work. As irreversible changes are always taking place in nature, the energy which is not available for work (unavailable energy) is continually increasing. Thus the energy of the universe is becoming more and more unavailable and we say that the universe is *run down* to that extent.

(4) All physical operations in the universe are inherently irreversible. This means that for every such operation performed, some of the energy becomes unavailable for useful work because it is added to the universe in the form of heat by way of conduction, radiation or through friction. A distant future may be visualized when, because of irreversibility, all energies existing in different forms will be converted into heat energy. Further, all temperature difference between various bodies in the universe will be evened out due to convection. Thus no heat engine will be able to transform heat into work as no flow of heat would be possible due to uniformity of temperature throughout the universe. At that stage a state of stagnation or *inaction* will prevail upon the universe but the entropy of the universe will be maximum at that time. This state



towards which we appear to be heading has been described as the *heat death* of the universe.

(5) The addition of heat leads to the increase of entropy of the substance. Again the addition of heat increase the molecular agitation or the *disorderliness* of the molecules in a substance. Thus every increase of entropy is accompanied by an increase of disorderliness of the molecules of the substance. Hence, according to Boltzmann, *the entropy is a measure of disorder or randomness in the molecular arrangement of a substance, an increase in entropy implying a transition from a less to a more disordered state.*

In the universe around us we commonly experience that the particles left to themselves spontaneously pass from an ordered to a disordered state; in other words *the disordered states are most probable in nature and the ordered states are least probable.*

The main properties of entropy may be summed up as follows:

- (i) *Net change in entropy in a reversible cycle is zero.*
- (ii) *Entropy increases during irreversible operations. There is no operation during which entropy decreases.*
- (iii) *Entropy remains constant during all adiabatic frictionless processes.*
- (iv) *Irrespective of whether temperature changes or not, entropy increases when heat is added and it decreases when heat is removed.*
- (v) *Entropy is always on the increase.*
- (vi) *Entropy is a measure of unavailability of heat energy for useful work.*
- (vii) *Entropy is a measure of the disorder or randomness in molecular arrangement of a substance.*

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*Example 21.2. Find the change in entropy when 1 gm of ice at  $0^{\circ}\text{C}$  changes to water at  $0^{\circ}\text{C}$  under 1 atmosphere pressure. Latent heat of fusion of ice = 80 cal. per gm.*

*Soln.*

80 cal. of heat must be supplied (added) to convert 1 gm. of water at  $0^{\circ}\text{C}$  to 1 gm of ice at  $0^{\circ}\text{C}$  under 1 atmosphere pressure.



Hence

$$\text{Change in entropy} = S_2 - S_1 = \frac{Q}{T}$$

Here  $Q = 80 \text{ cal.}$  ;

$$T = (0 + 273) = 273 \text{ K.}$$

$$\therefore S_2 - S_1 = \frac{80}{273} = 0.293 \text{ cal/K.}$$

**Example 21.3.** Find the difference in entropy between 1 gm of water at  $100^\circ\text{C}$  and 1 gm of steam at  $100^\circ\text{C}$  and at atmospheric pressure. (Latent heat of steam at  $100^\circ\text{C} = 540 \text{ cal/gm}$ ).

**Soln.**

540 calories of heat will have to be supplied (added) to convert 1 gm of water at  $100^\circ\text{C}$  to 1 gm of steam at  $100^\circ\text{C}$  at atmospheric pressure.

Hence

$$\text{Change in entropy} = S_2 - S_1 = \frac{Q}{T}$$

Here  $Q = 540 \text{ cal.}$

$$T = (100 + 273) = 373 \text{ K.}$$

$$\therefore S_2 - S_1 = \frac{540}{373} = 1.448 \text{ cal/K.}$$

**Example 21.4.** Calculate the increase in entropy when 1 gm of ice at  $-10^\circ\text{C}$  is converted into steam at  $100^\circ\text{C}$ . specific heat of ice = 0.5, latent heat of ice = 80 cal/gm. latent heat of steam = 540 cal/gm.

**Soln.**

(i) increase in entropy when the temperature of 1 gm of ice increases from  $-10^\circ\text{C}$  to  $0^\circ\text{C}$ .

$$dS = \int_{T_1}^{T_2} \frac{dQ}{T} \quad (dQ = m.s.dT)$$

$$= \int_{T_1}^{T_2} \frac{msdT}{T} = ms \int_{T_1}^{T_2} \frac{dT}{T}$$

$$= ms \ln \frac{T_2}{T_1}$$

$$= m \times s \times \ln \frac{T_2}{T_1}$$

Here  $m = 1 \text{ gm}$ ,  $s = 0.5$ ,

$$T_1 = (-10 + 273) = 263\text{K}$$

$$\text{and } T_2 = (0 + 273) = 273\text{K.}$$

$$\therefore dS = 1 \times 0.5 \times \ln \frac{273}{263}$$

$$= 0.01865 \text{ cal/K.}$$

(ii) increase in entropy when 1 gm of ice at  $0^\circ\text{C}$  is converted into water at  $0^\circ\text{C}$ .

$$dS = \frac{Q}{T} = \frac{80}{273}$$

$$= 0.293 \text{ cal/K} \quad (\text{Example 21.2})$$

(iii) increase in entropy when the temperature of 1 gm of water is raised from  $0^\circ\text{C}$  to  $100^\circ\text{C}$ .

$$dS = \int_{T_1}^{T_2} \frac{dQ}{T}$$

$$= m.s.\ln \frac{T_2}{T_1}$$

$$= m.s.\ln \frac{T_2}{T_1}$$

$$T_1 = (0 + 273) = 273\text{K}$$

$$T_2 = (100 + 273) = 373\text{K.}$$

$$s = 1, m = 1$$

$$\therefore dS = 1 \times 1 \times \ln \frac{373}{273}$$

$$= 0.312 \text{ cal/K.}$$

(iv) increase in entropy when 1 gm of water at  $100^\circ\text{C}$  is converted into steam at  $100^\circ\text{C}$ .

$$dS = \frac{dQ}{T} = \frac{540}{373}$$

$$= 1.448 \text{ cal/K.} \quad (\text{Example 21.3}).$$

So, the total increase in entropy

$$= 0.01865 + 0.293 + 0.312 + 1.448$$

$$= 2.07165 \text{ cal/K.}$$

**Example 21.5.** One kilogram of water initially at  $0^\circ\text{C}$  is heated to  $100^\circ\text{C}$ . Find the increase in its entropy.

If this water is then mixed with an equal amount of water at  $0^\circ\text{C}$ , find the net increase in the entropy of the system, the temperature of the mixture being  $50^\circ\text{C}$ . Take the specific heat throughout as unity and consider the entropy of water at  $0^\circ\text{C}$  to be zero.

**Soln.**

As pointed out in Art. 21.2, the entropy of water at  $0^\circ\text{C}$  is assumed to be zero.

Then the increase in entropy when 1 kg of water at  $0^\circ\text{C}$  is heated to  $100^\circ\text{C}$  or the entropy of water at  $100^\circ\text{C}$  is

$$\int_{S_1}^{S_2} dS = \int_{T_1}^{T_2} \frac{dQ}{T} \quad dQ = m \cdot s \cdot dT$$

$$\text{or, } S_2 - S_1 = \int_{T_1}^{T_2} ms \frac{dT}{T}$$



$$= m.s.\ln \frac{T_2}{T_1}$$

$$m = 1000 \text{ gm}, s = 1$$

$$T_1 = 273\text{K}, T_2 = 373\text{K}$$

$$\therefore S_2 - S_1 = 1000 \times 1 \times \ln \frac{373}{273}$$

$$= 312 \text{ cal/K.}$$

Before mixing,

entropy of hot water ( $S_2$ ) = 312 cal/K

entropy of cold water ( $S_1$ ) = 0

Total entropy of 2 kg of water = 312 cal/K.

After mixing, we get 2 kg of water at 50°C or 323K.

$\therefore$  Entropy of the mixture

$$= \int_{T_1}^{T_2} ms \frac{dT}{T} = \int_{273}^{323} ms \frac{dT}{T}$$

$$= m.s.\ln \frac{323}{273} \quad (m = 2000 \text{ gm}, s = 1)$$

$$= 336 \text{ cal/k.}$$

Hence, there has been an increase in entropy = (336 - 312) cal/K

$$= 24 \text{ cal/K} = 0.024 \text{ kcal/K.}$$

**Example 21.6.** 0.5 kg of a gas at 30 kg/cm<sup>2</sup> suffers an increase of temperature from 727°C to 1127°C without change of pressure. Find the change in entropy.  $C_p = 0.25$ ,  $C_v = 0.18$ .

**Soln.**

Change in entropy when the gas is heated at constant pressure (Art. 21.7) is

$$S_2 - S_1 = m C_p \ln \frac{T_2}{T_1}$$

Here  $m = 0.5 \text{ kg}$ ,  $C_p = 0.25$ ,

$$T_1 = (273 + 727) = 1000\text{K}$$

$$T_2 = (273 + 1127) = 1400\text{K}.$$

$$\begin{aligned} \therefore S_2 - S_1 &= 0.5 \times 0.25 \times \ln \frac{1400}{1000} \\ &= 0.042 \text{ kcal/K.} \end{aligned}$$

**Example 21.7.** Calculate the increase in entropy of 2gm of  $O_2$  when its temperature is raised from  $20^\circ\text{C}$  to  $100^\circ\text{C}$  and its volume is also doubled.

**Soln.**

The change in entropy of a gas when both its temperature and volume change during heating is given by (Art. 21.7)

$$S_2 - S_1 = m \left( C_v \ln \frac{T_2}{T_1} + \frac{r}{J} \ln \frac{V_2}{V_1} \right)$$

where  $r$  is the gas constant for 1 kg of oxygen. Now for oxygen,

$$m = 2 \text{ gm} = 2 \times 10^{-3} \text{ kg.}$$

$$C_v = 0.19 \text{ kcal/kg/K.}$$

$$J = 4200 \text{ joule/kcal.}$$

$$\begin{aligned} R &= \text{universal gas constant for one kg-molecule of a gas} \\ &= 8320 \text{ joules/kg-mole-K.} \end{aligned}$$

For oxygen, molecular weight  $M = 32$ .

$$\therefore R = M.r$$

$$\text{or, } r = \frac{R}{M} = \frac{8320}{32} = 260 \text{ joules/kg} \cdot \text{K}.$$

$$T_1 = (20 + 273) = 293\text{K}$$

$$T_2 = (100 + 273) = 373\text{K}$$

$$\frac{V_2}{V_1} = 2.$$

$$\begin{aligned} \therefore S_2 - S_1 &= (2 \times 10^{-3}) \times \left[ 0.19 \ln \frac{373}{293} + \frac{260}{4200} \cdot \ln 2 \right] \\ &= 0.177 \times 10^{-3} \text{ kcal/K.} \end{aligned}$$

**Example 21.8.** One gm-molecule of a gas expands isothermally to four times its volume. Calculate the change in entropy in terms of the gas constant.

**Soln.**

$$\text{Work done} = \int_{V_1}^{V_2} P dV$$

$$= \int_{V_1}^{V_2} RT \frac{dV}{V} \quad \left( P = \frac{RT}{V} \text{ from } PV = RT \right)$$

$$= RT \ln \frac{V_2}{V_1}$$

$$\text{Here } \frac{V_2}{V_1} = 4$$

$$\therefore W = RT \ln 4$$

Both  $W$  and  $R$  are in units of work.

$$\text{Now } dQ = dU + \frac{W}{J}$$

But at constant temperature  $dU = 0$



$$\therefore dQ = \frac{W}{J} = \frac{RT \ln 4}{J}$$

$$\text{Hence gain in entropy} = \frac{dQ}{T}$$

$$= \frac{RT \ln 4}{JT}$$

$$= 1.387 \frac{RT \ln 4}{J} \text{ cal/K.}$$

**Example 21.9.** 10 gm of steam at  $100^\circ\text{C}$  is blown into 90 gramme of water at  $0^\circ\text{C}$ , contained in a calorimeter of water equivalent 10 gms. The whole of the steam is condensed. Calculate the increase in the entropy of the system.

**Soln.**

Heat lost by steam = Heat gained by water – calorimeter system.

$$m_1 L + m_1 s (T_1 - T_2) = (m_2 S + W) (T - T_2)$$

where  $m_1 = 10 \text{ gm}$ ,  $s = 1$

$$L = 540 \text{ cal/gm}, W = 10 \text{ gm}$$

$$T_1 = 100^\circ\text{C} = 373\text{K.}$$

$$T_2 = 0^\circ\text{C} = 273\text{K}$$

and  $T$  is the final temperature of the mixture.

$$\therefore 10 \times 540 + 10 (373 - T) = (90 + 10) (T - 273)$$

$$\text{or, } T = 331.2\text{K.}$$

(i) Change in entropy when 10 gm of steam  $100^\circ\text{C}$  is condensed into water at  $100^\circ\text{C}$

$$= \frac{dQ}{T}$$

$$T = 100^\circ\text{C} = 373\text{K}$$

$$dQ = m \times L = 10 \times 540$$

$$= - \frac{10 \times 540}{373}$$

$$= -14.47 \text{ cal/K.}$$

(minus sign indicates decrease in entropy).

(ii) Change in entropy when 10 gm of water at 373K is cooled to water at 331.2K

$$= \frac{dQ}{T} = ms \int_{T_1}^{T_2} \frac{dT}{T} \quad T_1 = 373K$$

$$= ms. \ln \frac{T_2}{T_1} \quad T_2 = 331.2K$$

$$= 10 \times 1 \times \ln \frac{331.2}{373}$$

$$= -1.188 \text{ cal/K.}$$

(iii) Change in entropy when the temperature of water and calorimeter rises from 273K to 331.2K.

$$= \frac{dQ}{T} = (ms + W) \int_{T_1}^{T_2} \frac{dT}{T}$$

$$= (ms + W) \times \ln \frac{331.2}{273}$$

$$= (90 + 10) \times \ln \frac{331.2}{273}$$

$$= +19.32 \text{ cal/K.}$$

Net change in entropy of the system

$$= 19.32 - 14.47 - 1.188$$

$$= +3.662 \text{ cal/K.}$$

Hence the net increase in the entropy of the system

$$= 3.662 \text{ cal/K.}$$

**Example 21.10.** The melting point of lead is  $327^\circ\text{C}$  and the latent heat of its fusion at  $327^\circ\text{C}$  is 5.86 cal. per gm. Calculate the change in entropy when 4 gm-molecules of lead are fused. (Atomic weight of lead = 207).

**Soln.**

Since lead is mono-atomic, 4 gm-molecules of lead are equal to  $4 \times 207 = 828$  gms of lead. Therefore, the amount of heat that must be supplied to fuse 4 gm-molecules, i.e., 828 gms. of lead is

$$Q = 5.86 \times 828 = 4852 \text{ cal.}$$

Melting point of lead,

$$T = (273 + 327) = 600\text{K.}$$

$$\therefore \text{Change in entropy} = \frac{Q}{T} = \frac{4852}{600}$$

$$= 8.1 \text{ cal/K.}$$

**Example 21.11.**  $4.5 \text{ m}^3$  of air at  $1.2 \text{ kg/cm}^2$  and  $100^\circ\text{C}$  are compressed to a volume of  $0.5 \text{ m}^3$  and a pressure of  $40 \text{ kg/cm}^2$ . Heat is then added at constant volume till the pressure becomes  $64 \text{ kg/cm}^2$ .

Determine the change in entropy during each of these operations stating whether the change is an increase or decrease. Assume  $C_p = 0.238$  and  $C_v = 0.169$ .

**Soln.**

(i) The general relation for change in entropy (Art. 21.8) is

$$S_2 - S_1 = m \left( C_v \ln \frac{T_2}{T_1} + \frac{R}{J} \ln \frac{V_2}{V_1} \right).$$

$$\text{Now } \frac{R}{J} = C_p - C_v$$

$$= 0.238 - 0.169 = 0.069.$$

$$\text{Also, } \frac{P_1 V_1}{T_1} = \frac{P_2 V_2}{T_2}$$

$$\text{or, } T_2 = \frac{P_2 V_2}{P_1 V_1} \times T_1$$



$$= \frac{40 \times 0.5}{1.2 \times 4.5} \times 373 = 1381 \text{ K.}$$

$$\therefore S_2 - S_1 = m \left( 0.169 \ln \frac{1381}{373} + 0.069 \ln \frac{0.5}{4.5} \right)$$

$$= + 0.068 \text{ m kcal/K.}$$

(ii) when heating is carried out at constant volume, the change in entropy,

$$S_2 - S_1 = m C_v \frac{T_2}{T_1}$$

Since volume is constant,

$$\frac{P_2}{P_1} = \frac{T_2}{T_1}; \quad \text{or, } T_2 = \frac{P_2}{P_1} \times T_1$$

$$= \frac{64}{40} \times 1381 = 2210 \text{ K.}$$

$$\therefore S_2 - S_1 = m \times 0.169 \ln \frac{2210}{1381}$$

$$= + 0.0794 \text{ m kcal/K.}$$

**Example 21.12.** Calculate the change in entropy of 1 kg of air which is heated at constant pressure from an initial temperature of  $327^\circ\text{C}$  to a final temperature of  $1727^\circ\text{C}$ . The specific heat of air at constant pressure is given by

$$C_p = \alpha + \beta T + \gamma T^2 \text{ kcal/kg} - \text{K.}$$

where  $\alpha = 0.22$ ,  $\beta = 0.34 \times 10^{-4}$  and  $\gamma = -0.29 \times 10^{-8}$

**Soln.**

$$\int dS = \int \frac{dQ}{T}$$

$$\text{Now } dQ = m C_p dT$$

$$= C_p dT \quad (\text{for 1 kg}).$$

$$\therefore dQ = (\alpha + \beta T + \gamma T^2) dT$$

$$\begin{aligned} \therefore \int_{S_1}^{S_2} dS &= \int_{T_1}^{T_2} \left( \frac{\alpha + \beta T + \gamma T^2}{T} \right) dT \\ &= \int_{T_1}^{T_2} \left( \frac{\alpha}{T} + \beta + \gamma T \right) dT \end{aligned}$$

$$\therefore S_2 - S_1 = \left[ \alpha \ln T + \beta T + \frac{\gamma T^2}{2} \right]_{T_1}^{T_2}$$

$$= \left[ \alpha \ln T + \beta T + \frac{\gamma T^2}{2} \right]_{600}^{2000}$$

$$= \alpha \ln \frac{2000}{600} + \beta (2000 - 600) + \frac{\gamma}{2} (2000^2 - 600^2)$$

$$= 0.22 \times 1.2176 + 0.34 \times 10^{-6} \times 1400 - 0.29 \times 10^{-8} \times 364 \times 10^4$$

$$= 0.2646 + 0.000476 - 0.01056$$

$$= 0.255 \text{ kcal/K.}$$

### EXERCISES

- [1] Explain fully what is meant by entropy? Show that the entropy remains constant in an isolated reversible process and increase in an irreversible process.
- [2] Define entropy. Discuss its physical significance. Show the entropy remains constant in a reversible adiabatic process and increases in an irreversible process.
- [3] Explain entropy. Derive an expression for the entropy of  $m$  gms of a perfect gas.
- [4] Explain the terms entropy and isentropics. Are all adiabatics isentropics?

Derive an expression for the change of entropy of a gm-molecule of a gas during an isothermal expansion.

[5] Explain what do you mean by entropy. Prove that (i) the entropy of the universe is always increasing and heading towards a maximum; (ii) the change of entropy of a reversible cycle is zero.

[6] Draw the P-V and T-S diagrams for a Carnot cycle. Deduce the efficiency of the cycle using the T-S diagram.

The entropy of a substance is a measure of its state of disorder – discuss fully.

[7] Define entropy. Obtain expressions for the change in entropy of a gas when it is heated (i) at constant volume; (ii) at constant pressure and (iii) in a general manner.

[8] Calculate the change in entropy when 10 gms of ice at  $0^{\circ}\text{C}$  is converted into water at the same temperature. [2.93 cal/K].

[9] Calculate the change in entropy when 5 kg of water at  $100^{\circ}\text{C}$  is converted into steam at the same temperature. [7240 cal/K].

[10] Calculate the change in entropy when 150 kcal of heat are supplied (a) at  $27^{\circ}\text{C}$  and (b) at  $327^{\circ}\text{C}$ . [(a) 0.5 kcal/K (b) 0.25 kcal/K].

[11] Find the difference in entropy between 10 gm. of water at  $100^{\circ}\text{C}$  and 10 gms. of steam at  $100^{\circ}\text{C}$ . [14.48 cal/K].

[12] Find the change in entropy when 1 gm of ice at  $0^{\circ}\text{C}$  is gradually changed to 1 gm of dry steam at  $100^{\circ}\text{C}$  at one atmosphere pressure. Assume latent heat of fusion of ice at  $0^{\circ}\text{C} = 80$  cal/gm.; specific heat of water between  $0^{\circ}\text{C}$  and  $100^{\circ}\text{C} = 1$ , and latent heat of steam at  $100^{\circ}\text{C} = 540$  cal/gm. [2.053 cal/K].

[13] 50 gms of water at  $0^{\circ}\text{C}$  is mixed with an equal mass of water at  $83^{\circ}\text{C}$ . Calculate the resultant increase in entropy. [0.806 cal/K].

[14] Calculate the change in entropy when 50 gms of water at  $15^{\circ}\text{C}$  is mixed with 80 gms of water at  $40^{\circ}\text{C}$ . Specific heat of water may be assumed to be equal to 1. [0.115 cal/K].

[15] 1 gm of water at  $20^{\circ}\text{C}$  is converted into ice at  $-10^{\circ}\text{C}$  at constant pressure. Heat capacity for 1 gm of water is 4.2 J/gm-K and that of ice is 2.1 J/gm-K. Heat of fusion of ice at  $0^{\circ}\text{C} = 335$  J/gm. Calculate the total change in the entropy of the system. [-1.60224 J/K].

[16] Calculate the change in entropy when 200 gms of solid mercury at its melting point is raised to a temperature of  $40^{\circ}\text{C}$ . [melting point of mercury =  $-39^{\circ}\text{C}$ , latent heat of fusion = 3 cal/gm; mean specific heat over the required range of temperature = 0.335 cal/gm- $^{\circ}\text{C}$ ]. [4.515 cal/K].



- [17] Calculate the increase in entropy of 10 gms of hydrogen when its temperature is raised from  $27^{\circ}\text{C}$  to  $77^{\circ}\text{C}$  and its volume is also doubled. ( $C_v = 5$  cal,  $R = 2$ ).  $[5 \times 2.3026 [5 \log \frac{350}{300} + 2 \log 2]]$ .
- [18] 10g of water at  $60^{\circ}\text{C}$  is mixed with 30g of water of  $20^{\circ}\text{C}$ . Will the entropy of the system increase or decrease? Calculate the change.  $[+0.063 \text{ cal/K}^{-1}$ ; so there is a net increase in entropy.]
- [19] Calculate the change in entropy when 1gm. atom of solid tin at its melting point ( $232^{\circ}\text{C}$ ) is raised to a temperature of  $272^{\circ}\text{C}$ . Given : at. wt. of tin = 118, latent heat of fusion = 14 cal/g, mean specific heat over the temperature change = 0.064.  $[22.04 \text{ cal/}^{\circ}\text{C}]$ .

## CHAPTER XXII

# THERMODYNAMIC RELATIONS AND THERMODYNAMIC FUNCTIONS

*Introduction – Two important relations of partial differentiation – Maxwell's thermodynamic relations – Clausius – Clapeyron equation – Thermodynamic relations with heat capacities – Thermodynamic functions – Re-derivation of Maxwell's thermodynamic relations.*

## 22.1 Introduction

A thermodynamic system, as we have already seen, possesses a number of properties such as pressure, temperature, heat capacity, compressibility, internal energy, entropy and so on. Of these, some like temperature, pressure, heat capacity, compressibility can be determined in the laboratory. On the other hand, some like the internal energy and entropy cannot be measured in any laboratory. The problem, however, could be solved if these could be related with properties that are measurable. It is here that the thermodynamic relations come to the rescue. We shall see that the internal energy and entropy could be expressed as partial derivatives involving extensive and intensive variables. A thermodynamic relation may be said to be an exact rule obtainable by general thermodynamic reasoning and is applicable to almost all system. Thermodynamic relations are in fact many and there are many ways of deriving them. The most convenient technique, however, is to use partial differentiation.

## 22.2 Two important relations of partial differentiation

If  $f(x, y, z) = 0$ , that is, if  $x$ ,  $y$  and  $z$  are connected by a functional equation so that each is expressible as a function of the other two, then

$$1 \cdot \left( \frac{\partial z}{\partial x} \right)_y = 1 / \left( \frac{\partial x}{\partial z} \right)_y$$

and similar relations, and

$$\text{II. } \left( \frac{\partial x}{\partial y} \right)_z \left( \frac{\partial y}{\partial z} \right)_x \left( \frac{\partial z}{\partial x} \right)_y = -1$$

**Proof.**

We have

$$x = x(y, z) \text{ and } z = z(x, y)$$

$$\therefore dx = \left( \frac{\partial x}{\partial y} \right)_z dy + \left( \frac{\partial x}{\partial z} \right)_y dz \quad (\text{i})$$

$$\text{and } dz = \left( \frac{\partial z}{\partial x} \right)_y dx + \left( \frac{\partial z}{\partial y} \right)_x dy \quad (\text{ii})$$

Eliminating  $dz$  from (i) and (ii), we get

$$dx = \left( \frac{\partial x}{\partial y} \right)_z dy + \left( \frac{\partial x}{\partial z} \right)_y \left[ \left( \frac{\partial z}{\partial x} \right)_y dx + \left( \frac{\partial z}{\partial y} \right)_x dy \right] \quad (\text{iii})$$

Now,  $dx$  and  $dy$  are independent of each other. Hence equating the co-efficients of  $dx$  and  $dy$  from both sides of (iii), we get

$$\left( \frac{\partial x}{\partial z} \right)_y \left( \frac{\partial z}{\partial x} \right)_y = 1$$

$$\text{or, } \left( \frac{\partial z}{\partial x} \right)_y = 1 / \left( \frac{\partial x}{\partial z} \right)_y$$

which is relation I.

$$\text{Also } \left( \frac{\partial x}{\partial y} \right)_z + \left( \frac{\partial x}{\partial z} \right)_y \left( \frac{\partial z}{\partial y} \right)_x = 0$$

Using I, we get

$$\left( \frac{\partial y}{\partial z} \right)_x \left( \frac{\partial z}{\partial x} \right)_y \left( \frac{\partial x}{\partial y} \right)_z + 1 = 0$$

$$\text{or, } \left( \frac{\partial x}{\partial y} \right)_z \left( \frac{\partial y}{\partial z} \right)_x \left( \frac{\partial z}{\partial x} \right)_y = -1$$

which is relation II.



### 22.3 Maxwell's thermodynamic relations

Maxwell derived the relations by combining the first and the second law of thermodynamics. For a PVT system undergoing an infinitesimal reversible process, we have from the first law of thermodynamics

$$dQ = dU + dW$$

$$= dU + pdV$$

$$\text{or, } dU = dQ - pdV \quad (i)$$

From the second law of thermodynamics, we have

$$dS = \frac{dQ}{T}$$

$$\text{or, } dQ = T \cdot dS \quad (ii)$$

Substituting the value of  $dQ$  as given by eqn. (ii) in eqn. (i), we get

$$dU = T dS - pdV \quad (iii)$$

Eqn. (iii) involves five state functions :  $p$ ,  $V$ ,  $T$ ,  $S$  and  $U$ . Of these any two, say pressure  $p$  and temperature  $T$ , can be used to determine the thermodynamic state of the system. Then, each of  $U$ ,  $S$  and  $V$  will be functions of  $p$  and  $T$ . But, for the sake of generality, let the independent variables be designated as  $x$  and  $y$  so that

$$U = U(x, y); S = S(x, y) \text{ and } V = V(x, y) \quad (iv)$$

From the set of equations (iv) :

$$dU = \left( \frac{\partial U}{\partial x} \right)_y dx + \left( \frac{\partial U}{\partial y} \right)_x dy$$

$$dS = \left( \frac{\partial S}{\partial x} \right)_y dx + \left( \frac{\partial S}{\partial y} \right)_x dy$$

$$\text{and } dV = \left( \frac{\partial V}{\partial x} \right)_y dx + \left( \frac{\partial V}{\partial y} \right)_x dy$$

Substituting these values in eqn. (iii).

$$\left(\frac{\partial U}{\partial x}\right)_y dx + \left(\frac{\partial U}{\partial y}\right)_x dy = T \left[ \left(\frac{\partial S}{\partial x}\right)_y dx + \left(\frac{\partial S}{\partial y}\right)_x dy \right] - p \left[ \left(\frac{\partial V}{\partial x}\right)_y dx + \left(\frac{\partial V}{\partial y}\right)_x dy \right]$$

or,  $\left(\frac{\partial U}{\partial x}\right)_y dx + \left(\frac{\partial U}{\partial y}\right)_x dy = \left[ T \left(\frac{\partial S}{\partial x}\right)_y - p \left(\frac{\partial V}{\partial x}\right)_y \right] dx + \left[ T \left(\frac{\partial S}{\partial y}\right)_x - p \left(\frac{\partial V}{\partial y}\right)_x \right] dy$

Comparing the co-efficients of  $dx$  and  $dy$ , we get

$$\left(\frac{\partial U}{\partial x}\right)_y = T \left(\frac{\partial S}{\partial x}\right)_y - p \left(\frac{\partial V}{\partial x}\right)_y \quad (v)$$

$$\left(\frac{\partial U}{\partial y}\right)_x = T \left(\frac{\partial S}{\partial y}\right)_x - p \left(\frac{\partial V}{\partial y}\right)_x \quad (vi)$$

Differentiating eqn. (v) with respect to  $y$  keeping  $x$  fixed and eqn. (vi) with respect to  $x$  keeping  $y$  fixed.

$$\frac{\partial^2 U}{\partial y \partial x} = \left(\frac{\partial T}{\partial y}\right)_x \left(\frac{\partial S}{\partial x}\right)_y + T \frac{\partial^2 S}{\partial y \partial x} - \left(\frac{\partial p}{\partial y}\right)_x \left(\frac{\partial V}{\partial x}\right)_y - p \frac{\partial^2 V}{\partial y \partial x}$$

and  $\frac{\partial^2 U}{\partial x \partial y} = \left(\frac{\partial T}{\partial x}\right)_y \left(\frac{\partial S}{\partial y}\right)_x + T \frac{\partial^2 S}{\partial x \partial y} - \left(\frac{\partial p}{\partial x}\right)_y \left(\frac{\partial V}{\partial y}\right)_x - p \frac{\partial^2 V}{\partial x \partial y}$

Since  $dU$  is a perfect differential, the above two expressions are equal, i.e.,

$$\frac{\partial^2 U}{\partial x \partial y} = \frac{\partial^2 U}{\partial y \partial x}$$

The order of differential is immaterial. The change in internal energy brought about by changing  $V$  and  $T$  whether  $V$  is changed by

$dV$  first and  $T$  by  $dT$  later or *vice versa* is the same. Also, since  $dS$  and  $dV$  are perfect differentials,

$$\begin{aligned} \left(\frac{\partial T}{\partial y}\right)_x \left(\frac{\partial S}{\partial x}\right)_y + T \frac{\partial^2 S}{\partial y \partial x} - \left(\frac{\partial p}{\partial y}\right)_x \left(\frac{\partial V}{\partial x}\right)_y - p \frac{\partial^2 V}{\partial y \partial x} \\ = \left(\frac{\partial T}{\partial x}\right)_y \left(\frac{\partial S}{\partial y}\right)_x + T \frac{\partial^2 S}{\partial x \partial y} - \left(\frac{\partial p}{\partial x}\right)_y \left(\frac{\partial V}{\partial y}\right)_x - p \frac{\partial^2 V}{\partial x \partial y} \end{aligned}$$

Simplifying, we get

$$\left(\frac{\partial T}{\partial y}\right)_x \left(\frac{\partial S}{\partial x}\right)_y - \left(\frac{\partial p}{\partial y}\right)_x \left(\frac{\partial V}{\partial x}\right)_y = \left(\frac{\partial T}{\partial x}\right)_y \left(\frac{\partial S}{\partial y}\right)_x - \left(\frac{\partial p}{\partial x}\right)_y \left(\frac{\partial V}{\partial y}\right)_x \quad (\text{vii})$$

Eqn. (vii) is the *parent equation* from which Maxwell's relations immediately follow.

#### First relation :

Taking  $T$  and  $V$  as independent variables and putting  $x = T$  and  $y = V$ , we get

$$\frac{\partial T}{\partial x} = 1 \quad ; \quad \frac{\partial V}{\partial y} = 1$$

$$\frac{\partial T}{\partial y} = 0 \quad ; \quad \frac{\partial V}{\partial x} = 0$$

Substituting these values in eqn. (vii), we get

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V$$

$$\text{But } dS = \frac{\partial Q}{T}$$

$$\therefore \left(\frac{\partial Q}{\partial V}\right)_T = T \left(\frac{\partial p}{\partial T}\right)_V \quad (22.1a)$$

For any thermodynamic system, the increase of entropy per unit increase of volume, at constant temperature, is equal to the increase of pressure per unit increase of temperature in an isochoric process.



**Second relation :**

Taking  $T$  and  $p$  as independent variables, *i.e.*, putting  $x = T$  and  $y = p$ , we get

$$\begin{aligned} \frac{\partial T}{\partial x} &= 1 & ; & & \frac{\partial p}{\partial y} &= 1 \\ \text{and } \frac{\partial T}{\partial y} &= 0 & ; & & \frac{\partial p}{\partial x} &= 0 \end{aligned}$$

Substituting these values in eqn. (vii), we get

$$\begin{aligned} \left( \frac{\partial S}{\partial p} \right)_T &= - \left( \frac{\partial V}{\partial T} \right)_p \\ \text{or, } \left( \frac{\partial Q}{\partial p} \right)_T &= - T \left( \frac{\partial V}{\partial T} \right)_p \end{aligned} \quad (22.1b)$$

The decrease of entropy per unit increase of pressure in an isothermal process is equal to the increase of volume per unit increase of temperature for an isobaric process.

**Third relation :**

Taking entropy and volume as independent variables, *i.e.*,  $x = S$  and  $y = V$ , we get

$$\begin{aligned} \frac{\partial S}{\partial x} &= 1 & ; & & \frac{\partial V}{\partial y} &= 1 \\ \frac{\partial S}{\partial y} &= 0 & ; & & \frac{\partial V}{\partial x} &= 0 \end{aligned}$$

Substituting these values in eqn. (vii), we get

$$\begin{aligned} \left( \frac{\partial T}{\partial V} \right)_S &= - \left( \frac{\partial p}{\partial S} \right)_V \\ \text{or, } \left( \frac{\partial T}{\partial V} \right)_S &= - T \left( \frac{\partial p}{\partial Q} \right)_V \end{aligned} \quad (22.1c)$$

Increase of temperature per unit increase of volume in a reversible adiabatic (isentropic) process is equal to the decrease of pressure per unit increase of entropy in an isochoric process.

#### Fourth relation :

Taking entropy and pressure as independent variables, i.e.,  $x = S$  and  $y = p$ , we have

$$\frac{\partial S}{\partial x} = 1 \quad ; \quad \frac{\partial p}{\partial y} = 1$$

$$\frac{\partial S}{\partial y} = 0 \quad ; \quad \frac{\partial p}{\partial x} = 0$$

Substituting these values in eqn. (vii), we get

$$\left( \frac{\partial T}{\partial p} \right)_S = \left( \frac{\partial V}{\partial S} \right)_p$$

$$\text{or, } \left( \frac{\partial T}{\partial p} \right)_S = T \left( \frac{\partial V}{\partial Q} \right)_p \quad (22.1d)$$

Increase of temperature per unit increase of pressure in a reversible adiabatic (isentropic) process is equal to the increase of volume per unit increase of entropy in an isobaric process.

#### Fifth relation :

Taking  $p$  and  $V$  as independent variables, i.e.,  $x = p$  and  $y = V$ , we get

$$\frac{\partial p}{\partial x} = 1 \quad ; \quad \frac{\partial V}{\partial y} = 1$$

$$\frac{\partial p}{\partial y} = 0 \quad ; \quad \frac{\partial V}{\partial x} = 0$$

Substituting these values in eqn. (vii), we get

$$\left(\frac{\partial T}{\partial p}\right)_V \left(\frac{\partial S}{\partial V}\right)_p - \left(\frac{\partial T}{\partial V}\right)_p \left(\frac{\partial S}{\partial p}\right)_V = 1 \quad (22.1e)$$

#### Sixth relation :

Taking T and S as independent variables, *i.e.*, putting  $x = T$  and  $y = S$ , we get

$$\frac{\partial T}{\partial x} = 1 = \frac{\partial S}{\partial y}$$

$$\frac{\partial T}{\partial y} = 0 = \frac{\partial S}{\partial x}$$

Substituting these values in eqn. (vii), we get

$$\left(\frac{\partial p}{\partial T}\right)_S \left(\frac{\partial V}{\partial S}\right)_T - \left(\frac{\partial p}{\partial S}\right)_T \left(\frac{\partial V}{\partial T}\right)_S = 1 \quad (22.1f)$$

Eqns. 22.1a – 22.1f are the important thermodynamic relations. While solving a particular problem, the appropriate equation is used.

### 22.4 Clausius – Clapeyron equation

At first sight it may appear that Maxwell's thermodynamic relations are just the product of mathematical exercises out of the first and second law of thermodynamics. Then what is the *significance* of having these relations? This can be understood by applying these relations to physical processes.

Let us take the first relation of Maxwell :

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V$$

Multiplying both sides by T, we get

$$T \left(\frac{\partial S}{\partial V}\right)_T = T \left(\frac{\partial p}{\partial T}\right)_V$$

$$\text{or, } \left(\frac{\partial Q}{\partial V}\right)_T = T \left(\frac{\partial p}{\partial T}\right)_V \quad (i)$$



Expressed in language, heat absorbed per unit volume in isothermal expansion is  $T$  times the rate of increase of pressure with temperature in an isochoric process.

Let us apply eqn. (i) to the equilibrium between two states of the same substance, say *phase transition* such as vapourisation of a liquid or melting of a solid. Now imagine a liquid in equilibrium with its vapour contained in a cylinder. The pressure is the saturated vapour pressure which is a function of temperature only and is independent of the amount of liquid and vapour present. If the system be allowed to expand at constant temperature, the vapour pressure will remain constant; only some additional liquid, say of mass  $dm$ , will evaporate to fill the extra space with vapour. Then

$$\delta Q = L \, dm$$

where  $L$  is the latent heat of evaporation per unit mass.

The corresponding volume change will be

$$\partial V = (v_{\text{vap}} - v_{\text{liq}}) \, dm$$

$v_{\text{vap}}$  and  $v_{\text{liq}}$  being the specific volumes of the vapour and the liquid respectively.

$$\therefore \left( \frac{\partial Q}{\partial V} \right)_T = \frac{L}{v_{\text{vap}} - v_{\text{liq}}} = T \left( \frac{\partial P}{\partial T} \right)_v \quad (\text{ii})$$

If, however, the volume is kept constant the temperature increases slightly, the liquid will evaporate till the mixture attains a new equilibrium state and a new saturated vapour pressure so that

$$\left( \frac{\partial P}{\partial T} \right)_v = \left( \frac{\partial P}{\partial T} \right)_{\text{sat}} \quad (\text{iii})$$

From (ii) and (iii), we get

$$\left( \frac{\partial P}{\partial T} \right)_{\text{sat}} = \frac{L}{T (v_{\text{vap}} - v_{\text{liq}})} \quad (22.3)$$

Eqn. (22.3), which is one of the most important relations in thermodynamics, is the well-known *Clapeyron equation*. It gives the rate at which the vapour pressure must change with temperature for

the two phases to coexist in equilibrium. The effect of pressure on the boiling point can be investigated with the help of this equation.

Since  $v_{\text{vap}}$  is always greater than  $v_{\text{liq}}$ , the right side of eqn. (22.3) is always positive, hence  $\left(\frac{\partial P}{\partial T}\right)_{\text{sat}}$  will also be positive. Thus, the *increase of pressure raises the boiling point and conversely*. This explains why food can be easily cooked in a pressure cooker and why it is difficult to cook food at high altitudes.

The Clapeyron equation as given by eqn. (22.3) is derived for evaporation. But it is equally valid for any phase transition : solid-liquid, liquid-vapour and solid-vapour transitions that involves latent heat.

$$\therefore \left(\frac{\partial P}{\partial T}\right)_v = \frac{L}{T(v_2 - v_1)}$$

where  $V_1$  and  $V_2$  are the specific volumes in phase 1 and 2 respectively.

The Clapeyron equation holds in all phase transitions in which there is discontinuity in entropy and volume at the transition temperature. This type of phase transition is known as the *first order phase transition*. There are transitions in which there is no change in entropy and volume during a phase transition but discontinuities appear in specific heat at constant pressure and isobaric volume expansivity. Such transitions are called *second order phase transitions*.

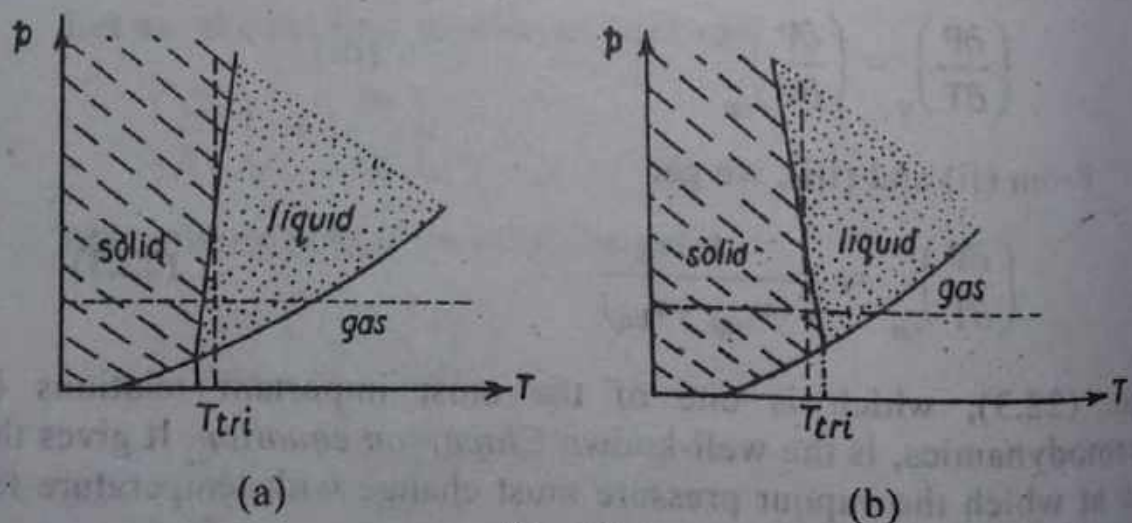


Fig. 22.1



A plot of all pressures and temperatures at which a phase transition takes place for a given substance is called a *phase diagram*. Phase diagrams for a simple compressible system and water are shown in Fig. 22.1(a) and (b) respectively. The curves give the unique relationship that must hold for the two phases to coexist. The point where the three curves intersect is called the *triple point*. All the three phases coexist at the triple point. For water, the triple point is ( $T_{tri} = 0.0075^\circ\text{C}$ ,  $p_{tri} = 4.58\text{mm}$ ).

The slope of the solid-liquid curve in Fig. 22.1 (a) is positive. So most substances expand on melting which means that  $\Delta V$  and hence  $dp/dT$  is positive. Hence, an increase of pressure will raise the melting point of this type of materials. However, as can be seen from Fig. 22.1(b), for water, the slope of the solid-liquid curve is negative. So water expands on freezing and its melting point is lowered by pressure. In this context, water is an exception and it is possible to pass from vapour  $\rightarrow$  solid  $\rightarrow$  liquid phase at  $T = T_{tri}$  by increasing the pressure isothermally.

**Thermal expansion** – To appreciate the significance of the second relation, let us apply it to thermal expansion. We have

$$\left(\frac{\partial S}{\partial p}\right)_T = \left(\frac{\partial V}{\partial T}\right)_p$$

$$\text{or, } \left(\frac{\partial Q}{\partial p}\right)_T = -T \left(\frac{\partial V}{\partial T}\right)_p = -TV\alpha \quad (22.4)$$

where  $\alpha = \frac{1}{V} \left(\frac{\partial V}{\partial T}\right)_p$  is the co-efficient of volume expansion.

For substances that expand on heating,  $\alpha > 0$ . So from eqn. (22.4)  $\left(\frac{\partial Q}{\partial p}\right)_T$  is negative. This implies that when such a substance is compressed isothermally, heat must be removed. In other words, these substances generate heat on isothermal compression.



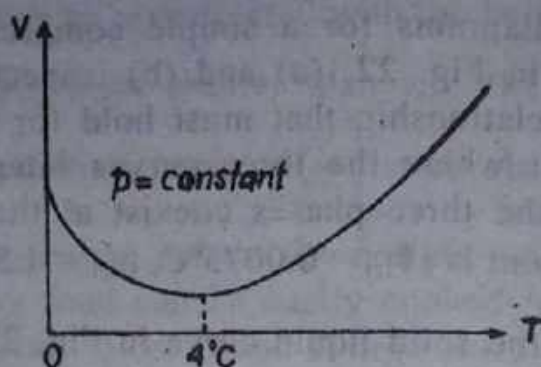


Fig. 22.2

For substances that contract on heating,  $\alpha < 0$ . The example is again water. When heated at  $0^\circ\text{C}$  isobarically at normal pressure, water shows the so-called *anomalous expansion*. Its specific volume decreases until  $4^\circ\text{C}$  (Fig. 22.2) beyond which however it increases. So the slope  $(\partial V / \partial T)_p$  in its  $V$ - $T$  diagram is negative at  $0^\circ\text{C}$ , so that  $(\partial Q / \partial p)_T$  is positive. So when water at  $0^\circ\text{C}$  is compressed isothermally, heat is to be added, *i.e.*, heat is absorbed. India rubber is another example.

Conversely, water below  $4^\circ\text{C}$  gets cooled on adiabatic compression although its internal energy is increased. So does India rubber. When stretched adiabatically, it shows heating effect, as expected, and can be easily felt by lips.

Maxwell's other two relations also find applications in adiabatic-isochoric and adiabatic-isobaric processes.

The importance of Maxwell's relations can be realised from the above discussions. Some examples have been worked out which would bear testimony to the importance of Maxwell's relations.

## 22.5 Thermodynamic relations with heat capacities

The relation involving the difference between two specific heats of a gas is of particular importance in physics and can be readily deduced using Maxwell's thermodynamic relations.

With  $T$  and  $V$  as independent variables, we may write

$$S = S(T, V)$$

$$\therefore dS = \left( \frac{\partial S}{\partial T} \right)_V dT + \left( \frac{\partial S}{\partial V} \right)_T dV$$

$$\therefore \left( \frac{\partial S}{\partial T} \right)_P = \left( \frac{\partial S}{\partial T} \right)_V + \left( \frac{\partial S}{\partial V} \right)_T \left( \frac{\partial V}{\partial T} \right)_P$$

Multiplying both sides by  $T$  and rearranging,

$$T \left( \frac{\partial S}{\partial T} \right)_P - T \left( \frac{\partial S}{\partial T} \right)_V = T \left( \frac{\partial S}{\partial V} \right)_T \left( \frac{\partial V}{\partial T} \right)_P$$

$$\text{or, } C_P - C_V = T \left( \frac{\partial p}{\partial T} \right)_V \left( \frac{\partial V}{\partial T} \right)_P \quad (22.2a)$$

using Maxwell's relation

Two other forms can also be obtained from eqn. (22.2a) using the important property of partial differentials, i.e., if  $f(p, V, T) = 0$

$$\left( \frac{\partial p}{\partial V} \right)_T \left( \frac{\partial V}{\partial T} \right)_P \left( \frac{\partial T}{\partial p} \right)_V = -1$$

$$\therefore \left( \frac{\partial p}{\partial T} \right)_V = - \left( \frac{\partial p}{\partial V} \right)_T \left( \frac{\partial V}{\partial T} \right)_P \quad (i)$$

$$\text{and } \left( \frac{\partial V}{\partial T} \right)_P = - \left( \frac{\partial p}{\partial T} \right)_V \left( \frac{\partial V}{\partial p} \right)_T \quad (ii)$$

Using eqns. (i) and (ii) successively in (22.2a), we get

$$C_P - C_V = -T \left( \frac{\partial p}{\partial V} \right)_T \left( \frac{\partial V}{\partial T} \right)_P^2 \quad (22.2b)$$

$$\text{and } C_P - C_V = -T \left( \frac{\partial V}{\partial p} \right)_T \left( \frac{\partial p}{\partial T} \right)_V^2 \quad (22.2c)$$

Eqns. 22.2a – 22.2c are often used to correlate or compute important physical properties of the system.

The following conclusions can be drawn from eqn. (22.2b) :



(i) For all known substances  $(\partial p / \partial V)_T$  is negative. But  $(\partial V / \partial T)_p^2$  is always positive. So  $C_p - C_v$  is always positive, i.e.  $C_p$  can never be less than  $C_v$ .

(ii) For liquids and solids  $(\partial V / \partial T)_p$  is rather small; so the difference  $C_p - C_v$  is also small. But when  $(\partial V / \partial T)_p = 0$ , as in the case of water at  $0^\circ\text{C}$ ,  $C_p - C_v = 0$ ; or,  $C_p = C_v$ .

(iii) As  $T \rightarrow 0$ ,  $C_p \rightarrow C_v$ . That is, as the absolute zero is approached, the two heat capacities tend to equalise – a fact confirmed by experiments.

### $C_p - C_v$ for a perfect gas

For a perfect gas

$$PV = RT$$

$$\left(\frac{\partial p}{\partial T}\right)_v = \frac{R}{V} \quad \text{and} \quad \left(\frac{\partial V}{\partial T}\right)_p = \frac{R}{P}$$

$$\text{But } C_p - C_v = T \left(\frac{\partial p}{\partial T}\right)_v \left(\frac{\partial V}{\partial T}\right)_p$$

$$= T \left(\frac{R}{V}\right) \left(\frac{R}{P}\right)$$

$$= \frac{TR^2}{PV} = \frac{TR^2}{RT} = R.$$

Hence, for a perfect gas,  $C_p - C_v = R$

where  $R$  is in heat units.

**Example 22.1.** Calculate the change in melting point of ice at  $0^\circ\text{C}$  when the pressure is increased by 2 atmos. How much pressure is required to lower the melting point by  $1^\circ\text{C}$ ? Given : latent heat of fusion  $80 \text{ cal g}^{-1}$  and the specific volumes of water and ice are respectively  $1.0001 \text{ c.c.}$  and  $1.0908 \text{ c.c.}$

**Soln.**



From Clausius – Clapeyron eqn.

$$\left(\frac{dp}{dT}\right)_{\text{sat}} = \frac{L}{T(v_w - v_i)}$$

Here

$$T = 273 \text{ K}$$

$$L = 80 \times 4.2 \times 10^7 \text{ erg.g}^{-1}$$

$$v_w = 1.0001 \text{ c.c.}$$

$$v_i = 1.0908 \text{ c.c.}$$

$$dp = 2 \text{ atmos.}$$

$$= 2 \times 1.013 \times 10^6 \text{ dynes cm}^{-2}$$

$$\therefore \frac{dp}{dT} = \frac{80 \times 4.2 \times 10^7}{273 (1.0001 - 1.0908)}$$

$$= -13.5 \times 10^7 \text{ dyne cm}^{-2} \text{K}^{-1}$$

$$\therefore dT = -\frac{2 \times 1.013 \times 10^6}{13.5 \times 10^7}$$

$$= -0.015 \text{ K} = -0.015^\circ\text{C}.$$

Thus the melting point of ice falls with pressure and the drop in temperature per atmosphere is  $0.0075^\circ\text{C}$ .

$\therefore$  Pressure required to lower the melting point by  $1^\circ\text{C}$

$$= \frac{1}{0.0075} = 133 \text{ atmos.}$$

**Example 22.2.** Show that the ratio of the adiabatic to isothermal elasticity is  $\gamma$ , the ratio of the two specific heats.

**Soln.**

$$\text{adiabatic elasticity, } E_S = -V \left( \frac{\partial p}{\partial V} \right)_S$$

$$\text{isothermal elasticity, } E_T = -V \left( \frac{\partial p}{\partial V} \right)_T$$

$$\therefore \frac{E_S}{E_T} = \frac{(\partial p / \partial V)_S}{(\partial p / \partial V)_T}$$

$$= \frac{(\partial p / \partial T)_S (\partial T / \partial V)_S}{(\partial p / \partial S)_T (\partial S / \partial V)_T}$$

$$= \frac{(\partial S / \partial V)_p (\partial p / \partial S)_V}{(\partial T / \partial V)_p (\partial p / \partial T)_V}$$

(Using Maxwell's relations)

$$= \frac{(\partial S / \partial T)_p}{(\partial S / \partial T)_v} = \frac{C_p}{C_v} = \gamma.$$

**Example 22.3.** Show that for a Van der Waals' gas

$$C_p - C_v = \frac{R \left( p + \frac{a}{V^2} \right)_s}{p - \frac{a}{V^2} + \frac{2ab}{V^3}}$$

**Soln.**

For one mole of a Van der Waals gas, we have

$$p = \frac{RT}{V-b} - \frac{a}{V^2}$$

$$\therefore \left( \frac{\partial p}{\partial T} \right)_v = \frac{R}{V-b} \quad (i)$$

Rearranging Van der Waals equation, we get

$$pV + \frac{a}{V} - pb - \frac{ab}{V^2} = RT$$

Differentiating the above equation with respect to  $T$ , keeping  $P$  fixed,

$$p \left( \frac{\partial V}{\partial T} \right)_p - \frac{a}{V^2} \left( \frac{\partial V}{\partial T} \right)_p + \frac{2ab}{V^3} \left( \frac{\partial V}{\partial T} \right)_p = R$$

$$\therefore \left( \frac{\partial V}{\partial T} \right)_p = \frac{R}{p - \frac{a}{V^2} + \frac{2ab}{V^3}} \quad (ii)$$

But, from Maxwell's thermodynamic relations, we have

$$C_p - C_v = T \left( \frac{\partial p}{\partial T} \right)_v \left( \frac{\partial V}{\partial T} \right)_p$$

$$= T \cdot \frac{R}{V-b} \cdot \frac{R}{P - \frac{a}{V^2} + \frac{2ab}{V^3}} \quad \text{using (i) and (ii)}$$

$$= \frac{R \left( P + \frac{a}{V^2} \right)}{P - \frac{a}{V^2} + \frac{2ab}{V^3}}$$

$$\left( \because \text{from Van der Waals eqn. } P + \frac{a}{V^2} = \frac{RT}{V-b} \right)$$

**Example 22.4.** Using Maxwell's thermodynamic relations, prove that the ratio of the adiabatic to the isochoric pressure coefficient of expansion is equal to  $\gamma/(\gamma-1)$ .

**Soln.**

Adiabatic pressure coefficient of expansion,

$$\beta_s = \frac{1}{P} \left( \frac{\partial P}{\partial T} \right)_s \quad \text{(i)}$$

Isochoric pressure coefficient of expansion,

$$\beta_v = \frac{1}{P} \left( \frac{\partial P}{\partial T} \right)_v \quad \text{(ii)}$$

$$\therefore \frac{\beta_s}{\beta_v} = \frac{\frac{1}{P} (\partial P / \partial T)_s}{\frac{1}{P} (\partial P / \partial T)_v} \quad \text{(iii)}$$

$$= \frac{(\partial P / \partial T)_s}{(\partial P / \partial T)_v} = \frac{1}{(\partial T / \partial P)_s \left( \frac{\partial P}{\partial T} \right)_v}$$

$$\text{But } (\partial T / \partial P)_s = (\partial V / \partial S)_P \quad \text{(iv)}$$

$$\therefore \frac{\beta_s}{\beta_v} = \frac{1}{(\partial V / \partial S)_P (\partial P / \partial T)_v} \quad \text{(v)}$$



$$\begin{aligned}
 &= \frac{1}{(\partial V/\partial T)_p (\partial T/\partial S)_p (\partial P/\partial T)_v} \\
 &= \frac{(\partial S/\partial T)_p}{(\partial V/\partial T)_p (\partial P/\partial T)_v} \\
 &= \frac{T (\partial S/\partial T)_p}{T (\partial V/\partial T)_p (\partial P/\partial T)_v} \\
 &= \frac{C_p}{C_p - C_v} = \frac{C_p/C_v}{C_p/C_v - 1} = \frac{\gamma}{\gamma - 1} \quad \text{(vi)}
 \end{aligned}$$

## 22.6 Thermodynamic Functions

The condition of a system may be specified by means of certain selected variables, such as  $P$ ,  $V$ ,  $T$ , or  $S$ . Certain relations exist between these variables. The two laws of thermodynamics provide two more relations given by  $dU = dQ - dW$  and  $dQ = TdS$ . For a complete knowledge of the system some other relation – such as equation of state of a fluid, will be necessary. But in general, when such an equation of state is not available, certain functions of the variables may be introduced for the purpose of simplifying the mathematical operations connected with a change in the condition of the system. These functions, called *thermodynamic functions*, are discussed below.

### 1. Internal energy

According to first law of thermodynamics, internal energy is defined by the equation

$$\begin{aligned}
 dQ &= dU + dW \\
 &= dU + pdV \\
 \text{or, } dU &= dQ - pdV \\
 &= TdS - pdV \quad \text{(i)}
 \end{aligned}$$

Internal energy is therefore a function of entropy and volume so that we can write

$$U = U(S, V)$$

$$\therefore dU = \left( \frac{\partial U}{\partial S} \right)_V dS + \left( \frac{\partial U}{\partial V} \right)_S dV \quad (ii)$$

Equating the coefficients of  $dS$  and  $dV$  from eqns. (i) and (ii) we get

$$\left( \frac{\partial U}{\partial S} \right)_V = T \quad (iii)$$

$$\left( \frac{\partial U}{\partial V} \right)_S = -P \quad (iv)$$

## 2. Helmholtz free energy function

In classical mechanics, the amount of work done by a mechanical system is equal to the decrease of energy  $E$  of the system, i.e.,

$$\Delta W = -\Delta E \quad (i)$$

And from the first law of thermodynamics, we have

$$\Delta W = -\Delta U + \Delta Q \quad (ii)$$

where  $\Delta Q$  is the amount of heat energy exchanged between the system and the surroundings and  $\Delta U$  is the change in internal energy of the system.

Our aim is to investigate the physical conditions of the system under which it is possible to define thermodynamic functions which play a role analogous to that of energy in mechanics.

Let us consider a thermodynamic system  $K$  which undergoes a transformation from an initial state  $i$  to a final state  $f$ . During the transformation from  $i \rightarrow f$  let the system be in thermal contact with the surroundings whose temperature is  $T_K$ . The transformation process is therefore accomplished under isothermal condition.

From Clausius' theorem of inequality, we have

$$\int_i^f \frac{dQ}{T} \leq S(f) - S(i) \quad (iii)$$

Since the temperature is constant throughout the process, eqn. (iii) gives



$$\int_i^f dQ \leq T [S(f) - S(i)]$$

$$\text{or, } \Delta Q \leq T [S(f) - S(i)]$$

$$\text{or, } (\Delta U + \Delta W) \leq T [S(f) - S(i)]$$

$$\text{or, } U(f) - U(i) + \Delta W \leq T [S(f) - S(i)]$$

$$\therefore \Delta W \leq [U(i) - TS(i)] - [U(f) - TS(f)] \quad (\text{iv})$$

$$\text{Introducing } U - TS = F \quad (\text{v})$$

$$\text{we get } U(i) - TS(i) = F(i) \quad (\text{va})$$

$$U(f) - TS(f) = F(f) \quad (\text{vb})$$

The temperature  $T$  is the same at the initial and the final state.

The inequality equation given by (iv) then becomes

$$\Delta W \leq F(i) - F(f) \quad (\text{vi})$$

Now  $F(f) - F(i) = \Delta F$  is the change in  $F$  during the process  $i \rightarrow f$ .  
Therefore

$$\Delta W \leq -\Delta F \quad (\text{vii})$$

It may, therefore, be concluded that the work done by a system undergoing an isothermal transformation is less than or equal to the decrease in  $F$ , according as the process is irreversible or reversible. The function  $F$ , defined by the equation

$$F = U - TS \quad (\text{viii})$$

is known as the *free energy* of the system after Helmholtz. It is also referred to as *Helmholtz free energy* or *Helmholtz function*. In a reversible isothermal transformation, the work performed by the system is exactly equal to the decrease in free energy. Free energy,  $F$ , thus plays the role of *energy* of a mechanical system.

Eqn. (v) or (vii) shows that the internal energy is made up of two parts : (i) the *free energy*  $F$  which is *available* for work in reversible isothermal processes and (ii) the *latent* or *bound energy*  $TS$ , which is unavailable for useful work. As entropy increases, available energy decreases.



### Significance of free energy :

The significance of free energy  $F$  as a *potential function* in thermodynamics can be realised if we recall that for a stable state of mechanical equilibrium, the potential energy of the system is minimum.

Let the system  $K$  undergoing isothermal transformation from  $i \rightarrow f$  be kept inside an enclosure whose walls are rigid but diathermic. The system can thus exchange heat with the surroundings but no external work can either be performed or absorbed. The system is then said to be dynamically insulated but in perfect thermal contact with the surroundings. Dynamical insulation means  $V = \text{const}$ . Thus the transformation is an *isothermal isochoric* one.

$$\therefore \Delta W = 0 \quad (\text{ix})$$

From eqn. (vi), we get

$$F(f) \leq F(i) \quad (\text{x})$$

Thus for a system undergoing an *isothermal-isochoric* transformation, the free energy in the final state can never exceed that in the initial state. So the *state of minimum free energy* for such a system is the *stable state of equilibrium* having definite set of values for temperature and pressure. Any further transformation will cause increase of free energy in violation of relation (x). The correspondence between the state of stable equilibrium and minimum potential energy of a mechanical system makes one convinced that the free energy of a thermodynamic system undergoing isothermal-isochoric transformation plays the role of *potential energy* in thermodynamics. Hence the Helmholtz free energy  $F$  is defined as *thermodynamic potential at constant volume*.

Since  $F$  involves a combination of  $V$ ,  $T$  and  $S$ , all of which are functions of state,  $F$  must also be a *function of state* and  $\delta F$  is an exact differential.

From relation (viii)

$$F = U - TS$$

For an infinitesimal reversible process, we have for a PV-T system

$$\delta F = \delta U - T\delta S - S\delta T$$

$$= -p\delta V - S\delta T \quad (\text{xi})$$

$$(\because T\delta S = \delta U + p\delta V)$$

Eqn. (xi) defines the dependence of  $F$  on the independent variations of  $T$  and  $V$ . thus  $T$  and  $V$  make the natural pair for Helmholtz free energy and we can write

$$F = F(T, V)$$

$$\therefore dF = \left(\frac{\partial F}{\partial T}\right)_V dT + \left(\frac{\partial F}{\partial V}\right)_T dV \quad (\text{xii})$$

Equating the co-efficients of  $dT$  and  $dV$  from (xi) and (xii), we get

$$P = -\left(\frac{\partial F}{\partial V}\right)_T \quad (\text{xiii})$$

$$S = -\left(\frac{\partial F}{\partial T}\right)_V \quad (\text{xiv})$$

Thus, once  $F$  is known for the system under consideration, a complete information about its thermal properties can be obtained. As can be seen from eqn. (xiii), an increase in volume decreases the free energy; the rate of fall being greater the higher the pressure. Similarly eqn. (xiv) points to the fact that as the temperature increases, the free energy decreases; the rate of decrease being greater the higher the entropy of the system. This explains why as temperature increases, the rate of fall of  $F$  is maximum for gases and minimum for solids.

It may further be noted that

(1) There is a correspondence between  $U$  and  $F$  for an adiabatic and isothermal process which are as follows:

For a reversible isothermal process  $\Delta W = -\Delta F$  and for an adiabatic process, from the first law  $\Delta W = -\Delta U$ .

(2) As will be seen later, the Helmholtz free energy is particularly important in providing a link between thermodynamics and statistical mechanics.



### Gibbs' free energy function

There exists thermodynamic transformations for which the temperature and pressure do not change but are equal to those of the surroundings during the transformation.

Consider a thermodynamic system K, undergoing such a typical *isothermal-isobaric* process causing a transition of the system from an initial state *i* to a final state *f*, the temperature and pressure remaining constant throughout the process. If  $V(i)$  and  $V(f)$  are the initial and final volumes of the system respectively, then the work done by it during the transformation is

$$\Delta W = P [V(f) - V(i)]$$

As the transformation is isothermal, referring to Clausius' eqn. of inequality (eqn. .... Art. ...), we get

$$P [V(f) - V(i)] \leq [U(i) - TS(i)] - [U(f) - TS(f)]$$

$$\text{or, } [PV(f) - TS(f) + U(f)] \leq [PV(i) - TS(i) + U(i)] \quad (i)$$

Introducing a function

$$G = U - TS + PV \quad (ii)$$

We obtain from eqn. (i)

$$G(f) \leq G(i) \quad (iii)$$

$$\text{or, } \Delta G \leq 0 \quad (iv)$$

In analogy with free energy *F*, the function *G* introduced and defined by eqn (ii) is called the *thermodynamic potential at constant pressure*. The function *G* is named *Gibbs free energy* after Willard Gibbs. From eqn. (iii), we conclude that for a system undergoing isothermal-isobaric transformation, Gibbs free energy in the final state cannot exceed that in the initial state. This implies that *the state of minimum G is the state of stable equilibrium* at a definite set of values of temperature and pressure.

Again, since *G* involves quantities which are all functions of state, *G* is also a *function of state* and *dG* is an exact differential. The relation (iv) implies that a system kept at a fixed temperature and pressure is defined completely by *G*. The relation further implies that a *spontaneous change will occur in the direction of*



decreasing Gibbs energy, while in a reversible change Gibbs energy will be conserved.

For an infinitesimal reversible process, we have from (iii)

$$\begin{aligned}\delta G &= \delta U - T\delta S - S\delta T + V\delta P + P\delta V \\ &= -S\delta T + V\delta P \quad (v)\end{aligned}$$

$$(\because T\delta S = \delta U + P\delta V)$$

G is thus a function of two independent variables, T and p. Hence we may write

$$G = G(T, P)$$

$$\therefore dG = \left(\frac{\partial G}{\partial T}\right)_P dT + \left(\frac{\partial G}{\partial P}\right)_T dP \quad (vi)$$

Equating the coefficients of dT and dP from (v) and (vi), we get

$$S = -\left(\frac{\partial G}{\partial T}\right)_P \quad (vii)$$

$$V = \left(\frac{\partial G}{\partial P}\right)_T \quad (viii)$$

which shows that increase in temperature at constant P decreases the Gibbs energy, while Gibbs energy increases with pressure at constant temperature.

#### 4. Enthalpy

Let us consider the Clausius inequality for an infinitesimal process.

$$\delta Q \leq T\delta S$$

$$\text{or, } \delta U + P\delta V \leq T\delta S$$

$$\text{or, } \delta U + P\delta V + V\delta P \leq T\delta S + V\delta P$$

$$\text{or, } \delta(U + PV) \leq T\delta S + V\delta P \quad (i)$$

Introducing a function

$$H = U + PV \quad (ii)$$

we have from (i)

$$\delta H \leq T\delta S + V\delta P$$

For an isentropic-isobaric process,

$$\delta H \leq 0 \quad (\text{iii})$$

The function  $H$  thus introduced and defined by eqn. (ii) is called *enthalpy* or *total heat function*. Since  $H$  involves quantities which are all functions of state,  $H$  is also a *function of state* and  $\delta H$  is an exact differential.

For an infinitesimal reversible process,

$$\delta H = \delta U + P\delta V + V\delta P$$

$$= T\delta S + V\delta P \quad (\text{iv})$$

$$(\because T\delta S = \delta U + P\delta V)$$

So  $H$  is a function of entropy and pressure so that we may write

$$H = H(S, P)$$

$$\therefore dH = \left(\frac{\partial H}{\partial S}\right)_P dS + \left(\frac{\partial H}{\partial P}\right)_S dP \quad (\text{v})$$

Equating the coefficients of  $dS$  and  $dP$  from (iv) and (v)

$$T = \left(\frac{\partial H}{\partial S}\right)_P \quad (\text{vi})$$

$$V = \left(\frac{\partial H}{\partial P}\right)_S \quad (\text{vii})$$

While eqn. (vi) implies that enthalpy is *heat content*, eqn. (vii) shows that in an isentropic process an increase of pressure results in an increase of enthalpy.

In the above discussions we have established the existence of four thermodynamic potentials :  $U(S, V)$ ,  $F(T, V)$ ,  $G(T, P)$  and  $H(S, P)$ . They are also called *characteristic functions* and are useful in ascertaining the state of stable thermodynamic equilibrium for reversible processes.



The importance of thermodynamic functions can hardly be overestimated in science and engineering. The function  $F$  plays an important role in many chemical reactions that take place isothermally and isochorically. The Gibbs function  $G$  is particularly important in connection with processes involving a change of phase. Sublimation, fusion, vaporisation all take place isothermally and isobarically. For the same reason, it is also of utmost importance in chemical reactions and finds some applications in engineering as well. Since isobaric processes are much more important than isochoric ones in engineering and chemistry, the enthalpy  $H$  is of greatest use in these two branches. Apart from this, one of its most important application in Joule - Thomson effect (*throttling - process*) that paved the way for liquefaction of gases.

### 22.7 Re-derivation of Maxwell's thermodynamic relations

Maxwell's thermodynamic relations will be re-derived in this section by introducing the thermodynamic potentials  $U$ ,  $F$ ,  $G$  and  $H$ .

We have

$$P = - \left( \frac{\partial F}{\partial V} \right)_T \quad (i)$$

$$\text{or, } S = - \left( \frac{\partial F}{\partial T} \right)_V \quad (ii)$$

Differentiating (i) with respect to  $T$  keeping  $V$  constant and (ii) w.r.t.  $V$  keeping  $T$  constant, we obtain

$$\left( \frac{\partial P}{\partial T} \right)_V = - \frac{\partial^2 F}{\partial T \partial V}$$

$$\text{and } \left( \frac{\partial S}{\partial V} \right)_T = - \frac{\partial^2 F}{\partial V \partial T}$$

$$\text{Since } \frac{\partial^2 F}{\partial T \partial V} = \frac{\partial^2 F}{\partial V \partial T}$$



$$\therefore \left( \frac{\partial P}{\partial T} \right)_V = \left( \frac{\partial S}{\partial V} \right)_T \quad \text{(iii)}$$

which is the first of the Maxwell's thermodynamic relations.

Again, we have

$$T = \left( \frac{\partial U}{\partial S} \right)_V \quad \text{(iv)}$$

$$\text{and } P = - \left( \frac{\partial U}{\partial V} \right)_S \quad \text{(v)}$$

Differentiating (iv) w.r.t.  $V$  keeping  $S$  constant and (v) w.r.t.  $S$  keeping  $V$  constant, we get

$$\left( \frac{\partial T}{\partial V} \right)_S = \frac{\partial^2 U}{\partial V \partial S}; \quad - \left( \frac{\partial P}{\partial S} \right)_V = \frac{\partial^2 U}{\partial S \partial V}$$

Since  $\partial U$  is a perfect differential, we can write

$$\left( \frac{\partial T}{\partial V} \right)_S = - \left( \frac{\partial P}{\partial S} \right)_V \quad \text{(vi)}$$

which is the third of Maxwell's relations, Similarly we have

$$T = \left( \frac{\partial H}{\partial S} \right)_P; \quad V = \left( \frac{\partial H}{\partial P} \right)_S \quad \text{(vii a, b)}$$

Differentiating (vii a, b) w.r.t.  $P$  and  $S$  respectively keeping respectively  $S$  and  $P$  constant,

$$\left( \frac{\partial T}{\partial P} \right)_S = \frac{\partial^2 H}{\partial P \partial S}; \quad \left( \frac{\partial V}{\partial S} \right)_P = \frac{\partial^2 H}{\partial S \partial P}$$

Since  $H$  is a function of state, we have  $\frac{\partial^2 H}{\partial P \partial S} = \frac{\partial^2 H}{\partial S \partial P}$

$$\therefore \left( \frac{\partial T}{\partial P} \right)_S = \left( \frac{\partial V}{\partial S} \right)_P \quad \text{(viii)}$$

which is the fourth of Maxwell's relation.

Finally, we have

$$S = - \left( \frac{\partial G}{\partial T} \right)_P \quad (\text{ix})$$

$$V = \left( \frac{\partial G}{\partial P} \right)_T \quad (\text{x})$$

Differentiating (ix) and (x) w.r.t. P and T respectively, keeping respectively T and P constant, we get

$$\left( \frac{\partial S}{\partial P} \right)_T = - \frac{\partial^2 G}{\partial P \partial T}$$

$$\left( \frac{\partial S}{\partial T} \right)_P = - \frac{\partial^2 G}{\partial T \partial P}$$

Again, since G is a function of state,

$$\frac{\partial^2 G}{\partial P \partial T} = \frac{\partial^2 G}{\partial T \partial P}$$

We therefore have

$$- \left( \frac{\partial S}{\partial P} \right)_T = \left( \frac{\partial S}{\partial T} \right)_P \quad (\text{xi})$$

Eqn. (xi) is the second of Maxwell's relations.

**Example 22.4.** Prove the following two relations :

$$(i) \quad U = \left\{ \frac{\partial(F/T)}{\partial(1/T)} \right\}_V$$

$$(ii) \quad F = \left\{ \frac{\partial(G/P)}{\partial(1/P)} \right\}_T$$

The symbols having their usual meanings.

**Soln.** (i) We have,  $F = U - TS$  and also  $S = - (\partial F / \partial T)_V$   
Combining the above two relations, we get

$$U = F - T \left( \frac{\partial F}{\partial T} \right)_v \quad (i)$$

$$= -T^2 \left[ \frac{\partial}{\partial T} \left( \frac{F}{T} \right) \right]_v$$

$$= \left[ \frac{\partial (F/T)}{\partial (1/T)} \right]_v \quad (ii)$$

$$\text{since } \partial (1/T) = -\frac{1}{T^2} dT$$

Eqn. (i) is known as **Gibbs - Helmholtz equation**.

(ii) We have,  $F = G - PV$  and  $V = [\partial G / \partial P]_T$

$$\therefore F = G - P \left( \frac{\partial G}{\partial P} \right)_T$$

$$= -P^2 \left[ \frac{\partial}{\partial P} \left( \frac{G}{P} \right) \right]_T$$

$$= \left[ \frac{\partial (G/P)}{\partial (1/P)} \right]_T$$

$$\text{since } \partial \left( \frac{1}{P} \right) = -\frac{1}{P^2} dP.$$

**Example 22.5.** Compute the slopes of the vaporisation and the sublimation curves for  $H_2O$  at  $0^\circ C$  from the given data and hence show that they are two different lines and not two parts of the same line. Given : latent heat of vaporisation = 607 calorie; latent heat of sublimation = 687 calorie. Assume  $v_g - v_l \approx v_g - v_s = 21 \times 10^4 \text{ c.c.}$

**Soln.** From Clapeyron's equation

$$\frac{dp}{dT} = \frac{L}{T (v_f - v_i)}$$

$$\frac{dp}{dT} = \frac{T (v_f - v_i)}{L}$$



∴ slope of vaporisation curve at 0°C is

$$\begin{aligned}\left(\frac{dp}{dT}\right)_{T=273} &= \frac{607 \times 4.2 \times 10^7}{273 \times 21 \times 10^4} \quad (\because v_g - v_l = 21 \times 10^4 \text{ c.c.}) \\ &= 4.45 \times 10^2 \text{ dyne/cm}^2/\text{°C.} \\ &= 0.337 \text{ mm of Hg/°C.}\end{aligned}$$

slope of the sublimation curve at 0°C is

$$\begin{aligned}\left(\frac{dp}{dT}\right)_{T=273} &= \frac{687 \times 4.2 \times 10^7}{273 \times 21 \times 10^4} \quad (\because v_g - v_l = 21 \times 10^4 \text{ c.c.}) \\ &= 5.03 \times 10^2 \text{ dyne/cm}^2/\text{°C.} \\ &= 0.381 \text{ mm of Hg/°C.}\end{aligned}$$

Sublimation curve is, therefore, steeper than the vaporisation curve at 0°C. So, they are *two different lines* – not two parts of the same line.

**Example 22.6.** Mercury melts at  $-38.87^\circ\text{C}$  at 1 atmospheric pressure, and its density is 13.59 g/c.c. The density of solid is 14.19 g/c.c. The heat of fusion is 2.33 cal/g. What would be the melting point of mercury at 1000 atmosphere?

**Soln.**

$$\text{Given: } dp = (1000 - 1) \text{ atmos} = 999 \times 1.013 \times 10^6 \text{ dynes/cm}^2.$$

$$L = 2.33 \text{ cal} = 2.33 \times 4.2 \times 10^7 \text{ erg.}$$

$$v_l = (1/13.59) \text{ c.c.} = 0.073 \text{ c.c.}$$

$$v_s = (1/14.19) \text{ c.c.} = 0.070 \text{ c.c.}$$

$$T = -38.87^\circ\text{C} = (-38.87 + 273) = 234.13 \text{ K.}$$

From Clapeyron's eqn., the change in the melting point is given by

$$dT = \frac{T(v_l - v_s)}{L} dp$$

$$= \frac{234.13 \times (0.073 - 0.070) \times 999 \times 1.013 \times 10^6}{2.33 \times 4.2 \times 10^7} \text{ K}$$

$$= 7.26 \text{ K} \quad \text{or,} \quad 7.26^\circ\text{C.}$$

$$\therefore \text{ Required melting point} = (-38.87 + 7.26) \\ = -31.6^\circ\text{C.}$$

**Example 22.7.** The vapor-exit tube of a pressure cooker has a radius of 2mm and is closed by a mass of 140 g fitted at its mouth. What is the boiling point of water inside the cooker? Given : latent heat of vaporisation of water = 540 cal and the specific volume of water vapour = 1674 c.c.

**Soln.**

The vapour from inside the cooker will be forced out of the exit tube when the vapour pressure exceeds the pressure due to the imposed mass.

According to the problem

$$\begin{aligned} \Delta p &= p_2 - p_1 \\ &= \frac{140 \times 980}{\pi \times (0.2)^2} \\ &= \frac{140 \times 980}{\pi \times 0.04} \\ &= 1.092 \times 10^6 \text{ dyne/cm}^2. \end{aligned}$$

From the relation

$$\frac{dp}{dT} = \frac{L}{T(v_f - v_i)}$$

$$\Delta p = \int_{p_1}^{p_2} dp = \int_{T_1}^{T_2} \frac{L}{T(v_f - v_i)} dT$$

$$\text{or,} \quad p_2 - p_1 = \int_{T_1}^{T_2} \frac{L}{T(v_f - v_i)} dT$$

Assuming that the latent heat remains constant with change in boiling point (which however is not exactly true)

$$p_2 - p_1 = \frac{L}{v_f - v_i} \ln \frac{T_2}{T_1}$$

$$\text{or, } \ln \frac{T_2}{T_1} = \frac{(p_2 - p_1)(v_f - v_i)}{L}$$

$$= \frac{1.092 \times 10^6 \times (1674 - 1)}{540 \times 4.2 \times 10^7}$$

$$= \frac{1.092 \times 1673}{540 \times 42} = 0.08$$

$$\therefore \frac{T_2}{T_1} = 1.084.$$

$$\therefore T_2 = T_1 \times 1.084 = 373 \times 1.084 = 404 \text{ K} = 131^\circ\text{C}.$$

$\therefore$  Boiling point of water inside the cooker =  $131^\circ\text{C}$ .

## EXERCISES

- [1] Derive Clausius Clapeyron's latent heat equation

$$\frac{dp}{dT} = \frac{L}{T(v_2 - v_1)}$$

where the symbols have their usual meanings.

- [2] Derive Maxwell thermodynamic relations.

- [3] Show that

$$(i) \left( \frac{\partial S}{\partial V} \right)_T = \left( \frac{\partial p}{\partial T} \right)_V$$

$$(ii) \left( \frac{\partial H}{\partial V} \right)_T = T \left( \frac{\partial p}{\partial T} \right)_V$$



$$(iii) \left( \frac{\partial H}{\partial p} \right)_T = -T \left( \frac{\partial V}{\partial T} \right)_p$$

$$(iv) \left( \frac{\partial T}{\partial V} \right)_S = -T \left( \frac{\partial p}{\partial S} \right)_V$$

$$(v) \left( \frac{\partial T}{\partial p} \right)_S = T \left( \frac{\partial V}{\partial H} \right)_p$$

$$(vi) \left( \frac{\partial T}{\partial p} \right)_V \left( \frac{\partial S}{\partial V} \right)_p - \left( \frac{\partial T}{\partial V} \right)_p \left( \frac{\partial S}{\partial p} \right)_V = 1$$

$$\text{and } (vii) \left( \frac{\partial p}{\partial T} \right)_S \left( \frac{\partial V}{\partial S} \right)_T - \left( \frac{\partial p}{\partial S} \right)_T \left( \frac{\partial V}{\partial T} \right)_S = 1$$

- [4] Show from the considerations of Maxwell's thermodynamic relations that

- (i) for a perfect gas

$$C_p - C_v = R$$

- and (ii) for a Van der Waals gas

$$C_p - C_v = R \left( 1 + \frac{2a}{RTV} \right)$$

- [5] Prove thermodynamically the following relations :

$$(i) C_p - C_v = \left[ \left( \frac{\partial U}{\partial V} \right)_T + P \right] \left( \frac{\partial V}{\partial T} \right)_p$$

$$(ii) \left( \frac{\partial U}{\partial V} \right)_T = T \left( \frac{\partial P}{\partial T} \right)_V - P$$

- [6] Show that

$$\frac{\left( \frac{\partial P}{\partial T} \right)_S}{\left( \frac{\partial P}{\partial T} \right)_V} = \frac{\gamma}{\gamma - 1}$$

- [7] Devise Maxwell's thermodynamic relations and hence prove that

$$C_p - C_v = T \left( \frac{\partial P}{\partial T} \right)_V \left( \frac{\partial V}{\partial T} \right)_P$$

- [8] Explain clearly why  $F$  defined by  $F = U - TS$  and  $G$  defined by  $G = U + pV - TS$  are called energies – Helmholtz and Gibbs free energies respectively. Show that the internal energy is made up of two parts – the free energy and the latent energy or available and unavailable energy for useful work.

- [9] Show that the entropy and pressure of a fixed  $T$  and a fixed  $V$  system are given by

$$S = - \left( \frac{\partial F}{\partial T} \right)_V ; \quad p = - \left( \frac{\partial F}{\partial V} \right)_T$$

- [10] Prove that

$$(i) \quad S = - \left( \frac{\partial G}{\partial T} \right)_P \quad \text{and} \quad V = \left( \frac{\partial G}{\partial P} \right)_T$$

where the symbols have their usual meanings.

- [11] What is meant by the first order phase transition? Establish the Clapeyron equation for a system which can have a first order phase transition.

- [12] Show that the enthalpy  $H = \left[ \frac{\partial(G/T)}{\partial(1/T)} \right]_V$

where  $G$  is the Gibbs energy.

- [13] Define the thermodynamic functions  $H$ ,  $F$  and  $G$  for a system with constant composition. Express  $G = G(T, P)$  and find  $S$ ,  $H$ ,  $V$ ,  $F$  and  $U$  in terms of  $G$  and its derivatives. Hence show that for a reversible isothermal change

$$\Delta G = \Delta H + T \frac{\partial}{\partial T} (\Delta G) \Big|_P$$

- [14] 1 c.c. of water occupies 1674 c.c. of space at a temperature of 373.16 K. Find the latent heat of vaporisation if  $\frac{dp}{dT} = 27.12 \text{ torr/K}$  (1 torr = 1 mm of Hg). [539 cal].

- [15] The melting point of lead under normal pressure is 600K. Calculate the change in its value when the pressure is increased to 100 atmos.



Given : density of lead in solid and liquid phases is 11.01 g/c.c. and 10.65 g/c.c. respectively, the latent heat of fusion is  $24.5 \times 10^7$  erg/g. [ $0.75^\circ\text{C}$ ].

- [16] Calculate under what pressure ice would freeze at  $-1^\circ\text{C}$ . Change of specific volume when 1g of water at  $0^\circ\text{C}$  freezes into ice at  $0^\circ\text{C}$  is 0.091 c.c. Given  $J = 4.2 \times 10^7$  erg.  $\text{cal}^{-1}$ ; 1 atmos =  $10^6$  dyne  $\text{cm}^{-2}$ ; latent heat of fusion of ice = 80 cal  $\text{g}^{-1}$ . [ $135.2$  atmos.]
- [17] Calculate the boiling point of water at an altitude of 300 m. Assume that boiling point of water at sea level is  $100^\circ\text{C}$ . Given :  $T = 300\text{K}$ ;  $R = 2$  cal. $\text{mol}^{-1}\text{K}^{-1}$ ;  $g = 9.8$   $\text{ms}^{-2}$ ; mol. wt. of air = 29 g  $\text{mol}^{-1}$  and latent heat of vaporisation of water at STP is 540 cal.  $\text{g}^{-1}$ . [ $\sim 99^\circ\text{C}$ ].
- [18] Calculate the change of melting point of naphthalene per atmospheric change of pressure, given melting point =  $80^\circ\text{C}$  and latent heat = 35.5 cal/g. Density of solid = 1.145 g/c.c. and density of liquid = 0.981 g/c.c. [ $0.035^\circ\text{C/atm.}$ ]
- [19] The density of iodine at the boiling point ( $185.3^\circ\text{C}$ ) is 3.71 g/c.c. and the latent heat of vaporisation is 40.9 cal/g. If the boiling point changes by  $1^\circ\text{C}$  for a change of pressure of 17 mm of mercury, what is the specific volume of the vapour? [ $161.87$  c.c./g].
- [20] One gramme of water vapour at  $100^\circ\text{C}$  and at atmospheric pressure occupies a volume of 1640 c.c. Latent heat = 536 cal/g. Prove that the vapour pressure of water at  $99^\circ\text{C}$  is about 733 mm of Hg.
- [21] The specific volumes of water and steam at  $100^\circ\text{C}$  and 760 mm of Hg pressure are 1 c.c. and 1601 c.c. respectively.  $L = 536$  cal/g. Find the change in the boiling point of water due to a pressure change of 1 cm of mercury. [ $0.35^\circ\text{C}$ ].
- [22] Given the following data regarding carbon tetrachloride : boiling point =  $77^\circ\text{C}$  at atmosphere;  $L = 46$  cal/g; density of liquid = 1.6 g/c.c.,  $dp/dT = 23$  mm of mercury/ $^\circ\text{C}$ . Find the specific volume of  $\text{CCl}_4$  - vapour at the boiling point. [ $180$  c.c.]
- [23] Prove that

$$(a) \left( \frac{\partial T}{\partial P} \right)_V \left( \frac{\partial S}{\partial V} \right)_P - \left( \frac{\partial T}{\partial V} \right)_P \left( \frac{\partial S}{\partial P} \right)_V = 1$$

$$(b) \left( \frac{\partial P}{\partial T} \right)_S \left( \frac{\partial V}{\partial S} \right)_T - \left( \frac{\partial P}{\partial S} \right)_T \left( \frac{\partial V}{\partial T} \right)_S = 1$$



## CHAPTER - XXIII

## HYGROMETRY

*Importance of hygrometry – Humidity : absolute and relative humidity – Dew-point – Sensation of dampness and dryness – Hygrometers – Humidity control – Solved problems – Exercises.*

**23.1 Importance of Hygrometry**

Hygrometry is that part of physics which deals with the measurement of the amount of water vapour present in a given volume of air *i.e.*, *humidity* of air. The amount of water vapour present in the atmosphere may be considered to be exceedingly small in comparison with the amount of oxygen, nitrogen, etc., present in it. Yet the importance of the presence of water vapour in the atmosphere cannot be ignored. Humidity affects weather, hence regular observation of the changes in atmospheric humidity helps us to predict weather. These changes sometimes lead to the condensation of water vapour in air or to the evaporation of suspended droplets of water. These in turn, affect air movement, formation of clouds and precipitation, *i.e.*, formation of rain, hoar frost, and snow etc.

A knowledge of humidity is useful not only for meteorologists but also for many other purposes such as air-conditioning of buildings. Strict control of humidity is essential for spinning mills. Humidity also plays an important role in the artificial seasoning of wood, manufacture of tobacco and in cold storage.

**23.2 Humidity : Absolute humidity and Relative humidity**

By the term *humidity* is meant the dampness or wetness of the atmosphere. It increases or decreases accordingly as the water vapour content of the atmosphere increases or decreases. *Absolute humidity* of air is measured by the quantity of water vapour present in one cubic metre of atmospheric air. In actual practice, the knowledge of absolute humidity is not very important. What is important is the saturation condition of air, *i.e.*, how far it is removed from the saturation limit. The degree of saturation of air

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determines its relative humidity. The *relative humidity* or *hygrometry state* of air is defined to be the ratio of the quantity of water vapour actually present in a certain volume of air at a particular temperature to the quantity of water vapour that would be required to saturate the same volume of air at the same temperature. So

$$R.H. = \frac{\text{mass of water vapour actually present in any volume of air at } \theta^{\circ}\text{C}}{\text{mass of water vapour necessary to saturate the same volume of air at } \theta^{\circ}\text{C}}$$

Suppose that the mass of water vapour actually present in a given volume of air be  $m$  gm and the mass of water vapour required to saturate the same volume of air at the same temperature be  $M$  gm. Then the relative humidity (R.H.) expressed as a percentage is given by

$$R. H. = \frac{m}{M} \times 100$$

Since the mass of a given volume of any substance is proportional to its density, hence densities can be substituted instead of masses. Relative humidity can therefore be defined as

$$R. H. = \frac{\text{density of water vapour present in air}}{\text{density of saturated water vapour at the same temperature}}$$

Further, assuming that Boyle's law holds for water vapour right up to saturation, the density of water vapour is proportional to its pressure. Hence the above expression for relative humidity may be written as

$$R.H. = \frac{\text{pressure of water vapour actually present in the air at } \theta^{\circ}\text{C}}{\text{pressure of water vapour necessary to saturate the air at } \theta^{\circ}\text{C}}$$

### 23.3 Dew-point

If moist air is cooled progressively, a stage is reached when the moisture it contains is sufficient to saturate it. Any further cooling will make the water vapour condense in the form of dew on a cold surface. This temperature is called the *dew-point* and is defined as *the temperature at which the amount of water vapour actually present in air is sufficient to saturate it.*



It can be seen that the partial vapour pressure at room temperature becomes the saturation vapour pressure at dew-point. If  $f$  denotes the saturation vapour pressure at the dew-point and  $F$  the saturation vapour pressure at the temperature of the air (atmospheric temperature), then the relative humidity may be written as

$$\text{R.H.} = \frac{f}{F} \times 100$$

From the above considerations, when we say that the relative humidity is 64%, we mean that the amount of water vapour actually present in a given volume of atmosphere is 0.64 times of that would be required to saturate the same volume of atmosphere at that temperature.

### 23.4 Sensation of dampness and dryness

The feeling of dampness or dryness depends upon the factor of relative humidity. For example, two rooms have the same temperature but different humidities. Then the room having greater relative humidity will give a sensation of dampness more than that in the other room. Dampness or dryness of the air is judged by the rate at which evaporation goes on and this, in turn, depends on how far the air is from saturation state *i.e.*, how much more vapour it can absorb. A low relative humidity means that the air can take up more moisture and this encourages evaporation and the air will appear dry. A high value of relative humidity, on the other hand, discourages evaporation and the air will appear damp. This is why wet clothes are found to dry up more quickly if relative humidity of the atmosphere is low.

The rate of evaporation of water from our skin depends upon the relative humidity of the surrounding air or atmosphere. If the atmosphere is saturated with water vapour, then the evaporation is slow and we experience a feeling of mugginess as in a hot overcrowded room or on a sultry day. However, if the atmosphere is unsaturated with water vapour, then evaporation is fast, resulting in the dryness of our skins. Hence for the comfort and working efficiency of human beings, it is essential to maintain the relative humidity between certain reasonable limits.



**Example 23.1.** Air at  $20^{\circ}\text{C}$  contains 0.012 gm of water vapour per litre. What is the relative humidity? The saturated vapour pressure of water at  $20^{\circ}\text{C}$  is 1.74 cm of mercury and a litre of steam at  $100^{\circ}\text{C}$  and 76 cm of mercury weighs 0.606 gm.

**Soln.**

Saturated vapour pressure (S.V.P.) at  $20^{\circ}\text{C}$  is given. So if we can find partial or actual vapour pressure at  $20^{\circ}\text{C}$ , relative humidity can be determined.

Applying the general gas equation  $PV = mRT$  to conditions at  $20^{\circ}\text{C}$  and  $100^{\circ}\text{C}$ , we get

$$p \times 1 = 0.012 \times r \times 273$$

$$76 \times 1 = 0.606 \times r \times 373$$

$$\therefore p = 76 \times \frac{0.012}{0.606} \times \frac{273}{373}$$

$$= 1.18 \text{ cm of mercury.}$$

$$\therefore \text{relative humidity at } 20^{\circ}\text{C}$$

$$= \frac{p}{P} = \frac{1.18}{1.74} = 0.68 \text{ or } 68\%.$$

**Example 23.2.** The temperature of air in a closed space is observed to be  $15^{\circ}\text{C}$  and the dew point  $8^{\circ}\text{C}$ . If the temperature falls to  $10^{\circ}\text{C}$ , how are (i) the dew point and (ii) the pressure of the aqueous vapour in the air affected? Pressure of aqueous vapour at  $7^{\circ}\text{C} = 7.48 \text{ mm}$  and at  $8^{\circ}\text{C} = 8.02 \text{ mm}$ .

**Soln.**

Actual vapour pressure at  $15^{\circ}\text{C}$  is equal to the saturation vapour pressure at dew-point i.e., at  $8^{\circ}\text{C}$ , the value of which is given as 8.02 mm.

$$\therefore p = 8.02 \text{ mm}$$

$$T = 273 + 15 = 288 \text{ K}$$

$$T' = 273 + 10 = 273 \text{ K}$$

$$p' = ?$$

Applying Charles' law

$$\frac{p'}{T'} = \frac{p}{T}$$

$$p' = p \cdot \frac{T'}{T} = \frac{8.02 \times 283}{288} = 7.88 \text{ mm of Hg.}$$

- (ii)  $\therefore$  the new vapour pressure at  $10^\circ\text{C} = 7.88 \text{ mm.}$   
 (i) the new dew-point will be that temperature at which the saturation vapour pressure would be  $7.88 \text{ mm.}$

Now S.V.P. at  $7^\circ\text{C} = 7.48 \text{ mm}$

S.V.P. at  $8^\circ\text{C} = 8.02 \text{ mm}$

Change in S.V.P. for  $1^\circ\text{C} = (8.02 - 7.48) = 0.54 \text{ mm}$

We have to find the change in temperature corresponding to a change of S.V.P. of  $(7.88 - 7.48) = 0.4 \text{ mm.}$

Now change in S.V.P. of  $0.54 \text{ mm} = \text{change in temp. of } 1^\circ\text{C.}$

$$\therefore \text{Change in S.V.P. of } (7.88 - 7.48) = 0.4 \text{ mm} = \left( \frac{1}{0.54} \times 0.4 \right)^\circ\text{C}$$

$$= 0.74^\circ\text{C.}$$

$$\therefore \text{the new dew-point} = 7 + 0.74 = 7.74 \text{ mm.}$$

**Example 23.3.** The temperature of air on a certain day is  $29^\circ\text{C}$  and the relative humidity is  $60\%$ . What fraction of the mass of water vapour in the air would condense if the temperature decreases to  $16^\circ\text{C}$ ? (S.V.P. of water at  $29^\circ\text{C} = 31.6 \text{ mm}$ ; at  $16^\circ\text{C} = 13.6 \text{ mm}$ ).

**Soln.**

Actual vapour pressure at  $29^\circ\text{C}$

$$= 0.6 \times \text{S.V.P. at } 29^\circ\text{C}$$

$$= 0.6 \times 31.6 = 18.96 \text{ mm.}$$

Since maximum vapour pressure at  $16^\circ\text{C}$  is  $13.6 \text{ mm}$ , it means that due to condensation of some water vapour on account of fall in temperature, reduction in vapour pressure  $= 18.96 - 13.6 = 5.36 \text{ mm.}$

Assuming that vapour pressure is directly proportional to mass of water vapour present, we get

$$\frac{\text{reduction in mass of water vapour}}{\text{actual mass of water vapour at } 29^\circ\text{C}} = \frac{\text{reduction in vapour pressure}}{\text{actual V.P. at } 29^\circ\text{C}}$$



$$= \frac{5.36}{18.96} = 0.283 \text{ or } 28.3\%$$

### 23.5 Hygrometers

Hygrometers are instruments used for measuring relative humidity. The hygrometers that will be discussed are

- (i) Chemical hygrometer
- (ii) Dew-point hygrometer (Regnault's hygrometer)
- (iii) Wet and Dry bulb hygrometer
- (iv) Hair hygrometer.

#### (i) Chemical hygrometer

In this hygrometer, the amount of water vapour ( $m$ ) actually present in a given volume of air and the amount ( $M$ ) required to saturate it are found. The relative humidity is then calculated by using the relation

$$\text{Relative humidity} = \frac{m}{M}$$

Fig. 23.1 shows a chemical hygrometer. Water is allowed to flow from the aspirator at about 500 c.c./minute. This draws air through the drying tubes C and D which contain either pumic stone soaked in conc. sulphuric acid ( $\text{H}_2\text{SO}_4$ ) or calcium chloride ( $\text{CaCl}_2$ ).

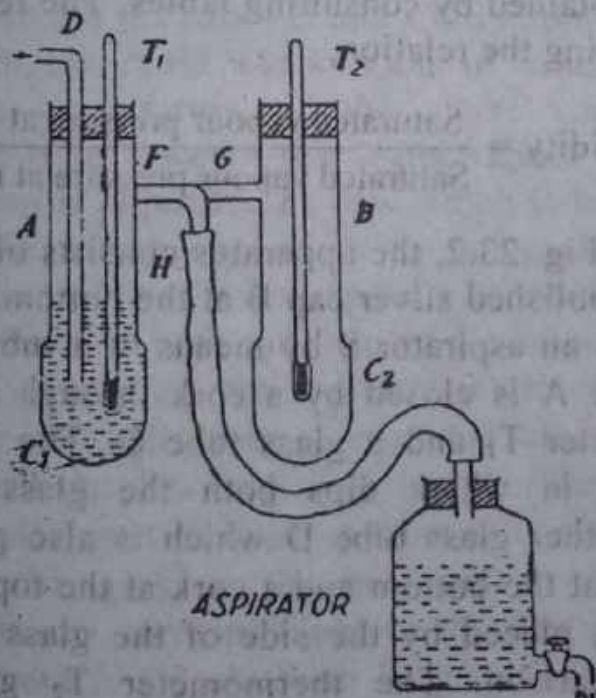


Fig. 23.1



Temperatures are measured by thermometers  $T_1$  and  $T_2$ . When a known volume of air has passed, the tubes C and D are disconnected and weighed. The increase in weight gives the weight of water vapour present in the given volume of air at the room temperature. The experiment is repeated with the same volume of air which has been saturated by passing it through water. The tubes C and D are weighed again and the increase in their weight noted. The ratio of the first increase in weight to the second gives the relative humidity.

During the experiment, all junctions are carefully sealed in order to avoid leakage and the trap B, which contains a suitable drying agent serves to prevent the water vapours passing from the aspirator A to the dry tubes C and D.

The method is direct and accurate but is slow and tedious. It gives the mean value of the relative humidity of air over the period of the experiment. Normally, this apparatus is used for standardizing other hygrometers.

### (ii) Regnault's Dew Point Hygrometer

In Regnault's dew-point hygrometer, commonly known as *Regnault's hygrometer*, the dew-point is found first and then the values of saturated vapour pressure at this dew-point and the room temperature are obtained by consulting tables. The relative humidity is calculated by using the relation,

$$\text{Relative humidity} = \frac{\text{Saturated vapour pressure at dew-point}}{\text{Saturated vapour pressure at room temp.}}$$

As shown in Fig. 23.2, the apparatus consists of a glass tube A fitted with a thin-polished silver cap B at the bottom and a side tube COE connected to an aspirator S by means of a rubber tubing. The mouth of the tube A is closed by a cork through which passes a delicate thermometer  $T_1$  and a glass tube G. The silver cap B is filled with ether in which dips both the glass tube and the thermometer. Another glass tube D which is also provided with a similar silver cap at the bottom and a cork at the top and carrying a thermometer  $T_2$  is placed by the side of the glass tube A for the purpose of comparison. The thermometer  $T_2$  gives the room temperature of the air. The aspirator is initially filled with water.

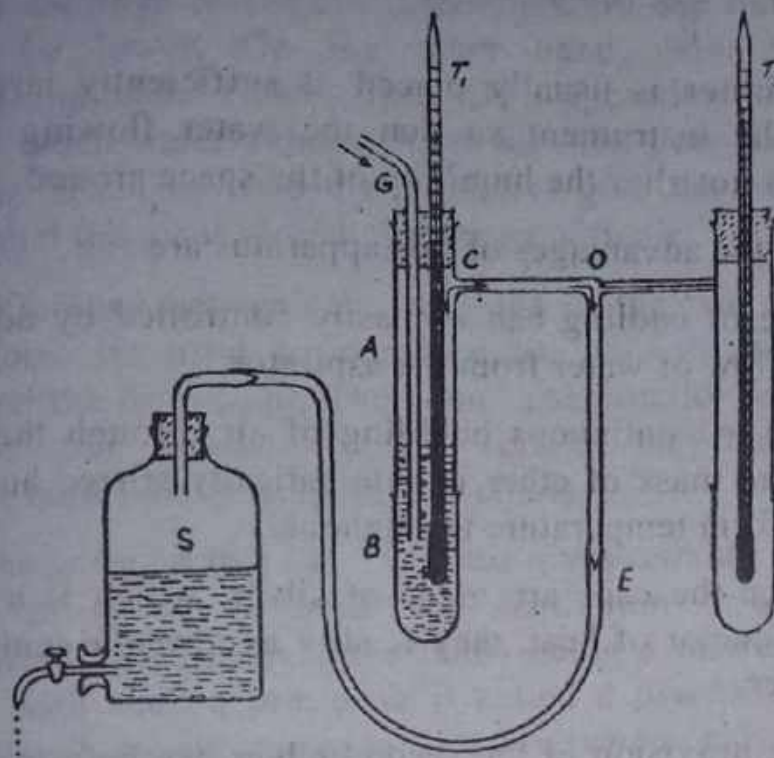


Fig. 23.2

To start the experiment, the aspirator is opened at the bottom. When water runs out slowly from the aspirator, a current of air is drawn in through the tubes G and COE bubbling through ether. The bubbling causes rapid evaporation of ether inside and steadily lowers the temperature of the silver cap B. Ultimately the temperature is so lowered that water vapour in the air surrounding the silver cap is deposited as dew which is indicated by the loss of lustre of the silver cap of A as compared to that of D. The temperature at which the dew appears is noted from the thermometer  $T_1$  from a distance with the help of a telescope. After cooling B a little below dew-point, the aspirator is closed thereby stopping the flow of air through ether. Hence the ether begins to warm up gradually and the dew begins to vanish. The temperature at which the film of dew just disappears from B is also noted. The mean of these two readings gives the dew-point at the time. The saturated vapour pressures corresponding to the dew-point and atmospheric temperature are found out from Regnault's chart and substituted in the equation.



$$\text{R.H.} = \frac{f}{F} \times 100 \%$$

The aspirator is usually placed at sufficiently large distance away from the instrument so that the water flowing out of the aspirator does not alter the humidity of the space around.

Some of the advantages of this apparatus are :

- (i) Rate of cooling can be easily controlled by adjusting the outflow of water from the aspirator.
- (ii) Due to continuous bubbling of air through the ether, the entire mass of ether is automatically stirred and it attains uniform temperature throughout.
- (iii) Since the caps are made of silver, which is a very good conductor of heat, they readily assume the temperature of ether.
- (iv) The provision of the second silver cap helps to detect the appearance and disappearance of dew on B more accurately. This is achieved by comparison of the brightness of the two silver caps.

### (iii) Wet and dry bulb hygrometer – the psychrometer.

Because of its simplicity of operation, this instrument is extensively used in meteorology and industry. The apparatus consists of two identical mercury thermometers mounted vertically side by side on the same framework which can be hung up against a vertical wall. The bulb of one of the thermometers is completely exposed to air and is called the *dry bulb*. This measures the temperature of the surrounding air. The bulb of the other thermometer is covered with muslin wick which is kept moist by dipping its free end into water contained in a small vessel. This is the *wet bulb* thermometer. Evaporation of water from the muslin takes place continuously at a rate depending upon the relative humidity of the atmosphere. This keeps the temperature of the wet bulb always lower than that of the dry bulb. The difference between the two temperatures indicates the humidity condition of the air. Lesser the humidity, the quicker is the evaporation and the more rapid is the cooling, hence lower the reading of the wet thermometer.



In other words, the drier the air, the greater is the difference between temperature readings of the two thermometers and hence the dew-point will be lower. On the other hand, less difference of temperature indicates slower rate of evaporation owing to the presence of much water vapour in the air. The dew-point in such a case is high. When the two thermometers give the same readings, this means that the air is already saturated with air.

The difference between the readings of the two thermometers can, therefore, be used to compute the values of the relative humidity and the dew-point. The actual computation is made easier because of the availability of certain empirical relations, hygrometric tables and psychrometric charts.

It has been found that the apparatus gives reliable readings only if suitable provision is made for ventilation – the necessary minimum wind velocity over the bulbs being 3 metres/second. An instrument with such a provision is called a *psychrometer*. In the case of one such psychrometer known as *Assmann psychrometer* the movement of air over the bulbs is maintained by clock-driven fans.

#### 4. Hair hygrometer

The hair hygrometer is based on the principle that hairs and some other fibres increase in length with increase in humidity. A fine hair *H* (Fig. 23.3) is fixed at one end *E* and passes over pulleys to

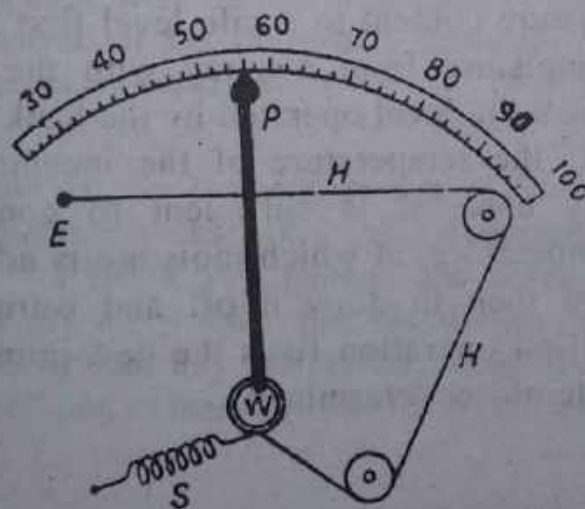


Fig. 23.3

a grooved wheel W. A spring S acting on W in the opposite direction keeps the hair taut. By this arrangement or by a lever system, a small extension of hair caused by the change in the humidity of the air is magnified to operate a pointer P moving over a scale which is graduated directly in terms of the relative humidity.

The special advantages of the instrument are that it is neat and compact, gives continuous readings and can operate at temperatures below  $0^{\circ}\text{C}$ . It is particularly suitable for use in cold storage or in cold climates and can be fitted with electrical contacts to operate an automatic control system. However, it must be checked from time to time since the hair acquires a permanent set when kept under tension, thus causing the instrument to over-read. Also great care must be taken to keep the hair free from grease etc. Otherwise the presence of grease will interfere with the absorption of moisture and render the readings of the instrument unreliable.

### 23.6 Humidity control

In many laboratory or industrial operations, it is necessary to keep the humidity at a certain desired level. Addition of moisture, when required, is usually accomplished by spraying water. Removal of moisture from incoming air is achieved either by cooling the air by passing it over pipes of brine fed from a refrigerator, or by using absorbing agents like silica gel. Silica gel can be subsequently freed from its moisture by strong heating. Often the most convenient method of keeping the relative humidity at or near a desired level is to reduce the moisture content to a safe level first and then control the addition of moisture from a spray with the help of a hair hygrometer in which the level operated by the hank of hair opens or closes a valve. If the temperature of the incoming air is to be controlled in any case, it is sufficient to cool the air to a predetermined temperature, at which moisture is added if necessary to saturate it, and then to draw it off and warm it to its final temperature. The first operation fixes the dew-point of the air, and its final state is, therefore, determined.

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## EXERCISES

- [1] Explain the terms "absolute humidity", "humidity" and "dew-point". How are these quantities related?
- [2] What is dew-point? How is it related to humidity? Describe Regnault's dew-point hygrometer.
- [3] Define relative humidity. Describe a wet and dry bulb hygrometer. How would you determine the relative humidity with its help.
- [4] Describe a chemical hygrometer and explain how you can use it to measure relative humidity. What are its main drawbacks?
- [5] Describe a hair hygrometer. How you can measure the relative humidity with it? Discuss the industrial applications of hygrometry.
- [6] The relative humidity in a closed room at  $15^{\circ}\text{C}$  is 60 per cent. If the temperature rises to  $20^{\circ}\text{C}$ , what will the relative humidity become? On what assumptions is your calculation based? Saturation vapour pressure of water vapour at  $15^{\circ}\text{C} = 12.67$  mm of mercury; at  $20^{\circ}\text{C} = 17.36$  mm. [45 per cent].
- [7] The dew-point on a certain day was  $12^{\circ}\text{C}$  while actual temperature was  $16^{\circ}\text{C}$ . If the saturated aqueous vapour pressure at  $12^{\circ}\text{C}$  and  $16^{\circ}\text{C}$  be 1.064 cm and 1.364 cm of mercury, find the relative humidity. [78 per cent].
- [8] The temperature of air on a certain day is  $23^{\circ}\text{C}$  and the relative humidity is 60%. What fraction of the mass of water vapour in the air would condense, if the temperature decreases to  $10^{\circ}\text{C}$ ? Saturated pressure of water vapour at  $23^{\circ}\text{C} = 21.1$  mm; at  $10^{\circ}\text{C} = 9.2$  mm. [0.27].
- [9] On a certain day the temperature was  $25^{\circ}\text{C}$ , pressure 755 mm of mercury and relative humidity 70%. Assuming density of dry air at N.T.P. =  $1.293 \times 10^{-3}$  gm/c.c. and density of water vapour relative to dry air = 0.62, find the density of moist air of the atmosphere on that day. Saturation vapour pressure of water at  $25^{\circ}\text{C} = 23.67$  mm of mercury. [0.001174 gm/c.c.].



## CHAPTER XXIV

## COMBINATION OF LENSES

*Deviation produced by a thin lens-Equivalent lens and equivalent focal length-Equivalent focal length of two lenses in contact-Equivalent focal length of two thin coaxial lenses separated by a finite distance-Restriction in equivalence-Power of a lens-Principal points and principal planes-Nodal points-Cardinal points-Solved problems-Exercises.*

## 24.1 Some formulae used in geometrical optics

Some formulae which represent refraction of light at single spherical surfaces and double spherical surfaces (lens) will be mentioned below. The students are already familiar with their derivation and as such they will not be derived here.

**Refraction at a single spherical surface :**

If light from a medium of refractive index  $\mu_1$  is incident on a spherical surface and is refracted into a medium of refractive index  $\mu_2$ , then the relation between the object distance  $u$ , the image distance  $v$  and the radius of curvature  $R$  of the spherical surface is given by

$$\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 \cos r - \mu_1 \cos i}{R} \quad (24.1)$$

where  $i$  is the angle of incidence and  $r$  is the angle of refraction.

If the angle of inclination  $\theta$  of the incident ray with the axis of the spherical surface is so small that  $\sin \theta = \theta$ , then both the angles of incidence and refraction are small, so that  $\cos i = \cos r \simeq 1$ . Eqn. (24.1) then becomes,

$$\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R} \quad (24.2)$$

If the first medium is air and the refractive index of the second medium is  $\mu$ , then

$$\frac{\mu}{v} - \frac{1}{u} = \frac{\mu - 1}{R} \quad (24.3)$$

However, if the angles of incidence and refraction are not small, then the correct relation is given by eqn. (24.1).

*Eqns. 24.1 – 24.3 express the relation governing refraction (i) at a convex spherical surface forming a real image of a real object, (ii) at a convex spherical surface forming a virtual image of a real object and (iii) at a concave spherical surface.*

### **Refraction at a lens :**

The relation governing refraction at a lens, placed is given by the equation

$$\frac{1}{v} - \frac{1}{u} = (\mu - 1) \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \quad (24.4)$$

where  $u$  = object distance;  $v$  = image distance;

$\mu$  = refractive index of the material of the lens,

$R_1$  = radius of curvature of the first surface of the lens, and

$R_2$  = radius of curvature of the second surface.

Another relation is given by the equation

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f} \quad (24.5)$$

where  $f$  is the focal length of the lens.

Eqn. (24.4) is usually referred to as the *lens maker's equation* while eqn. (24.5) is the *lens equation* and is applicable to a convex or a concave lens.

*Eqns. (24.4) and (24.5) are valid only for a thin lens and for paraxial rays i.e., when both the thickness and aperture of the lens are negligibly small compared to other measurable quantities involved i.e.,  $u$ ,  $v$ ,  $R_1$  and  $R_2$ .*

### **The first and second focal length of a lens :**

The second focal length or simply the focal length of a lens is the image distance when the object is at infinity, i.e.,  $v = f$  when  $u = \infty$ . Then from eqn. (24.4), we get

$$\frac{1}{f_2} = \frac{1}{f} = (\mu - 1) \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \quad (24.6)$$

The first focal length is the object distance when the image is at infinity i.e.,  $u = f_1$ , when  $v = \infty$ . Then from eqn. (24.4), we get

$$\frac{1}{f_1} = -(\mu - 1) \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \quad (24.7)$$



As can be seen from eqns. (24.6) and (24.7), when the medium on both sides of the lens is the same (as in the case when the lens is placed in air), the two focal lengths are equal in magnitude but opposite in sign.

*For a convex lens  $f_2$  is positive and  $f_1$  is negative.*

*For a concave lens  $f_1$  is positive while  $f_2$  is negative.*

While solving for numerical problems, proper signs for  $u$ ,  $v$ ,  $R_1$  and  $R_2$  are to be used.

## 24.2 The sign convention

The derivation of various formulae in geometrical optics are based upon measurement of various distances and angles from a suitably chosen axis and origin in every optical system. The distance of the object and the image from the refracting surface is a vector quantity and these distances must be represented with proper signs. It is, therefore, essential to adopt a convention of signs to ensure consistency in the derivations and use of various formulae. The following set of convention which agree with the usual convention of *cartesian system of co-ordinates* used in co-ordinate geometry as shown in Fig. 24.1 will be followed in this book.

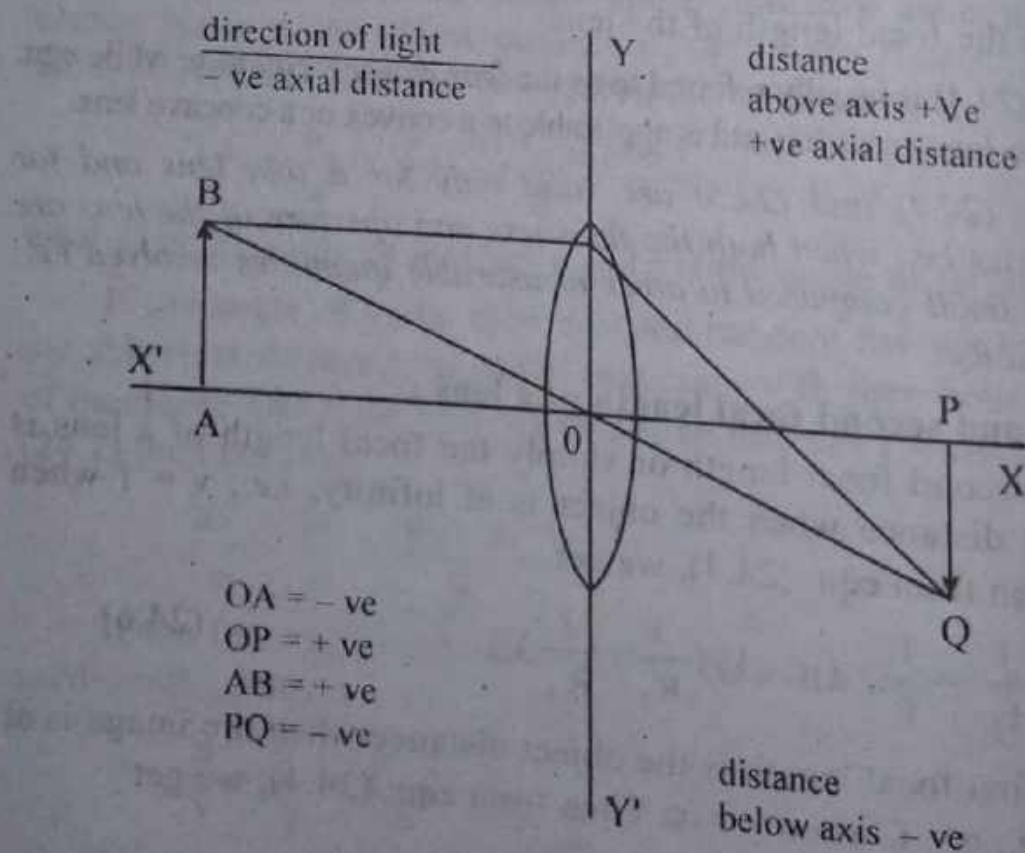


Fig. 24.1



- (i) All figures are to be drawn with the incident light travelling from left to right.
- (ii) The centre of the refracting surface is at the origin  $O$  and its axis is along  $XX'$ .
- (iii) All distances should be measured from the centre of the refracting surface i.e.,  $O$  in Fig. 24.1. Distances measured to the right are to be regarded as positive while those measured to the left are to be regarded as negative.
- (iv) Distances measured upward and normal to the  $x$ -axis are taken as positive, while downward normal distances are taken as negative.

In Fig. 24.1  $AB$  represents an object, its distance  $OA$  is negative. If  $PQ$  represents the image, the image distance  $OP$  is positive. The size of the object  $AB$  is positive while the size of the image  $PQ$  is negative.

#### convention for angles

(i) If an anticlockwise turn through the slope angle that a ray makes with the axis brings a ruler from the principal axis direction into coincidence with that of the ray, then the slope angle is to be considered positive.

(ii) All angles of incidence and refraction are to be considered positive when a ruler from a normal direction at the point of incidence on the surface must be rotated in the anticlockwise direction through the angle to bring it into coincidence with the ray.

It is advisable to memorize the convention of signs at this very stage which would make it easier to grasp and to derive consistent relations in this branch of optics.

### 24.3 Deviation produced by a thin lens

A lens might be considered as composed of a central glass plate over which a number of truncated prisms of different refracting angle are placed with their bases turned either towards the axis (in case of a convex lens) or away from the axis (in case of a concave lens). If a beam of rays parallel to the axis be incident on such a combination, then all the emergent rays will be deviated towards the

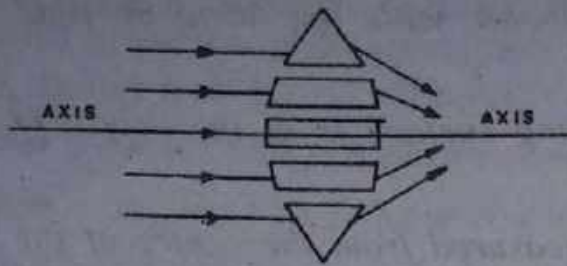


Fig. 24.2

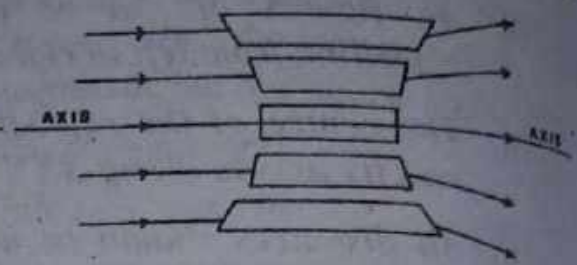


Fig. 24.3

bases of the prisms. As the bases of all the prisms are directed towards the axis in case of a convex lens, the emergent rays are deviated towards the axis thus making a convex lens *converging* (Fig. 24.2) In the case of a concave lens all the bases are turned away from the axis; hence the emergent rays are deviated away from the axis, thus making a concave lens *diverging* (Fig. 24.3).

Whether the rays are deviated towards the axis or away from the axis, the function of a lens is to deviate the incident rays of light. It is, therefore, necessary to find the deviation produced by a particular portion of the lens. In the following discussion it will be shown that *the deviation of any ray due to refraction through a thin lens is independent of the angle of incidence and depends upon (i) the height of the point above the principal axis where the deviation occurs and (ii) value of the focal length of the lens.*

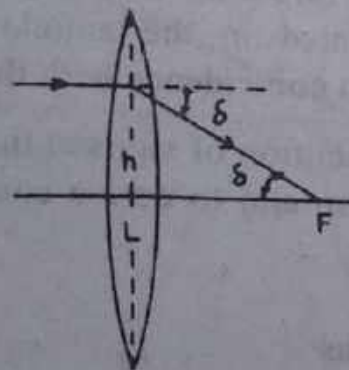


Fig. 24.4

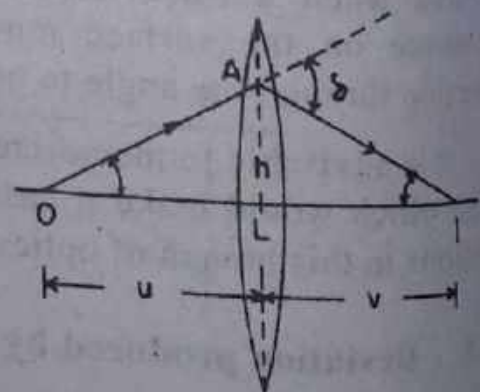


Fig. 24.5

Let a ray of monochromatic light parallel to the principal axis be incident on a thin lens at height  $h$  above the axis and let  $f$  be the focal length of the lens. As the ray is parallel to the principal axis,



after refraction it will pass through the second focus (Fig. 24.4). The deviation  $\delta$  suffered by the ray is given by

$$\tan \delta = \frac{h}{f}$$

In the paraxial region,  $\delta$  is small;

Hence,  $\tan \delta = \delta$

$$\text{or, } \delta = \frac{h}{f} \quad (24.6)$$

We shall next consider a luminous point object  $O$  and its corresponding image  $I$ . (Fig. 24.5). Then deviation suffered by the ray  $OA$  incident at  $A$  is given by  $\delta = \angle AOL + \angle AIL$ . In the paraxial region, since inclinations of the incident and refracted rays to the principal axis are small we can write, taking due account of sign convention.

$$\angle AOL = \tan \angle AOL = \frac{h}{-u}$$

$$\text{and } \angle AIL = \tan \angle AIL = \frac{h}{v}$$

$$\text{or, } \delta = \angle AOL + \angle AIL = \frac{h}{-u} + \frac{h}{v} = h \left[ \frac{1}{v} - \frac{1}{u} \right]$$

$$= h \cdot \frac{1}{f} = \frac{h}{f} \quad (24.7)$$

We, therefore, conclude that *all rays incident at the same point of the lens, from whatever directions, undergo the same deviation on refraction through it*. This statement is true for a divergent lens as well.

#### 24.4 Equivalent lens and equivalent focal length

A single lens is said to be *equivalent* to a number of lenses when this single lens produces an image of a given object at the same place and of same size as that formed by the number of lenses together. The focal length of such a single lens is known as the *equivalent focal length*.



### 24.5 Equivalent focal length of two lenses in contact (by the method of deviation).

$L_1$  and  $L_2$  are two thin convex lenses of focal lengths  $f_1$  and  $f_2$ . They are placed in contact with each other on the same common axis (Fig. 24.6). Let a parallel ray strike the first lens at a height  $h_1$  above the principal axis. The ray is refracted and falls on the second lens. Since the lenses are thin and are in contact with each other, the ray emerging from the first lens after refraction may be assumed to fall on the second lens at the same height  $h_1$  above the principal axis. Let the deviations produced by the first and the second lens be  $\delta_1$  and  $\delta_2$ . Since the deviations produced are in the same direction, the total deviation  $\delta = \delta_1 + \delta_2$ .

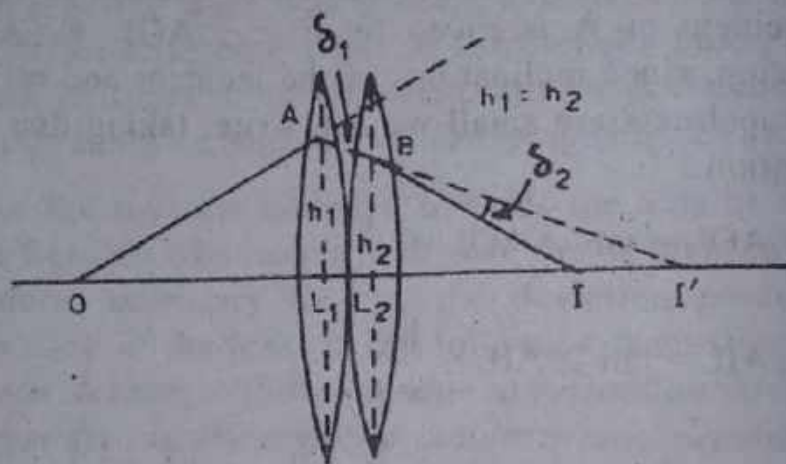


Fig. 24.6

$$\text{Now } \delta_1 = \frac{h_1}{f_1} \quad \text{and} \quad \delta_2 = \frac{h_2}{f_2}$$

$$\therefore \delta = \delta_1 + \delta_2 = \frac{h_1}{f_1} + \frac{h_1}{f_2}$$

If the combination is replaced by their equivalent lens of focal length  $f$  then the deviation

$$\delta = \frac{h_1}{f}$$

Comparing two values of  $\delta$ , we obtain,

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} \quad (24.8)$$

### 24.5 Equivalent focal length of two thin coaxial lenses separated by a finite distance (by the method of deviation).

Two thin convergent lenses,  $L_1$  and  $L_2$  of respective focal lengths  $f_1$  and  $f_2$  are placed coaxially in air separated by a distance  $d$  (Fig. 24.7). Let a ray of monochromatic light, parallel to the common axis be incident on the first lens  $L_1$  at a point A, a height  $h_1$  above the axis. After refraction through the lens, the ray is directed towards its second principal focus  $F_1$ . Then the deviation produced by the first lens is given by

$$\delta_1 = \frac{h_1}{f_1}$$

The emergent ray from the first lens, before reaching the point  $F_1$ , meets the second lens at a height  $h_2$  above the axis and suffers further refraction through it, finally meeting the axis at the point F. The deviation  $\delta_2$  produced as a result of refraction through the second lens, is given by

$$\delta_2 = \frac{h_2}{f_2}$$

Since the deviations  $\delta_1$  and  $\delta_2$  are in the same direction, the total deviation  $\delta$  suffered by the incident ray due to refraction through the combination of lenses is given by

$$\delta = \delta_1 + \delta_2 \quad (24.9)$$

The ray IA, incident on the optical system, is parallel to the principal axis; hence F, the point where the final emergent ray BF intersects the principal axis, is the (second) principal focus of the optical system. IA and BF when produced intersects at the point  $E_2$ . Let  $E_2H_2$  be a plane through  $E_2$  transverse to the principal axis. Then the optical system may be replaced by a single lens of focal length

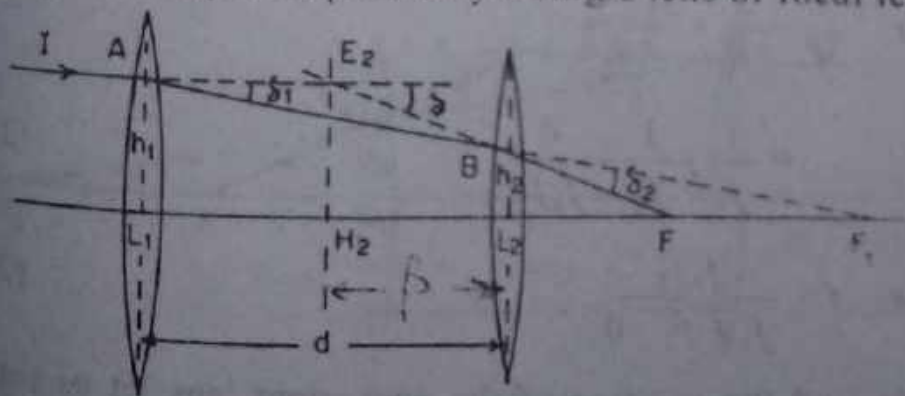


Fig. 24.7

$f = H_2F$  and placed in the position  $E_2H_2$ . Under this condition rays parallel to the principal axis, after refraction through this *hypothetical* lens, would also come to focus at  $F$ . Accordingly, this lens is termed as the *equivalent lens* for parallel rays incident on the first lens.  $H_2F$  is called the *equivalent focal length* of the combination and the plane  $E_2H_2$  along which the equivalent lens must be placed, is called the (*second*) *principal plane* of the optical system. The deviation  $\delta$ , produced as a result of refraction through the equivalent lens, is given by

$$\delta = \frac{h_1}{f}$$

Now  $\delta = \delta_1 + \delta_2$

Substituting the values of  $\delta$ ,  $\delta_1$  and  $\delta_2$ , we get,

$$\frac{h_1}{f} = \frac{h_1}{f_1} + \frac{h_2}{f_2} \quad (24.10)$$

From similar triangles  $AL_1F_1$  and  $BL_2F_1$ , we get,

$$\frac{AL_1}{L_1F_1} = \frac{BL_2}{L_2F_1} \quad ; \quad \text{or, } \frac{h_1}{f_1} = \frac{h_2}{f_1 - d}$$

$$\text{or, } h_2 = \frac{h_1(f_1 - d)}{f_1} \quad (24.11)$$

Substituting this value of  $h_2$  in equation (24.10).

$$\frac{h_1}{f} = \frac{h_1}{f_1} + \frac{h_1(f_1 - d)}{f_1 f_2}$$

$$\text{or, } \frac{1}{f} = \frac{1}{f_1} + \frac{f_1 - d}{f_1 f_2}$$

$$= \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2} \quad (24.12)$$

$$\text{or, } f = \frac{f_1 f_2}{f_1 + f_2 - d} \quad (24.13)$$

To find the position of the equivalent lens let us suppose that the distance of the equivalent lens from the second lens  $L_2$  is  $L_2H_2 = \beta$ .



From the similar triangles  $E_2H_2F$  and  $BL_2F$  we have

$$\frac{H_2F}{L_2F} = \frac{E_2H_2}{BL_2}; \quad \text{or, } \frac{f}{f - \beta} = \frac{h_1}{h_2} \quad (24.14)$$

Owing to the reason that the equivalent lens is located from the given position of the second lens, we should assign sign to  $\beta$  according to the adopted sign convention, with reference to lens positions as origins. Accordingly  $\beta$  is assigned negative sign since  $E_2H_2$  is to the left of  $L_2$ . Eqn. (24.14), therefore, becomes

$$\frac{f}{f - (-\beta)} = \frac{h_1}{h_2}; \quad \text{or, } \frac{f}{f + \beta} = \frac{h_1}{h_2}$$

But from eqn. (24.11)

$$\frac{h_1}{h_2} = \frac{f_1}{f_1 - d}$$

$$\therefore \frac{f}{f + \beta} = \frac{f_1}{f_1 - d}; \quad \text{or, } ff_1 - fd = ff_1 + f_1\beta$$

$$\text{or, } \beta = - \frac{f \cdot d}{f_1} \quad (24.15)$$

The equivalent focal length deduced above is the *second* focal length of the combination. Let us now refer to Fig. 24.8. A ray  $F'A$ , originating from the point  $F'$  on the common, principal axis of the combination, emerges parallel to the axis after refraction through combination.  $F'$  by definition is the first principal focus of the optical system. The incident ray  $F'A$  and the final emergent ray  $BD$

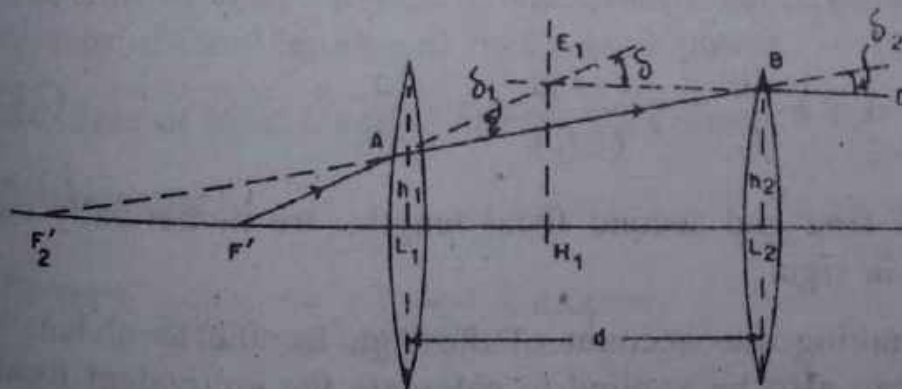


Fig. 24.8

when produced intersects at  $E_1$ . This gives another position  $E_1H_1$  of the equivalent lens and a lens with first focal length  $H_1F'$  and placed along  $E_1H_1$  will replace the optical system.  $E_1H_1$  is called the *first principal plane* of the optical system. When the medium on both sides of the lenses is the same, the first focal length  $f'$  will be numerically equal to the second focal length  $f$  but is negative, according to convention of signs, since  $f'$  is to the left of  $H_1$ . To find out the position of the first principal plane, let us denote the distance of the first principal plane  $E_1H_1$  from the first lens by  $\alpha$ . Since  $\alpha$  is to the right of  $L_1$ , according to sign convention  $\alpha$  is positive. To evaluate  $\alpha$ , we get from the similar triangles  $E_1H_1F'$  and  $AL_1F'$ .

$$\begin{aligned} \frac{h_1}{h_2} &= \frac{L_1F'}{H_1F'} = \frac{H_1F' - L_1H_1}{H_1F'} \\ &= \frac{(-f') - \alpha}{-f'} = \frac{f' + \alpha}{f'} \end{aligned} \quad (24.16)$$

Again, from the similar triangles  $AL_1F'_2$  and  $BL_2F'_2$ , we get

$$\begin{aligned} \frac{h_1}{h_2} &= \frac{L_1F'_2}{L_2F'_2} = \frac{L_2F'_2 - L_1L_2}{L_2F'_2} \\ &= \frac{(-f'_2) - d}{-f'_2} = \frac{f'_2 + d}{f'_2} \end{aligned} \quad (24.17)$$

where  $f'_2 = L_2f'_2$  is the first focal length of the second lens. Comparing eqns. (24.16) and (24.17), we get

$$\begin{aligned} \frac{f'_2 + d}{f'_2} &= \frac{f' + \alpha}{f'} \\ \text{or, } \alpha &= + \frac{f'd}{f'_2} = + \frac{(-f)d}{(-f_2)} = + \frac{fd}{f_2} \end{aligned} \quad (24.18)$$

since the first and second focal lengths are numerically equal but opposite in sign.

By taking due account of the sign for the focal lengths, eqn (24.12) can also be applied to calculate the equivalent focal length of two thin concave lenses separated by a distance. However, the same can be obtained in a manner similar to that described in Art. 24.5.



### 24.6 Restriction in equivalence

It should be clearly noted that a single lens will be truly equivalent to two lenses only for objects situated at infinity. If the object is situated at a finite distance, then the equivalence will hold good for one pair of object and image distances only. The equivalence will hold good neither with regard to size nor with regard to the position of the image for any other pair of points. Strictly speaking, a single lens is equivalent to two lenses only in a restricted sense. There cannot be any single lens which is equivalent to two lenses as regards the position and magnification of the image for all distances of the object.

### 24.7 Power of a Lens

The *power* of a lens is defined as the ability of the lens to converge a beam of light and is measured by the amount of convergence produced to a parallel beam of light. Since a *convex* lens produces *convergence* its power is taken as *positive* while the power of a *concave* lens is taken as *negative* as it produces *divergence*. Again a convex lens of small focal length produces a large converging effect to a beam of light while a convex lens of large focal length produces a small converging effect to the same beam. Thus a convex lens of small focal length has high power and a convex lens of large focal length has low power. *Power can, therefore, be taken as the reciprocal of the focal length.*

$$\text{or, } P = \frac{1}{f}$$

The unit in which power is measured is called *diopetre (D)*. A convex lens of focal length one metre has a power = + 1 diopetre and a convex lens of focal length 2 metres has a power = +  $\frac{1}{2}$  diopetre.

Mathematically,

$$\begin{aligned} \text{Power} &= \frac{1}{\text{focal length in metres}} \text{ diopetre} \\ &= \frac{100}{\text{focal length in cms}} \text{ diopetre} \end{aligned}$$



### Power of combination of lenses

If two lenses of focal lengths  $f_1$  and  $f_2$  are in contact then their equivalent focal length  $F$  is given by

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}$$

Power  $P$  of the combination will then be given by

$$P = P_1 + P_2 \quad (24.19)$$

where  $P_1$  and  $P_2$  are the powers of the individual lenses.

When the lenses are placed coaxially and separated by a distance  $d$  then the relations for equivalent focal length and equivalent power are

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}$$

$$\text{Hence, } P = P_1 + P_2 - dP_1P_2 \quad (24.20)$$

### 24.8 Principal points and principal planes

In the case of thin lenses the distances are always measured from the centre of the lens. The distance from the centre of the lens to either focus gives the focal length. But if a thick lens is used the distance from the centre to one focus is different from the distance to the other focus and neither distance equals the focal length of the lens. Also if an attempt is made to determine the focal length, say by the  $u$ - $v$  method, there is no fixed point from which these distances can be measured. To overcome this difficulty, Gauss in 1841 proved that any number of coaxial refracting systems can be treated as one unit and the simple formulae for thin lenses can be applied to this unit or to a thick lens provided the distances are measured from two theoretical parallel planes, fixed with respect to the refracting system. In case of a thick lens, the points of intersection of these planes with the axis are called the *principal or Gauss points*. The point at a distance  $\alpha$  from the first surface is called the *first principal point* and similarly the point at a distance  $\beta$  from the second surface is called the *second principal point*. Planes drawn perpendicular to the axis and passing through these points are called *first and second principal planes* respectively.

In case of the combination of lenses (optical system), the planes perpendicular to the lens axis at the two positions of the equivalent lens

are known as the principal planes and the points of intersection of these planes with the axis ( $H_1$  and  $H_2$ ) are termed as the first and second principal points respectively.

When the media on either side of the two lenses are similar, the first and the second principal focal lengths are equal, otherwise they will be different.

**Note :** The relation  $\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$  holds good usually for a thin lens.

The lenses used in the laboratory are generally thick. The above relation will also hold good for thick lenses provided the distances on either side of the lens are measured from the principal planes. In the case of an equi-convex lens of glass, having a refractive index of approximately 1.5, the planes are situated at a distance  $t/3$  inside the lens where  $t$  is the thickness of the lens. Thus it is advisable that while calculating the focal length of thick lens, the student should add one third of the thickness of the lens ( $t/3$ ) to the observed values of object and image distance measured from the surface of the lens.

## 24.9 Nodal points

If the medium on one side of the lens is different from that on the other, it becomes necessary to consider two more points on the axis. Like the principal points these are also a pair of conjugate points so that if a ray be allowed to pass through or travel towards the first point  $N_1$  (Fig. 24.9) then its corresponding emergent ray from the lens will be parallel to the incident ray and will appear to diverge from or pass through the second point  $N_2$ .  $N_1$  and  $N_2$  are respectively called the *first and second nodal points*. Planes drawn perpendicular to the principal axis of the lens and passing through the nodal points are known as *nodal planes*.

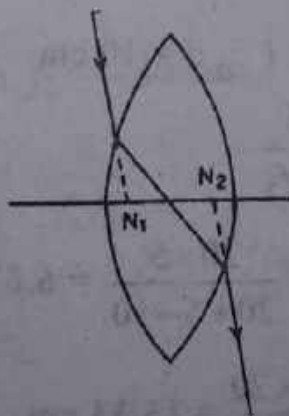


Fig. 24.9



### 24.10 Cardinal points

In case of refraction through a thick lens or a system of coaxial lenses, there are actually six points which are important. These are the *two principal foci, the two principal points and the two nodal points*. These six points taken together are known as the *cardinal points* of an optical system. If the medium on both sides of the lenses are same, the positions of the nodal points coincide with the positions of the principal points. In that case one has to consider four points instead of six points in case of refraction through an optical system.

**Example 24.1.** Two thin convex lenses of focal lengths 20 cm and 5 cm are placed 10 cm apart. Calculate the equivalent focal length and find the positions of the principal points of the combination.

**Soln.**

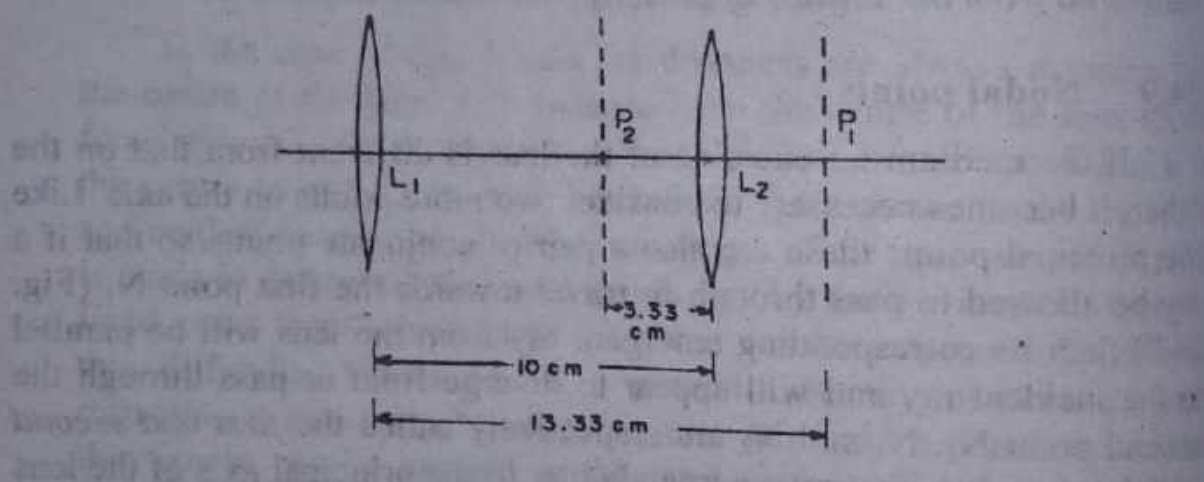


Fig. 24.10

$$f_1 = +20 \text{ cm}, f_2 = +5, d = 10 \text{ cm}$$

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}$$

$$\text{or, } f = \frac{f_1 f_2}{f_1 + f_2 - d} = \frac{20 \times 5}{20 + 5 - 10} = 6.67 \text{ cm}$$

$$\alpha = + \frac{d f}{f_2} = + \frac{10 \times 20}{5 \times 3} = 13.33 \text{ cm}$$



$$\beta = -\frac{df}{f_1} = -\frac{10 \times 20}{3 \times 20} = -3.33 \text{ cm}$$

The first principal point  $P_1$  is to the right of the first lens and is at a distance of 13.33 cm from it.

The second principal point  $P_2$  is 3.33 cm to the left of the second lens.

**Example 24.2.** Find the equivalent focal length and power of a thin concave lens of focal length 15 cm and a thin convex lens of focal length 10 cm placed coaxially in contact with each other.

**Soln.**

$$f_1 = -15 \text{ cm}, f_2 = +10 \text{ cm},$$

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} = -\frac{1}{15} + \frac{1}{10} = \frac{1}{30};$$

$$\text{or, } f = 30 \text{ cm}$$

$$\text{Power} = \frac{100}{f(\text{cm})} \text{ diopetre}$$

$$= \frac{100}{30} \text{ diopetre}$$

$$= 3.33 \text{ diopetre.}$$

**Example 24.3.** A thin convex lens of focal length 20 cm is placed coaxially with a concave lens of focal length 30 cm at a distance of 15 cm from each other. Find the focal length, power and position of the equivalent lens giving the same magnification.

**Soln.**

$$f_1 = +20 \text{ cm}, f_2 = -30 \text{ cm } d = 15 \text{ cm}$$

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}$$

$$\text{or, } f = \frac{f_1 f_2}{f_1 + f_2 - d}$$

$$= \frac{(+20)(-30)}{(20) + (-30) - (15)} = 24 \text{ cm}$$

$$P = \frac{100}{24} \text{ D} = 4.166 \text{ D.}$$

The distance of the equivalent lens from the second lens is given by

$$\beta = -\frac{df}{f_1} = -\frac{15 \times 24}{20} = -18 \text{ cm.}$$

The equivalent lens is located 18 cm to the left of the second (concave) lens.

**Example 24.4.** Two convex lenses of focal lengths 20 cm and 5 cm are 10 cm apart. Calculate the power of such a combination.

**Soln.**

$$P_1 = \frac{100}{20} \text{ D,} \quad P_2 = \frac{100}{5} \text{ D,} \quad d = \frac{10}{100} = 0.1 \text{ metre.}$$

$$\begin{aligned} P &= P_1 + P_2 - dP_1P_2 \\ &= 5 + 20 - 0.1 \times 5 \times 20 \\ &= 25 - 10 = 15 \text{ dioptres.} \end{aligned}$$

**Example 24.5.** Two similar thin convex lenses of focal lengths 10 cm each are co-axial and 5 cm apart. Find the equivalent focal length and the positions of the principal points. Also find the position of the object for which the image is formed at infinity.

**Soln.**

$$f_1 = f_2 = 10 \text{ cm,} \quad d = 5 \text{ cm}$$

$$f = \frac{f_1 f_2}{f_1 + f_2 - d} = \frac{10 \times 10}{10 + 10 - 5} = 6.67 \text{ cm}$$

$$\alpha = +\frac{df}{f_2} = +\frac{6.67 \times 5}{10} = +\frac{10}{3} = +3.33 \text{ cm}$$

$$\beta = -\frac{df}{f_1} = -\frac{6.67 \times 5}{10} = -3.33 \text{ cm}$$

Fig. 24.11 shows the position of the lenses and the principal points.

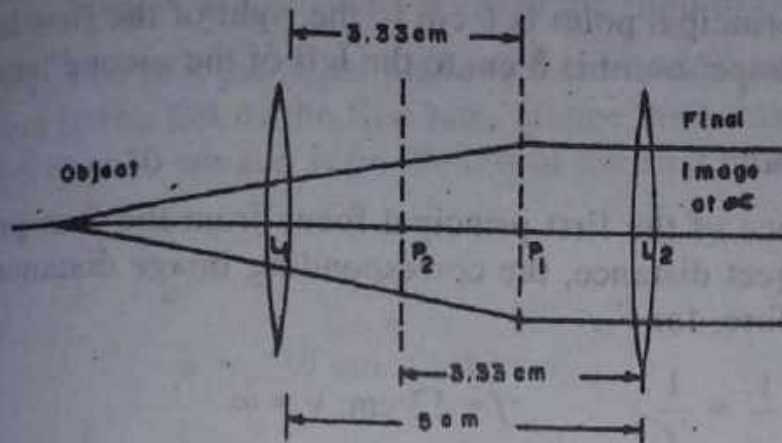


Fig. 24.11

All object and image distances must be measured from the principal points. Let  $u$  be the object distance for which the image is at infinity i.e.,  $v = \infty$

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}, \quad \text{or, } -\frac{1}{u} = \frac{15}{100}$$

$$\text{or, } u = -6.67 \text{ cm.}$$

$u$  is -ve. Hence the object is 6.67 cm to the left of the first principal plane. Since the first principal plane is 3.33 cm to the right of the first lens, the object must be placed  $(6.67 - 3.33) = 3.34$  cm to the left or in front of the first lens.

**Example 24.6.** Two thin converging lenses of focal lengths 15 cm and 20 cm are placed coaxially 10 cm apart. An object is placed at a distance of 24 cm from the first lens. Find (i) the position of the focal points and principal points and (ii) the position of the image.

**Soln.**

$$f_1 = 15 \text{ cm, } f_2 = 20 \text{ cm, } d = 10 \text{ cm}$$

$$f = \frac{f_1 f_2}{f_1 + f_2 - d} = \frac{15 \times 20}{15 + 20 - 10} = 12 \text{ cm}$$

$$\alpha = + \frac{df}{f_2} = + \frac{12 \times 10}{20} = + 6 \text{ cm}$$

$$\beta = - \frac{df}{f_1} = - \frac{12 \times 10}{15} = - 8 \text{ cm}$$



The first principal point is 6 cm to the right of the first lens and the second principal point is 8 cm to the left of the second lens.

### First focal point

The distance of the first principal focus from the first principal point is the object distance, the corresponding image distance being infinity. Therefore, for  $F_1$ ,

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}, \quad f = 12 \text{ cm}, v = \infty$$

$$\text{or, } -\frac{1}{u} = \frac{1}{12}, \quad \text{or, } u = -12 \text{ cm}$$

$u$  is -ve. Since the first principal point is 6 cm to the right of the first lens, the first focal point is  $(12 - 6) = 6$  cm to the left of the first lens.

### Second focal point

The distance of the second principal focus from the second principal point is the image distance, the corresponding object distance being infinity. Therefore, for  $F_2$

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f} \quad u = \infty, \quad f = 12 \text{ cm}$$

$$\text{or, } \frac{1}{v} = \frac{1}{12}, \quad \text{or, } v = 12 \text{ cm}$$

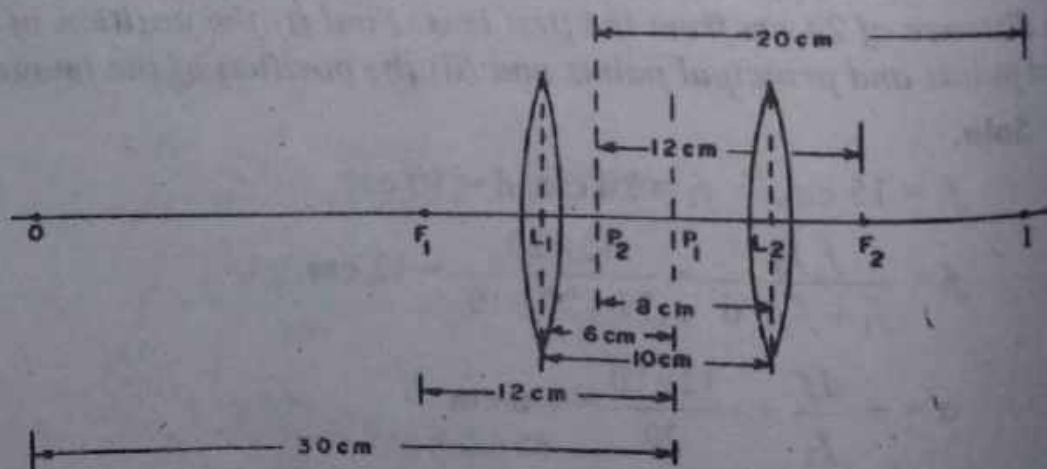


Fig. 24.12

$v$  is +ve. Thus  $F_2$  is  $(12 - 8) = 4$  cm to the right of the second lens.

(ii) The first principal plane is 6 cm to the right and the object is 24 cm to the left of the first lens. Hence the actual object distance is  $(24 + 6) = 30$  cm and is on the left of the first principal plane.

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

$$u = -30 \text{ cm}, f = 12 \text{ cm}$$

$$\therefore \frac{1}{v} + \frac{1}{30} = \frac{1}{12}, \quad \text{or, } v = 20 \text{ cm}$$

$v$  is +ve. Thus  $F_2$  is  $(12 - 8) = 4$  cm to the right of the second lens.

(ii) The first principal plane is 6 cm to the right and the object is 24 cm to the left of the first lens. Hence the actual object distance is  $(24 + 6) = 30$  cm and is on the left of the first principal plane.

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

$$u = 30 \text{ cm}, f = 12 \text{ cm}$$

$$\therefore \frac{1}{v} + \frac{1}{30} = \frac{1}{12}, \quad \text{or, } v = 20 \text{ cm}$$

$v$  is +ve. Since the second principal plane is 8 cm to the left of the second lens, the image is  $(20 - 8) = 12$  cm to the right of the second lens.

**Example 24.7.** A thin convex lens and a thin concave lens each of 50 cm focal length are coaxially situated and separated by 10 cm. Find the equivalent focal length and power of the combination. Also find the position and nature of the final image formed of an object placed 30 cm from the concave lens.

**Soln.**

$$f_1 = +50 \text{ cm} \quad f_2 = -50 \text{ cm}, d = 10 \text{ cm}$$

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{2}{f_1 f_2}$$

$$= \frac{1}{50} - \frac{1}{50} - \frac{10}{50 \times (-50)}$$

$$\text{or, } f = +250 \text{ cm.}$$

$$\alpha = + \frac{f \cdot d}{f_2} = + \frac{10 \times 250}{-50} = -50 \text{ cm}$$

$$\beta = - \frac{f \cdot d}{f_1} = - \frac{10 \times 250}{50} = -50 \text{ cm}$$

$$\text{Power} = \frac{100}{250} \text{ D} = +0.4 \text{ dioptre.}$$

The object is placed 30 cm to the left of the concave lens i.e., 20 cm to the left of the convex lens. Since the first principal plane is 50 cm to the left of the convex lens (first lens), the object is actually placed 30 cm to the right of the first principal plane.

Hence  $u = +30 \text{ cm}$ ,  $f = +250 \text{ cm}$

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}, \quad \text{or, } \frac{1}{v} - \frac{1}{30} = \frac{1}{250}$$

$$\text{or, } v = +26.8 \text{ cm}$$

$v$  is +ve. Hence the image is formed at a point 26.8 cm to the right of the second principal point. This means that the image is  $(50 - 26.8) = 23.2 \text{ cm}$  on the left of the concave lens or  $(23.2 - 10) = 13.2 \text{ cm}$  on the left of the convex lens. *The image is, therefore, virtual.*

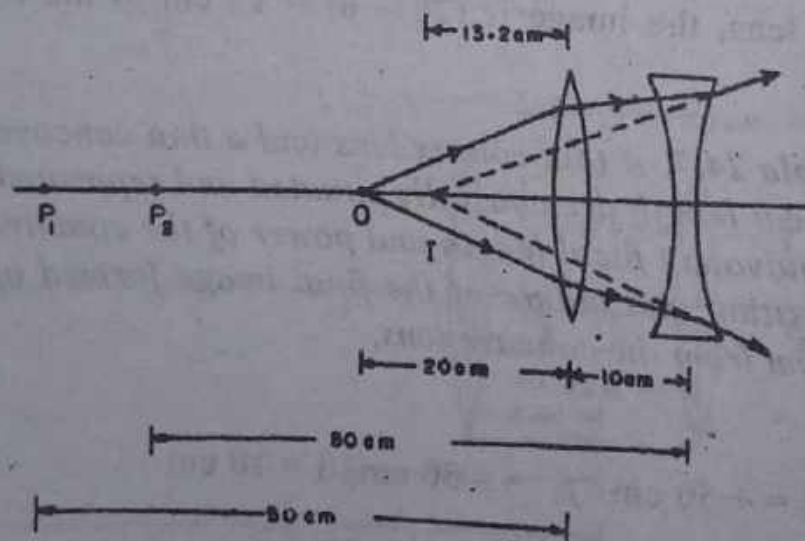


Fig. 24.13



**Example 24.8.** Two thin converging lenses each of 30 cm focal length are set coaxially 10 cm apart. An image of an upright pole 100 metres distant and 5 metres high is formed by the combination. Find the position of the focal planes and the image. Also find the size of the image.

**Soln.**

**Equivalent focal length**

$$f_1 = +30 \text{ cm}, f_2 = +30 \text{ cm} \quad d = 10 \text{ cm}$$

$$f = \frac{f_1 f_2}{f_1 + f_2 - d}$$

$$= \frac{30 \times 30}{30 + 30 - 10} = 18 \text{ cm}$$

**Position of the principal points**

$$\alpha = +\frac{f \cdot d}{f_2} = +\frac{18 \times 10}{30} = +6 \text{ cm}$$

$$\beta = -\frac{f \cdot d}{f_1} = -\frac{18 \times 10}{30} = -6 \text{ cm}$$

**Focal points**

For the first principal focus,

$$v = \infty, f = 18 \text{ cm (see example 24.6)}$$

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}; \quad \text{or, } -\frac{1}{u} = \frac{1}{18}$$

$$\text{or, } u = -18 \text{ cm}$$

The first principal focus is  $(18 - 6) = 12 \text{ cms}$  to the left of the first lens.

For the second principal focus,

$$u = \infty, f = 18 \text{ cm}$$

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

$$\text{or, } \frac{1}{v} = \frac{1}{18} \quad \text{or, } v = 18 \text{ cm}$$

The second principal focus is  $(18 - 6) = 12$  cm to the right of the second lens.

### Position of the image

$$u = -(100 \times 100 + 6) \text{ cm} \\ = -10006 \text{ cm.}$$

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}; \text{ or, } \frac{1}{v} + \frac{1}{10006} = \frac{1}{18}$$

$$\text{or, } v \simeq 18 \text{ cm.}$$

$v$  is +ve; the image is formed approximately at the second principal focus. This is expected since compared to the focal length of the combination, the object may be regarded to be at infinity.

### Magnification produced

$$= \frac{v}{u} = \frac{18}{10006} = .0018$$

Hence, size of the image = object size  $\times$  magnification =  $5 \times 100 \times .0018 \simeq .90$  cm

**Example 24.9.** A concavo-convex lens has a refractive index of 1.50 and the radii of curvature of its surfaces are 15 cm and 30 cm. The concave surface is upwards and it is filled with a liquid of refractive index 1.60. Calculate the focal length of the liquid-glass combination.

**Soln.**

For the glass lens,

$$R = +15 \text{ cm, } R_2 = +30 \text{ cm, } \mu = 1.5$$

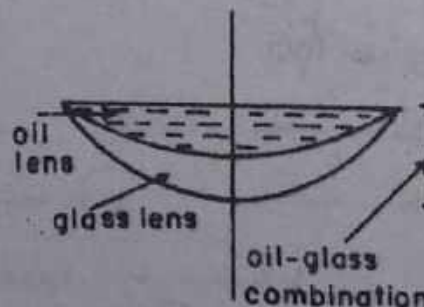


Fig. 24.14.

$$\frac{1}{f_{\text{glass}}} = (\mu - 1) \left( \frac{1}{R_1} - \frac{1}{R_2} \right)$$

$$= (1.5 - 1) \left( \frac{1}{15} - \frac{1}{30} \right)$$

$$\text{or, } f_{\text{glass}} = 60 \text{ cm}$$

For the liquid lens,  $R_1 = +30 \text{ cm}$ ,  $R_2 = \infty$ ,  $\mu = 1.6$

$$\frac{1}{f_{\text{liquid}}} = (\mu - 1) \left( \frac{1}{R_1} - \frac{1}{R_2} \right)$$

$$= (1.6 - 1) \times \frac{1}{30}$$

$$\text{or, } f_{\text{liquid}} = 50 \text{ cm}$$

For the liquid-glass combination,

$$\frac{1}{f} = \frac{1}{f_{\text{glass}}} + \frac{1}{f_{\text{liquid}}} = \frac{1}{60} + \frac{1}{50}$$

$$\text{or, } f = 27.3 \text{ cm}$$

**Example 24.10.** A plano-convex lens has a focal length of 20 cm and is made of glass of refractive index 1.5. What is the radius of curvature of its curved surface? Two such lenses are placed with their convex surfaces turned towards each other and the space between the two is filled with an oil of refractive index 1.7. Find the focal length of the combination.

**Soln.**

For the radius of curvature of the plano-convex lens, we have

$$\mu = 1.5, R_1 = ?, R_2 = \infty$$

$$\frac{1}{f} = (\mu - 1) \left( \frac{1}{R_1} - \frac{1}{R_2} \right)$$

$$\text{or, } \frac{1}{20} = (1.5 - 1) \left( \frac{1}{R_1} \right)$$

$$\text{or, } R_1 = 10 \text{ cm}$$



The combination is shown in the figure. It consists of two plano-concave lens of oil, the radius of curvature of each of the surface being 10 cm.

$$\mu = 1.7, R_1 = -10 \text{ cm} \quad R_2 = +10 \text{ cm}$$

$$\begin{aligned} \frac{1}{f_{\text{oil}}} &= (\mu - 1) \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \\ &= (1.7 - 1) \left( -\frac{1}{10} - \frac{1}{10} \right) \end{aligned}$$

$$\text{or, } f_{\text{oil}} = -7.15 \text{ cm}$$

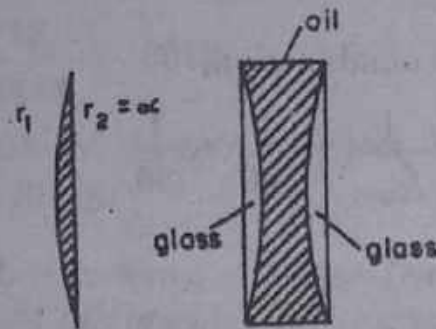


Fig. 24.14

The equivalent focal length of the combination is given by

$$\begin{aligned} \frac{1}{f} &= \frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_{\text{oil}}} \\ &= \frac{1}{20} + \frac{1}{20} - \frac{1}{7.15} \end{aligned}$$

$$\text{or, } f = -25 \text{ cm}$$

## EXERCISES

- [1] If a light ray is incident upon a lens of focal length  $f$  at a height  $h$  from the axis, then show that for small values of  $h$ , the deviation of the ray produced by the lens is  $\frac{h}{f}$  and is independent of the direction of the ray.

- [2] Explain what you mean by an equivalent lens and equivalent focal length. Find an expression for the focal length of two thin lenses placed in contact.

- [3] Two thin convex lenses of focal lengths  $f_1$  and  $f_2$  are placed coaxially in air a certain distance  $d$  apart. Show that their equivalent focal length  $f$  is given by

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{1}{f_1 f_2} d$$

- [4] Prove that for a combination of two coaxial thin convergent lenses of focal lengths  $f_1$  and  $f_2$ , separated by a distance  $d$  in air, the focal length  $f$  of the combination is given by

$$f = \frac{f_1 f_2}{f_1 + f_2 - d}$$

- [5] What is an equivalent lens? In what respect it is called an equivalent lens?

Derive an expression for the focal length of a system of two coaxial thin lenses placed in air and separated by a distance and calculate the position of the principal points.

- [6] What are cardinal points? Define them in relation to (i) a thick lens and (ii) a system of coaxial thin lenses. Explain the significance of the principal planes of a coaxial lens system.

- [7] What do you mean by the power of a lens? Calculate the power of two thin lenses of focal lengths  $f_1$  and  $f_2$  separated by a distance  $d$ .

- [8] Two thin convex lenses of focal lengths 5 cm and 2 cm are placed coaxially in air and separated by a distance of 3 cm. Find the equivalent focal length and the positions of the principal points.

$$[f = 2.5 \text{ cm. } \alpha = + 3.75 \text{ cm. } \beta = - 1.5 \text{ cm}]$$

- [9] Two thin converging lenses of powers 5 dioptres and 4 dioptres are placed coaxially in air 10 cm apart. Find the focal length of the combination. [14.3 cm].



- [10] A thin converging lens and a thin diverging lens are placed coaxially in air at a distance of 5 cm. If the focal length of each lens is 10 cm, find for the combination (i) the focal length, (ii) the power and (iii) position of the principal points.

[ $f = 20$  cm,  $P = +5$  dioptres,  $\alpha = -10$  cm,  $\beta = -10$  cm.]

- [11] An object is placed at a distance of 60 cm from a thin convex lens of focal length 20 cm. There is a second thin convex lens of 30 cm focal length 10 cm away from the first. Calculate the distance of the final image from the second lens. [12 cm to the right of the second lens].

- [12] Two identical thin convex lenses of focal lengths 8 cm each are placed coaxially in air separated by a distance of 4 cm. Find the equivalent focal length and the positions of the principal points. Also find the position of the object for which the image is formed at infinity. [ $f = 5.3$  cm,  $\alpha = +2.66$  cm,  $\beta = -2.6$  cm,  $u = -2.67$  cm from the first lens].

- [13] Two thin converging lenses of focal lengths 30 cm and 40 cm are placed coaxially 20 cm apart in air. An object is located at a distance of 48 cm from the first lens. Find (i) equivalent focal length, (ii) the position of the focal points and the principal points and (iii) the position of the image. Indicate these positions in a diagram.

[ $f = 245$  cm,  $\alpha = +12$  cm,  $\beta = -16$  cm,  $F_1 = 12$  cm to the left of the first lens,  $F_2 = 8$  cm to the right of the second lens,  $v = +24$  cm from the second lens].

- [14] Two thin converging lenses each of 20 cm focal length are set coaxially 5 cm apart. An upright pole 10 metres high is placed at a distance of 200 metres from the first lens. Find the position of the principal planes and the image. Also find the size of the image.

[ $f = 11.43$  cm,  $\alpha = +2.85$  cm,  $\beta = -2.85$  cm,  $F_1 = 8.58$  cm to the left of the first lens,  $F_2 = 8.58$  cm to the right of the second lens,  $v = +8.55$  cm from the second lens, size of the image = 0.57 cm.].

- [15] A concavo-convex lens has an index of refraction 1.5 and the radii of curvature of its surfaces are 10 cm and 20 cm. The concave surface is upwards and is filled with an oil of refractive index 1.60. Calculate the focal length of the oil-glass combination. [18.18 cm.].

- [16] Two thin converging lenses of focal lengths 3 cm and 4 cm are placed coaxially 2 cm apart in air. An object is placed 4 cm in front of the first lens. Find the position and nature of the image and its lateral magnification. [ $v = 2.86$  from the second lens, image is real, magnification = 0.86].



## CHAPTER XXV

## DEFECTS OF IMAGES : SPHERICAL AND CHROMATIC ABERRATIONS

*Aberrations – Spherical aberration of a lens – Reduction of spherical aberration – Coma – Abbe's sine condition – Aplanatic lens – Astigmatism – Curvature – Distortion – Dispersion by prism – Refraction through a prism – Dispersive power of an optical medium – Chromatic aberration in a lens – Axial chromatic aberration in a lens – Achromatism – Condition of achromatism of two lenses in contact – Achromatic doublet – Condition of achromatism of two lenses separated by a distance – Solved problems – Exercises.*

### 25.1 Aberrations

The simple equations connecting object and image distances, focal lengths etc. derived and discussed earlier, were based on the fundamental assumption that the rays were paraxial – the rays of light diverging from the object point were confined to a narrow cone of a small angular opening so as to allow us to replace the sine of the slope angle  $\theta$  of the ray by the angle itself in the following expansion :

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots \quad (25.1)$$

The paraxial condition led to the formation of a point image of a point object and this theory of image formation was called the *first order theory* because of the inclusion of only the first order term of the slope angle in the above expansion.

In practice, objects have finite dimensions and lenses are used to form images of points situated off the axis. Furthermore, considerations of brightness of the image demand the use of lenses with large apertures. All these lead to the conclusion that the rays, in reality, are not paraxial; they form a cone of such a wide angular opening that light from an object do not meet at a single point after refraction through the lens. Thus it becomes essential to take into account, in our discussions of the image formation, at least the first two terms of the expansion of  $\sin \theta$ . Due to the inclusion of the third

order term in  $\theta$ , this theory has come to be known as the *third order theory*. The *deviations from the actual size, shape and position of an image* as calculated from the earlier simple equations, are called *aberrations* produced by a lens. These aberrations may be classified as (i) *spherical aberration*, (ii) *coma* (iii) *astigmatism*, (iv) *curvature* and (v) *distortion*. The aberrations are also known as *Seidal aberrations* as the third order theory was developed by *Ludwig von Seidal* in 1855. These aberrations are present in the images formed by ordinary lenses, even when light of only one wavelength is employed. Consequently, they are also known as *monochromatic aberrations* (*monos* and *chroma* are greek words for *single* and *colour* respectively). Each of the five monochromatic aberrations is dependent upon the wavelength of light but, leaving aside spherical aberration, the variation is negligible in the others.

The focal length of a lens depends upon the refractive index of the lens material, which varies with the wavelength of light. Therefore, even in the first order theory, if the light diverging from the object is not monochromatic, a number of coloured images of different sizes will be formed by the lens at different positions. Thus another type of aberration, known as *chromatic aberration* because of its dependence on the wavelength, i.e., colour of light, is encountered.

A rigorous mathematical study of these aberrations is beyond our scope, but wherever possible simple mathematical deductions will be given while describing these aberrations. It must, however, be pointed out that lens aberrations are just the consequence of the refraction laws at the spherical surfaces and not due to defective construction of a lens such as the surfaces being not spherical, etc.

## 25.2 Spherical aberration of a lens

The presence of spherical aberration in the image formed by a lens, for which the ratio of the aperture to the focal length is relatively large, is illustrated in Fig. 25.1. O is a point object on the axis of the lens and rays coming from this object are incident on the lens at different heights from the axis. The lens may be imagined to be divided into a number of annular zones of increasing radius with the optical centre as the common centre. As the curvature of the lens surface increases with increasing distances from the axis of the lens,



each annular zone will have a different focal length; the greater the radius of the zone the smaller the focal length along the axis. This means that the rays suffering refraction at different zones will suffer

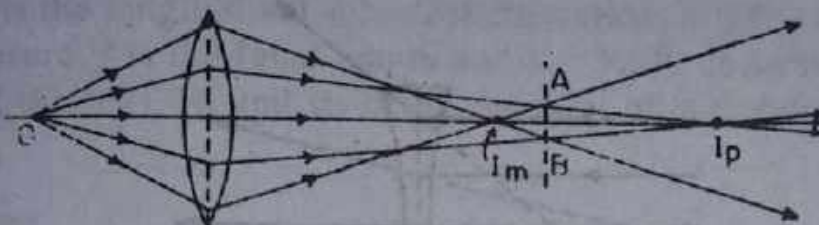


Fig. 25.1

deviation through different extent. The paraxial rays which suffer refraction at the zone close to the principal axis are least deviated and consequently come to focus at a point  $I_p$  farthest from the lens. The marginal rays on the other hand suffer refraction at the periphery of the lens and are the most deviated and come to focus at a point  $I_m$  closest to the lens. The rays which are refracted at any other zone in between these two extreme zones, in general come to focus at one image point between  $I_m$  and  $I_p$ . However, each annular zone will give rise to a different axial image point. The lens, therefore, does not produce one image point of the point object; the image is rather extended over the length  $I_m \sim I_p$ . The image is, therefore, not sharp at any point on the axis. This failure of the lens to provide a sharp image is known as *spherical aberration*. The cross-section of the refracted beam is everywhere circular and if a screen is placed perpendicular to the axis at any point between  $I_m$  and  $I_p$ , a circular patch of light will be obtained on the screen as the image. However, it will be seen from the figure that there is one plane,  $AB$ , at which the cross-section has least diameter. At positions on both sides of  $AB$ , the image patch has a larger diameter. This smallest cross-section is known as the *circle of least confusion* and the best image of the object  $O$  is obtained if a screen is placed perpendicular to the lens axis at this point. The spherical aberration produced by a concave lens for parallel rays is illustrated in Fig. 25.2.

The separation between the marginal focus  $I_m$  and the paraxial focus  $I_p$  is taken as a measure of the *longitudinal spherical aberration* of the lens, while the radius of the circle of least



confusion is taken as a measure of the *lateral or transverse spherical aberration*. The longitudinal spherical aberration is considered *positive* if the marginal focus  $I_m$  lies on the left of  $I_p$  and

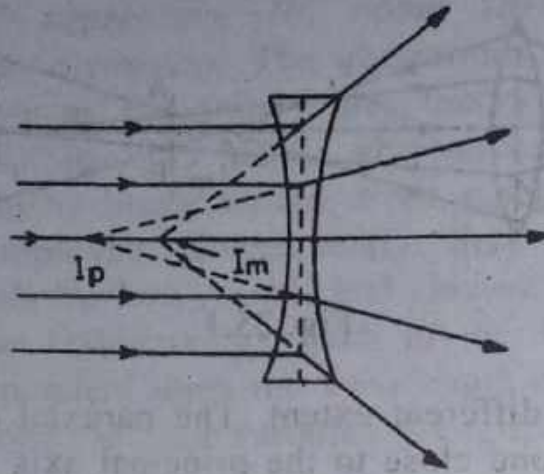


Fig. 25.2

is considered *negative* if  $I_m$  lies on the right of  $I_p$ . Accordingly, longitudinal spherical aberration produced by a convex lens is positive while that produced by a concave lens is negative. The spherical aberration produced by a lens varies approximately as the square of the height of the incident ray above the axis and it also depends upon the distance of the object point.

### 25.3 Reduction of spherical aberration

Spherical aberration produced by lenses can be reduced by the following methods :

(i) **Use of a stop** : Since the spherical aberration produced by a given lens depends on the radius of the lens aperture, it can be reduced by stopping down the lens aperture with the help of coaxial aperture stop. The stop used can be such as to permit either the paraxial rays or marginal rays of light to fall on the lens. However the image will appear less bright as the amount of light passing through the lens is reduced by the use of the stop.

(ii) **Use of crossed lens** : The longitudinal spherical aberration of a thin lens for parallel incident rays is given by

$$x = -\frac{h^2}{f} \left[ \frac{2 - 2\mu^2 + \mu^3 + k(\mu + 2\mu^2 - 2\mu^3) + k^2\mu^3}{2\mu(\mu-1)^2(1-k^2)} \right]$$

where  $x$  is the longitudinal spherical aberration,  $h$  is the radius of the lens aperture,  $f$  is the focal length and  $k = R_1/R_2$  is called the *shape factor* of the lens,  $R_1$  and  $R_2$  being the radii of curvature of the lens surfaces.

### Derivation

Since a lens is a double spherical surface, the formula will be derived for one surface and then for the other. Let  $PO$  be the curved surface under question having  $C$  as its centre of curvature (Fig. 25.3). A marginal ray of light from the source  $S$  strikes the surface at the point  $P$  and undergoes refraction in a direction which when extended backwards intersects the axis  $OS$  at point  $I$ . Let  $SO = u$ ,  $IO = v'$ ,  $OP = h$  and  $CO = r_1$ . If  $\mu$  be the refractive index of the material of the lens, then we have

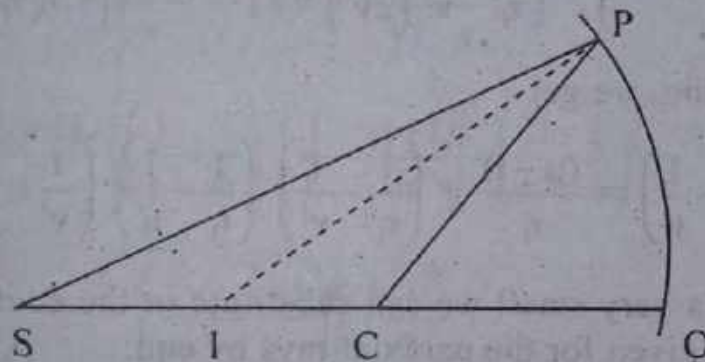


Fig. 25.3

$$\text{from } \Delta IPC, \frac{IC}{IP} = \frac{\sin \angle IPC}{\sin \angle ICP}$$

$$\text{and from } \Delta SPC, \frac{SC}{SP} = \frac{\sin \angle SPC}{\sin \angle SCP}$$

$$\text{Also, } \mu = \frac{\sin i}{\sin r} = \frac{\sin \angle SPC}{\sin \angle IPC}$$

$$\text{whence, } SC \cdot IP = \mu \cdot SP \cdot IC$$

Also from  $\Delta PCS$ ,

$$SP^2 = SC^2 + CP^2 + 2SC.CP \cos OCP$$

$$= (u - r_1)^2 + r_1^2 + 2r_1(u - r_1) \cos \frac{h}{r_1}$$

$$= u^2 - (u - r_1) \frac{h^2}{r_1}, \text{ neglecting higher powers of } h$$

$$\therefore SP = u \left[ 1 - \left( \frac{1}{r_1} - \frac{1}{u} \right) \frac{h^2}{2u} \right]$$

$$\text{Similarly, } IP = v' \left[ 1 - \left( \frac{1}{r_1} - \frac{1}{v'} \right) \frac{h^2}{2v'} \right]$$

Substituting these values in eqn. (i), we get

$$(u - r_1) v' \left\{ 1 - \left( \frac{1}{r_1} - \frac{1}{v'} \right) \frac{h^2}{2v'} \right\} = \mu(v' - r_1)u \left\{ 1 - \left( \frac{1}{r_1} - \frac{1}{u} \right) \frac{h^2}{2u} \right\}$$

Rearranging, we get

$$\left( \frac{\mu}{v'} - \frac{1}{u} \right) = \frac{(\mu - 1)}{r_1} + \left( \frac{1}{r_1} - \frac{1}{v'} \right) \left( \frac{1}{r_1} - \frac{1}{u} \right) \left( \frac{1}{v'} - \frac{\mu}{u} \right) \frac{h^2}{2}$$

Since  $h^2$  is very small we can substitute in the coefficient of  $h^2$  the value of  $v'$  given for the paraxial rays by eqn.

$$\frac{\mu}{v'} - \frac{1}{u} = \frac{\mu - 1}{r_1}$$

We then get

$$-\frac{\mu}{v'} + \frac{1}{u} = \frac{\mu - 1}{-r_1} + \frac{\mu - 1}{\mu^2} \left( \frac{1}{r_1} - \frac{1}{u} \right)^2 \left( \frac{1}{r_1} - \frac{\mu + 1}{u} \right)^2 \frac{h^2}{2} \quad (\text{ii})$$

Proceeding for the second surface in similar manner, we can get

$$-\frac{\mu}{v'} + \frac{1}{v} = \frac{\mu - 1}{-r_2} + \frac{\mu - 1}{\mu^2} \left( \frac{1}{r_2} - \frac{1}{v} \right)^2 \left( \frac{1}{r_2} - \frac{\mu + 1}{v} \right)^2 \frac{h^2}{2} \quad (\text{iii})$$



Subtracting eqn. (ii) from eqn. (iii), we get

$$\left(\frac{1}{v} - \frac{1}{u}\right) = (\mu - 1) \left(\frac{1}{r_1} - \frac{1}{r_2}\right) - \frac{\mu - 1}{\mu^2} \left\{ \left(\frac{1}{r_1} - \frac{1}{u}\right)^2 \left(\frac{1}{r_1} - \frac{\mu + 1}{u}\right) - \left(\frac{1}{r_2} - \frac{1}{v}\right)^2 \left(\frac{1}{r_2} - \frac{\mu + 1}{v}\right) \right\} \frac{h^2}{2} \quad (\text{iv})$$

Eqn. (iv) gives the distance  $v$  of the intersection of marginal ray with the axis (*i.e.*, image distance for marginal rays); that for the paraxial rays say  $V$  is given by

$$\frac{1}{V} - \frac{1}{u} = (\mu - 1) \left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

So the longitudinal spherical aberration  $x$  is given by

$$\begin{aligned} x = V - v &= \left(\frac{1}{v} - \frac{1}{V}\right) Vv = \left(\frac{1}{V} - \frac{1}{v}\right) V^2 \\ &= \frac{(\mu - 1)}{\mu^2} \left\{ \left(\frac{1}{r_1} - \frac{1}{u}\right)^2 \left(\frac{1}{r_1} - \frac{\mu + 1}{u}\right) - \left(\frac{1}{r_1} - \frac{1}{v}\right)^2 \left(\frac{1}{r_1} - \frac{\mu + 1}{v}\right) \right\} \frac{h^2 v^2}{2} \quad (\text{v}) \end{aligned}$$

In case of parallel rays *i.e.*, when the object is at infinity,  $u = \infty$  and  $v = f$ . Eqn. (v), which is a general expression for  $x$ , then reduces to

$$x = \frac{(\mu - 1)}{\mu^2} \left\{ \frac{1}{r_1^3} - \left(\frac{1}{r_2} - \frac{1}{f}\right)^2 \left(\frac{1}{r_2} - \frac{\mu + 1}{f}\right) \right\} \frac{h^2 f}{2} \quad (\text{vi})$$

Substituting  $\frac{r_1}{r_2} = k$  and  $\frac{1}{f} = (\mu - 1) \left(\frac{1}{r_1} - \frac{1}{r_2}\right)$

$$\text{We get } \frac{1}{f} = \frac{(\mu - 1)(1 - k)}{r_1} = \frac{(\mu - 1)(1 - k)}{kr_2} \quad (\text{vii})$$

Eliminating  $r_1$  and  $r_2$  between (vi) and (vii), we get

$$x = -\frac{h^2}{f} \left[ \frac{2 - 2\mu^2 + \mu^3 + k(\mu + 2\mu^2 - 2\mu^3) + k^2\mu^3}{2\mu(\mu - 1)^2(1 - k)^2} \right] \quad (25.2)$$

The ratio  $\frac{h}{f}$  is called the *relative aperture* of the lens.

It appears from eqn. 25.2 that when  $k$  is constant the ratio of longitudinal spherical aberration to the focal length is directly proportional to the square of the relative aperture of the lens. The lens is said to be *undercorrected* for spherical aberration when the focus of the marginal rays is nearer to the lens than the focus of the paraxial rays; otherwise it is called *overcorrected*. In the former case the longitudinal spherical aberration is called negative and in the latter positive.

For given value of  $h$ ,  $f$  and  $\mu$  we can always find a value of  $k$  for which spherical aberration is minimum. Thus differentiating eqn. 25.2 with respect to  $k$  and equating the result to zero the value of  $k$  corresponding to minimum spherical aberration is given by

$$k = \frac{R_1}{R_2} = \frac{\mu(2\mu - 1) - 4}{\mu(2\mu + 1)} \quad (25.3)$$

It is evident from eqn. (ix) that the spherical aberration for a single lens can never be made to vanish.

For a lens, the refractive index of whose material is  $\mu = 1.5$ ,  $k = -\frac{1}{6}$ . Thus the lens giving minimum spherical aberration must be *bi-convex* or *bi-concave*, as indicated by the minus sign, and the radius of curvature of its first surface (surface facing the rays) should be  $1/6^{\text{th}}$  the radius of curvature of its second surface. For a given focal length of the lens, one can therefore, control the spherical aberration by changing the value of  $k$ . This procedure is called the *bending of the lens* and a lens having the desired value of  $k$  for which spherical aberration is minimum is called *crossed lens*. It should, however, be emphasized that the spherical aberration cannot be completely removed by any bending of lens with spherical surfaces. However, spherical aberration can be further reduced in a lens of the same



focal length and radius of aperture by simply constructing it from a material of higher refractive index. As for example, for a lens with  $f = 100$  cm and radius of aperture  $h = 10$  cm,  $x = 1.07$  cm when  $\mu$  and  $k$  are 1.5 and  $-1/6$  respectively. If for the same values of  $f$  and  $h$ ,  $\mu$  is now taken to be equal to 2,  $k$  becomes equal to  $+1/5$  and the longitudinal spherical aberration reduces to 0.44 cm. A crossed lens ( $\mu = 1.5$  and  $k = -1/6$ ) is shown in Fig. 25.4. The physical reason for the minimum longitudinal spherical aberration to occur at  $k = -1/6$  is as follows. It has already been mentioned that for a convergent lens, longitudinal spherical aberration arises because the marginal rays are much more deviated than the paraxial rays. Therefore, spherical aberration will be minimum when the deviation of marginal rays are minimum. In the case of a prism the deviation is minimum when the incident and emergent rays make equal angles with its faces i.e., the total deviation suffered is equally divided between the two surfaces. As a lens may be regarded as a number of truncated prisms placed one above the other, the deviation of marginal rays will be minimum when they enter the first surface and leave the second surface at more or less equal angles. Thus, as a general rule, when a lens is so designed or used that the total deviation of a given ray is divided equally between the two refractions suffered at the two surface, spherical aberration will be minimum.

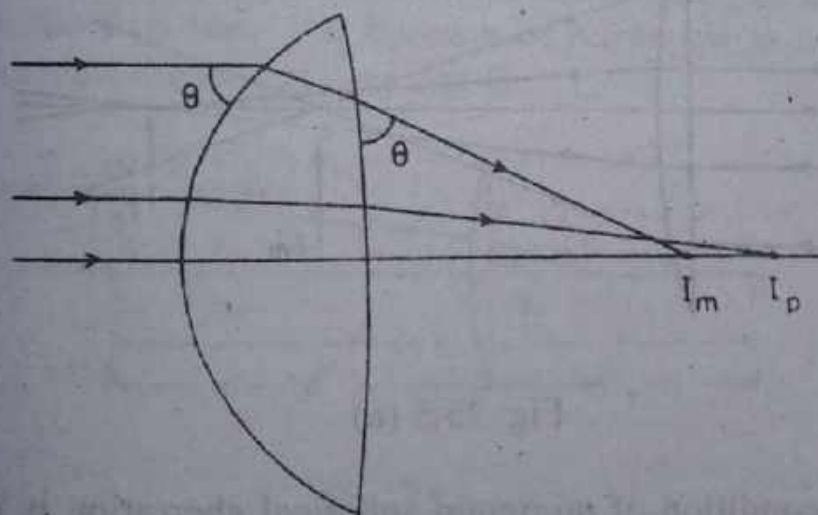


Fig. 25.4

Looking at Fig. 25.3, it is clear that for  $k = -\frac{1}{6}$  for the given lens, the deviations suffered at each of the surface are equal and the



paraxial and marginal rays of light come to focus with minimum of spherical aberration.

(iii) *Use of plano-convex lens* : Crossed lenses are expensive. For this reason, plano-convex lenses which are much cheaper to make, are employed in many optical instruments to reduce spherical aberration. When the curved surface of a plano-convex lens faces the light, the longitudinal spherical aberration is very nearly the same as that produced by a crossed lens of the same focal length and radius of the lens aperture. In reality the spherical aberration of the crossed lens is only 8% less than that of the plano-convex lens. However, if the plane side is turned towards the object the spherical aberration is very large. The reason is not far to seek. As already mentioned, spherical aberration will be minimum when a lens is so designed or used that the total deviation of a given ray is divided equally between the two refractions. When a plano-convex lens is so used that its plane surface faces the object as illustrated in Fig. 25.5 (a) the deviation of a given ray is simply produced at the curved surface

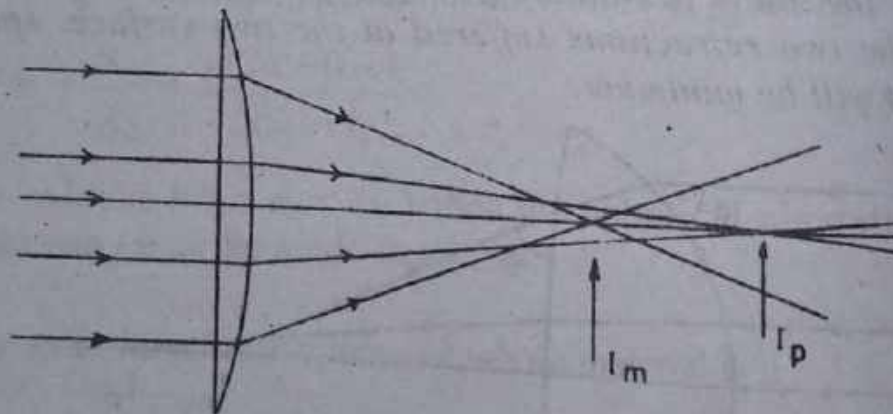


Fig. 25.5 (a)

and thus the condition of minimum spherical aberration is violated, which explains the presence of a large spherical aberration. On the other hand when the curved surface faces the light, the deviation is almost equally divided between the two surfaces, as shown in Fig. 25.5 (b) thus accounting for the presence of minimum spherical aberration.

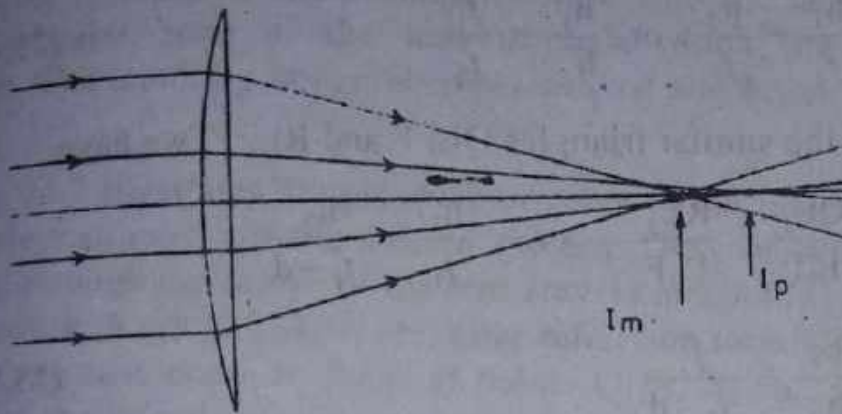


Fig. 25.5 (b)

**(iv) Use of two convergent lenses separated by a fixed distance :**

Spherical aberration can be minimised by extending the above principle of equal distribution of the deviation between the two surfaces of the lens to two convergent lenses separated by a distance. In this arrangement, two convergent lenses of focal lengths  $f_1$  and  $f_2$  are placed coaxially at a distance  $d$  apart as shown in Fig. 25.6.

Let a ray PQ parallel to the principal axis be incident on the first lens at a height  $h_1$  above the axis. In the absence of the second lens, the ray would have met the axis at the point F, as a result of refraction at the first lens. But because of refraction at the second lens, the ray meets the axis at the point G.

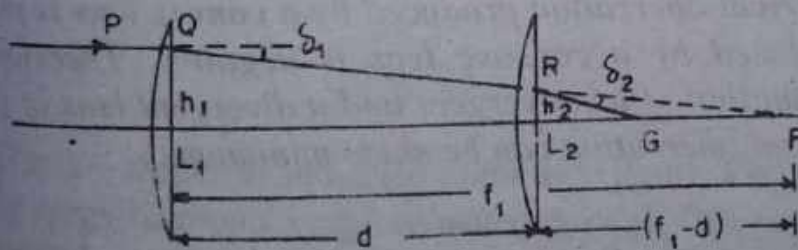


Fig. 25.6

Let  $\delta_1$  and  $\delta_2$  be the deviations produced by the first and second lens respectively. For minimum spherical aberration  $\delta_1 = \delta_2$

$$\text{Now } \delta_1 = \frac{h_1}{f_1} \quad \text{and } \delta_2 = \frac{h_2}{f_2}$$



$$\text{or, } \frac{h_1}{f_1} = \frac{h_2}{f_2}; \text{ or, } \frac{h_1}{h_2} = \frac{f_1}{f_2} \quad (25.4)$$

From the similar triangles  $QL_1F$  and  $RL_2F$ , we have

$$\frac{QL_1}{L_1F} = \frac{RL_2}{L_2F} \quad \text{or, } \frac{h_1}{f_1} = \frac{h_2}{f_1 - d};$$

$$\text{or, } \frac{h_1}{h_2} = \frac{f_1}{f_1 - d} \quad (25.5)$$

Equating the values of  $h_1/h_2$  from eqns. (25.4) and (25.5) we have

$$\frac{f_1}{f_2} = \frac{f_1}{f_1 - d}; \quad \text{or, } \frac{1}{f_2} = \frac{1}{f_1 - d}$$

$$\text{or, } f_1 - f_2 = d \quad (25.6)$$

Thus the spherical aberration for a combination of two convergent lenses is minimum when the distance between the two lenses is equal to the difference of their focal lengths. It is essential that the incident ray should suffer refraction first through the lens of longer focal length and then through the one of smaller focal length. *Spherical aberration can be further minimized by using plano-convex lenses instead of bi-convex lenses.*

(v) *Spherical aberration produced by a convex lens is positive and that produced by a concave lens is negative. Therefore, by suitable combination of a convergent and a divergent lens of proper shapes, spherical aberration can be made minimum.*

## 25.4 Coma

rays coming from an object point not situated on the axis of the lens suffers another type of aberration called *coma* (Greek *kome* meaning *hair*). Comatic aberration is similar to spherical aberration in that both are due to the failure of the lens to bring all rays from a point object to focus at the same point. Spherical aberration refers to object points situated on the axis whereas comatic aberration refers to object points situated off the axis. Comatic aberration arises from the fact that for these non-axial point objects there is either a



decrease or increase of lateral magnification with the height of the narrow circular zone of the lens through which the rays are refracted. The resulting image is *comet-shaped* and hence the name *coma*.

Fig. 25.7 illustrates the presence of coma in the image due to a point object situated off the axis of the lens. Ray of light getting refracted through the centre of the lens (ray 1) meets the screen XY at the point P. Rays 2, 2, 3, 3, etc. after refraction through the outer zones of the lens come to focus at points Q, R, S, etc. nearer the lens. Thus the lateral magnification for outer zones is less than that for the central zone and on the screen overlapping circular patches of gradually increasing diameter are formed. The resultant image of the

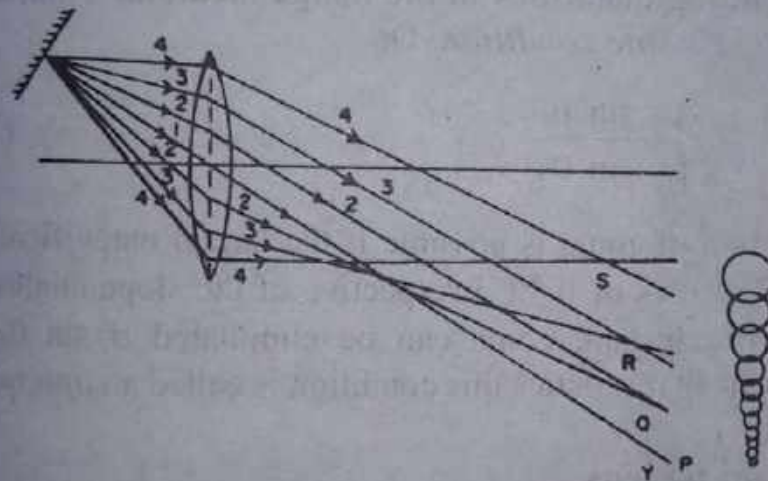


Fig. 25.7

point is comet-shaped as indicated in the side figure. Coma is the result of varying magnification for rays refracted through different zones of the lens and can be positive or negative. As can be seen in Fig. 25.7 for example, rays of light getting refracted through the outer zones come to focus at points nearer the lens. Hence the magnification of the image due to the outer zones is smaller than the inner zones and in this case coma is said to be *negative*. Coma is said to be *positive* when the magnification produced in an image due to the outer zones is greater.

Like spherical aberration comatic aberration produced by a single lens can also be corrected by properly choosing the radii of

curvature of the lens surfaces. Coma can be altogether eliminated for a given pair of object and image points whereas spherical aberration cannot be completely corrected. Further, a lens corrected for coma will not be free from spherical aberration and *vice versa*. Use of a stop or a diaphragm at the proper position eliminates coma.

### 25.5 Abbe's sine condition

Abbe, a German optician, showed that coma can be eliminated if a lens satisfies the condition

$$\mu_1 h_1 \sin \theta_1 = \mu_2 h_2 \sin \theta_2 \quad (25.7)$$

where  $\mu_1$ ,  $h_1$  and  $\theta_1$  refer to the refractive index of the object medium, height of the object above the axis and the slope angle of the incident ray of light respectively. Similarly  $\mu_2$ ,  $h_2$  and  $\theta_2$  refer to the corresponding quantities in the image medium. This condition is known as *Abbe's sine condition*. Or

$$\frac{h_2}{h_1} = \frac{\mu_1 \sin \theta_1}{\mu_2 \sin \theta_2} \quad (25.8)$$

Elimination of coma is possible if the lateral magnification  $h_1/h_2$  is the same for all rays of light, irrespective of the slope angles  $\theta_1$  and  $\theta_2$ . Since  $\mu_1/\mu_2$  is constant, coma can be eliminated if  $\sin \theta_1/\sin \theta_2$  is constant. A lens that satisfies this condition is called an *aplanatic lens*.

### 25.6 Aplanatic lens

A lens which is free from the defects of spherical aberration and coma is called an *aplanatic lens* and the pair of conjugate points free from these defects are called *aplanatic points*. Fig. 25.1 illustrates the *aplanatic lens* and aplanatic points. O and R are the centre and radius of curvature of the lens respectively, the refractive index of whose material is  $\mu$ . P is a point on the axis of the lens such that  $PO = \frac{R}{\mu}$ . It can be shown that all rays passing through the point

P appear to diverge from the point Q irrespective of the slope angle made by the incident rays. In the figure PA is the incident ray; the corresponding refracted ray appears to diverge from the point Q which, therefore, is the image of P. Let  $i$  and  $r$  be the angles of



incidence and refraction and  $\alpha$  and  $\beta$  the slope angles made by the incident and refracted rays.

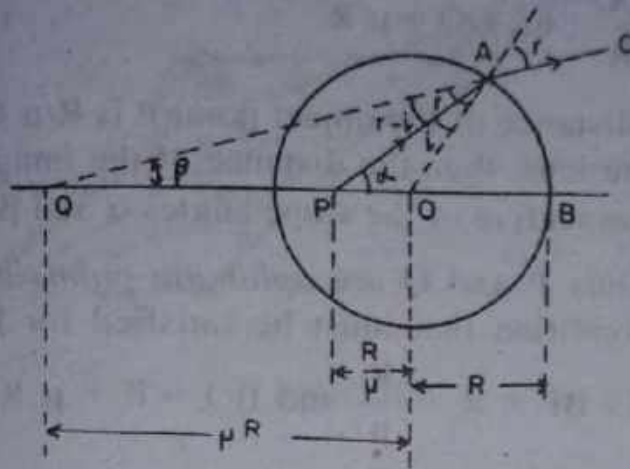


Fig. 25.8

$$\text{Now } \frac{\sin i}{\sin r} = \frac{1}{\mu} \quad (\text{i})$$

But in the triangle APO,

$$\frac{\sin i}{\sin \alpha} = \frac{PO}{AO} = \frac{R}{\mu R} = \frac{1}{\mu} \quad (\text{ii})$$

From eqns. (i) and (ii)

$$\frac{\sin i}{\sin r} = \frac{\sin i}{\sin \alpha} ; \text{ hence } \angle r = \angle \alpha \quad (\text{iii})$$

Again in the triangle APQ,

$$\alpha = \beta + (r - i) \quad (\text{iv})$$

Substituting  $\angle \alpha = \angle r$  in equation (iv)

$$r = \beta + r - i ; \quad \text{or, } i = \beta \quad (\text{v})$$

In the triangle AQO

$$\frac{\sin r}{\sin \beta} = \frac{\sin r}{\sin i} = \mu$$



$$\text{But } \frac{\sin r}{\sin i} = \frac{OQ}{OA} = \frac{OQ}{R}$$

$$\text{Hence } \mu = \frac{OQ}{R}; \text{ or, } OQ = \mu R \quad (\text{vi})$$

Thus, if the distance of the object point P is  $R/\mu$  from the centre of curvature of the lens, then the distance of the image point Q will always be  $\mu R$  irrespective of the slope angles  $\alpha$  and  $\beta$ .

Thus the points P and Q are *aplanatic points* and the lens is *aplanatic*. The condition that must be satisfied for P and Q to be aplanatic points is  $BP = R + \frac{R}{\mu}$  and  $BQ = R + \mu R$ , the distances being measured from the point B.

The high power microscope objective, called the oil immersion objective uses an aplanatic lens as the front lens. As it is not possible to place an object inside a solid spherical lens, the lens is ground a little and the object embedded in between a drop of oil and the lens surface. The oil should be chosen in such a manner that it has the same refractive index as that of the lens.

## 25.7 Astigmatism

When a lens is corrected for spherical aberration and coma, it will form sharp images of object points lying on or very near the axis. But the image of an object point lying at appreciable distance from the lens axis is not a point but a pair of mutually perpendicular lines, some distance apart. This aberration is known as *astigmatism*. Both coma and astigmatism are aberrations in the images of object points off the axis. The difference between astigmatism and coma, however, is that in coma the spreading of the image takes place in a plane perpendicular to the lens axis and in astigmatism the spreading takes place along the lens axis. *Astigmatism discussed here is different from that in defective vision.*

Fig. 25.9 illustrates the defect of astigmatism in the image of a point B situated off the axis. For the point B, far away from the axis, the lens is not perfectly symmetrical. The cone of the rays of light refracted through the *tangential or meridional* (vertical) plane BMN comes to focus at a point  $P_1$  nearer the lens. The cone of rays refracted

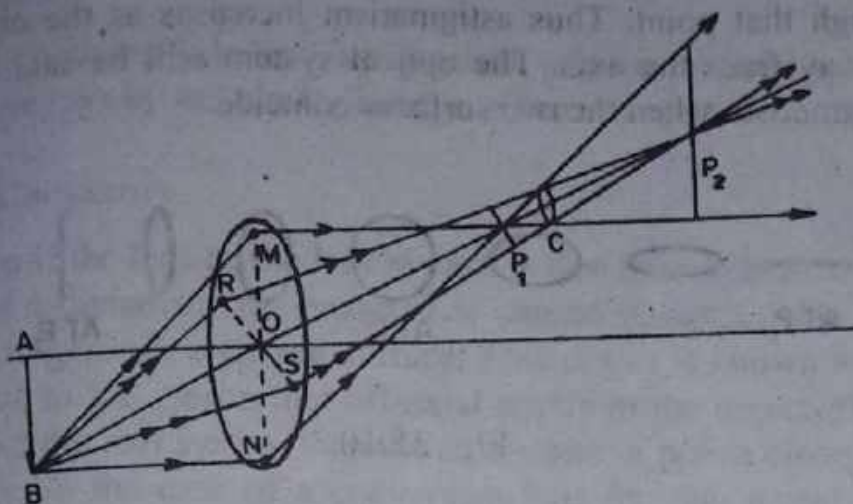


Fig. 25.8

through the *sagittal* (horizontal) plane BRS, comes to focus at a point  $P_2$  away from the lens. Since at the point  $P_1$ , the rays in the sagittal plane have not still focussed, one in fact has a focal line which is normal to the meridional plane. This focus line at  $P_1$  is called the *tangential focal line*. Similarly, since at  $P_2$ , the rays at the meridional plane have defocused, one obtains a focal line which lies in the tangential plane and is called the *sagittal focal line*. All rays pass through a horizontal line passing through  $P_1$  called the *primary image* and also through a vertical line passing through  $P_2$  called the *secondary image*. The cross-section of the refracted beam is elliptical which ends to a horizontal line at  $P_1$  and a vertical line at  $P_2$ . At some point between  $P_1$  and  $P_2$  the cross-section of the refracted beam is circular and this is called the *circle of least confusion*. If a screen is held perpendicular to the refracted beam between the points  $P_1$  and  $P_2$  the shape of the image at different positions is as shown in Fig. 25.10. The distance between  $P_1$  and  $P_2$  is a measure of astigmatism.

Each point on the extended object gives rise, in the manner described above, to its corresponding primary image, secondary image and the circle of least confusion. Their respective loci are surfaces of revolution, about the lens axis, paraboloidal in form and called the *primary image surface*, the *secondary image surface* and the *surface of best focus* respectively. These surfaces are tangential to one another at a point on the axis of the lens, the point being the paraxial image of the conjugate axial point A in the object space. The amount of astigmatism present corresponding to any object point is measured by the difference between the primary and secondary image surfaces measured along the principal



ray through that point. Thus astigmatism increases as the object point moves away from the axis. The optical system will be said to be free from astigmatism when the two surfaces coincide.

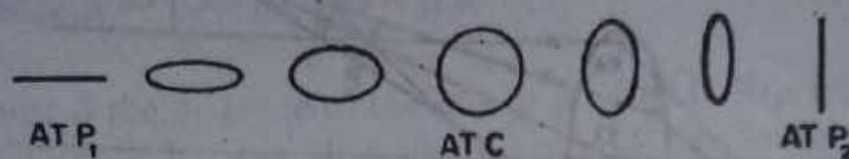


Fig. 25.10

### Reduction or elimination of astigmatism

(a) In case of a divergent lens the sagittal focus  $P_2$  is nearer the lens than the meridional focus  $P_1$  and astigmatism is said to be *negative*. For a convergent lens, the meridional focus  $P_1$  is nearer the lens than the sagittal focus  $P_2$  and the astigmatism is said to be *positive*. Fig. 25.11 illustrates positions of the primary and the secondary images for a convergent as well as a divergent lens.

Since astigmatism is positive in case of a convergent lens and negative in case of a divergent lens, by proper spacing of convergent and divergent lenses and suitable choice of their focal lengths, a combination may be designed in which the astigmatic differences compensate for one another to some extent and the images are formed on a single paraboloid surface, shown by the thick line. Such a combination is called an *astigmat*.

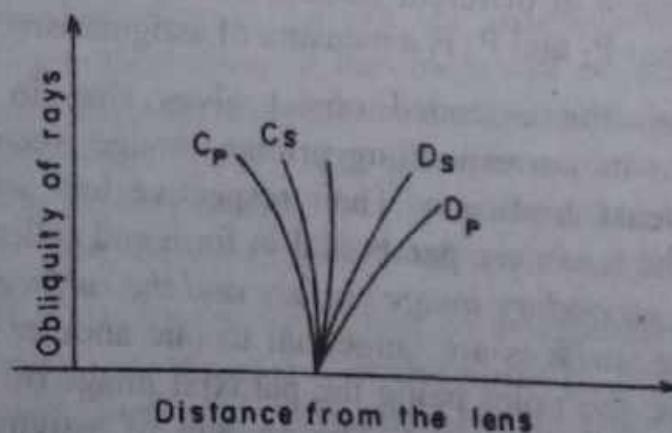


Fig. 25.11



(b) With a single lens, astigmatism may be reduced by cutting out the oblique rays by judicious placings of coaxial apertures called *stops*.

### 25.8 Curvature

Even if the lens or the lens system is free from spherical aberration, coma and astigmatism, the image of an extended object plane is not a flat one but, in general, a curved surface. This defect is known as *curvature* and is due to the reason that off-axial points in the object plane are far away from the lens system than the axial point or points close to the axis. Therefore, in the case of a convergent lens forming a real image, the image of an off-axial object point is formed closer to the lens than the image of an axial object point. Thus the marginal focal length is shorter than the paraxial focal length. This accounts for the curvature of image shown in Fig. 25.12. In a convergent system the radius of curvature of the image surface is negative and hence the curvature is regarded as *negative*. Since the converse holds true for a divergent system, the resultant curvature is regarded as *positive*. As a consequence of this, by a suitable combination of convergent and divergent systems the image can be flattened.

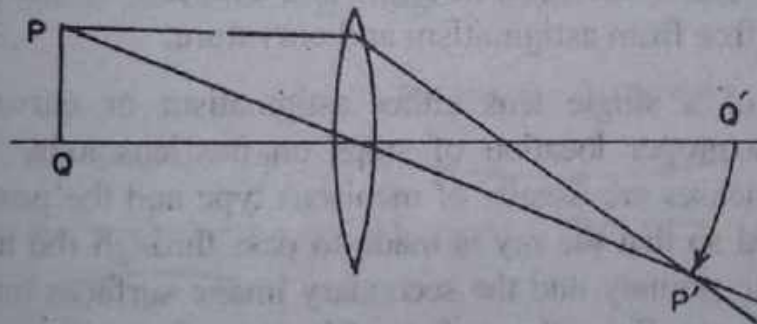


Fig. 25.12

From theoretical considerations it can be shown that for a system of these lenses, the curvature in the final image is given by

$$\frac{1}{R} = \sum \frac{1}{\mu_n f_n}$$

where  $R$  is radius of curvature of the final image,  $\mu_n$  and  $f_n$  are respectively the refractive index of the material and focal length of

the  $n^{\text{th}}$  lens. It follows from the relation that for no curvature i.e., for the image to be flat,  $R$  must be infinity.

$$\therefore \frac{1}{R} = \sum \frac{1}{\mu_n f_n} = \frac{1}{\infty} = 0$$

In the case of two lenses the above condition reduces to

$$\frac{1}{\mu_1 f_1} + \frac{1}{\mu_2 f_2} = 0, \quad \text{or, } \mu_1 f_1 + \mu_2 f_2 = 0$$

Since  $\mu_1$  and  $\mu_2$  are positive,  $f_1$  and  $f_2$  should be of opposite signs. This means that the above condition will be satisfied if one of the lenses is convex and the other concave. The condition holds good if the lenses are separated by a distance or placed in contact. The condition is known as *Petzwal's condition* for no curvature. If  $f_1$  refers to the convergent component, then for the combination to be also convergent,  $f_2$  must be greater than  $f_1$ . Therefore, for Petzwal's condition to be satisfied, it is essential that  $\mu_2$  must be less than  $\mu_1$ . Petzwal's condition can also be satisfied by forming a combination of convergent and divergent lenses made of the same glass, of equal focal lengths and separated by a distance less than the focal length of either of the lenses. Their combined focal length  $f$ , under such condition, is positive, that is, the combination is convergent free from astigmatism and curvature.

In case of a single lens either astigmatism or curvature can be minimized, by proper location of stops on the lens axis. To eliminate curvature, the lenses are usually of meniscus type and the position of front stop is adjusted so that the ray is made to pass through the lens in such a manner that the primary and the secondary image surfaces have equal and opposite curvatures. Since the surface of least confusion lies approximately midway between them, the image is flattened under these conditions. Astigmatism, however, is more pronounced in the outer parts of the field.

## 25.9 Distortion

Even if it is possible to eliminate or minimize spherical aberration, coma, astigmatism and curvature, there may remain another type of aberration termed *distortion*. This aberration arises because of variation in the magnification with lateral distance produced by a lens for object points at different distances from the lens axis. This aberration is not due



to lack of sharpness in the image and is of two types viz., (a) *pin-cushion distortion* and (b) *barrel-shaped distortion*. In the pin-cushion distortion, the magnification increases with the increase in the lateral distance. Owing to the disproportionately high magnification of the corners as compared with other points, the image of a *square like object* situated transverse to the axis will be of the form resembling a *pin-cushion*.

On the other hand, if the magnification decreases with increasing axial distance, the image of the diagonal is shortened relatively more than the images of the side of the square, that is, the opposite effect is produced. The resulting image of the side of the square is of the form resembling a *barrel* giving rise to a distortion known as barrel-shaped distortion. In Fig. 25.13 the *ideal image* is shown by the dotted line and the *distorted image* by the solid line.

### Reduction or elimination of distortion

A little amount of distortion may be present in the case of optical instruments intended mainly for visual observation. But photographic camera lens, where the magnification of the various regions of the object should be the same, must be completely free from distortion. A single thin lens, without any stop to limit the rays, is free from distortion practically for all object distances. But if stops are used, the resulting image is distorted. The distortion will

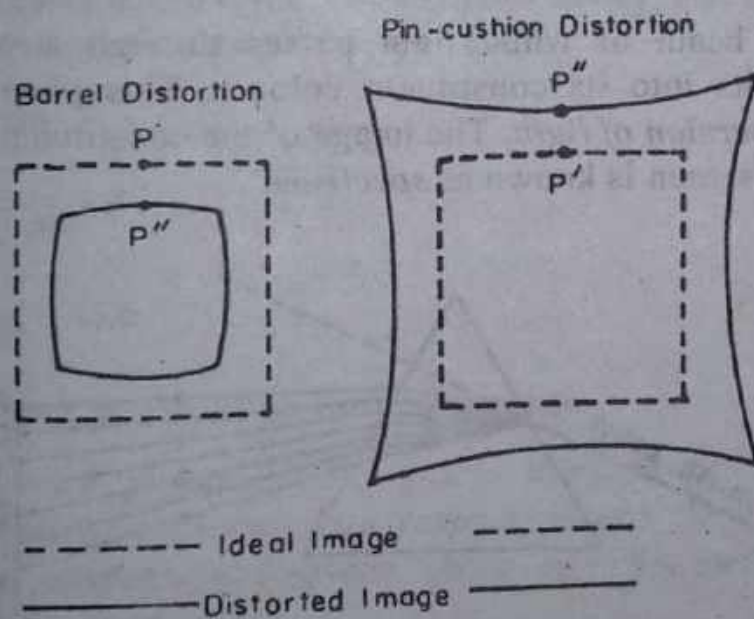


Fig. 25.12



be of barrel-shaped or of pin-cushion type depending on whether the stop is placed in between the object and the lens or the lens and the image.

To eliminate distortion, a stop is placed in between two symmetrical lenses, so that the pin-cushion distortion produced by the first lens is compensated by the barrel-shaped distortion produced by the second lens (Fig. 25.14). Camera and projection lenses are constructed in this manner.

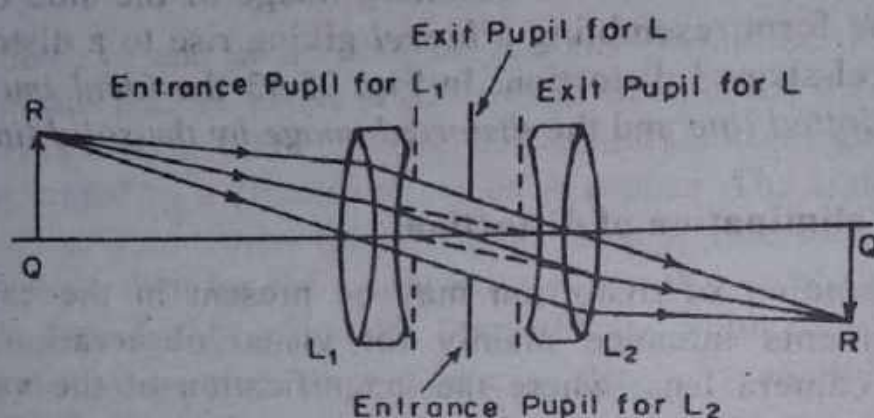


Fig. 25.14

### 25.10 Dispersion by prism

When a beam of white light passes through a prism (Fig. 25.15), it splits into its constituent colours. This phenomenon is known as *dispersion of light*. The image of the constituent colours as received on a screen is known as *spectrum*.

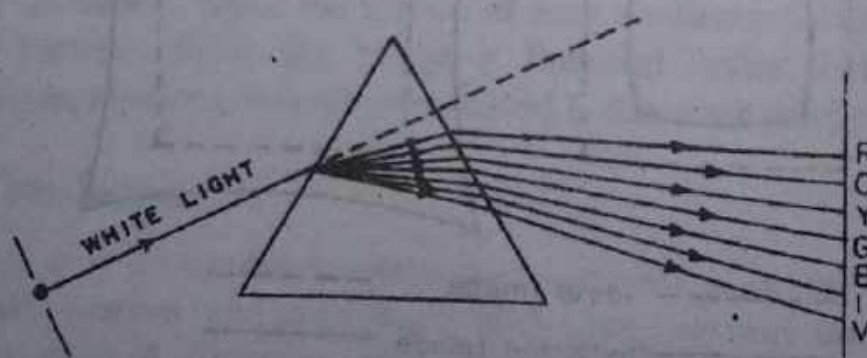


Fig. 25.15

The spectrum consists of both visible and invisible regions. The colours in the visible region are represented by VIBGYOR (violet, indigo, blue, green, yellow, orange and red). The region of the spectrum of wavelengths longer than red is called *infrared* and the region of wavelengths shorter than violet is called *ultraviolet*.

The refractive index for the material of a prism (or of a lens) depends on the wavelength (or colour) of light. The refractive index for violet ray of light is greater than that of red ray. Hence violet is deviated more than the red. A ray which has the average wavelength of two rays near the two ends of the spectrum is taken as the *mean ray* of these two colours. In spectroscopy, yellow ray is taken as the mean ray of the red and blue rays.

### 25.11 Refraction through a prism

The refractive index of the material of a prism for a particular colour is given by

$$\mu = \frac{\sin [(A + D)/2]}{\sin (A/2)}$$

where  $A$  is the angle of the prism and  $D$  is the angle of minimum deviation for the colour, say yellow. For a small angled prism,  $A$  is small and so is  $D$ . Writing  $\alpha$  and  $\delta$  for  $A$  and  $D$  respectively, and taking sines of the angles to be equal to the angles, we have

$$\mu = \frac{\sin \frac{\alpha + \delta}{2}}{\sin \frac{\alpha}{2}} = \frac{(\alpha + \delta)/2}{\alpha/2}$$

$$\text{or, } \delta = (\mu - 1) \alpha \quad (25.9)$$

### 25.12 Dispersive power of an optical medium

The dispersive power of an optical medium placed in air (strictly speaking in vacuum) with respect to any two colours is defined as the dispersion between those two colours per unit deviation of their mean colour.

Let  $\delta_r$ ,  $\delta_b$  and  $\delta$  be the angles of deviation produced in the red, blue and their mean colour yellow rays respectively by a prism (Fig. 25.16).

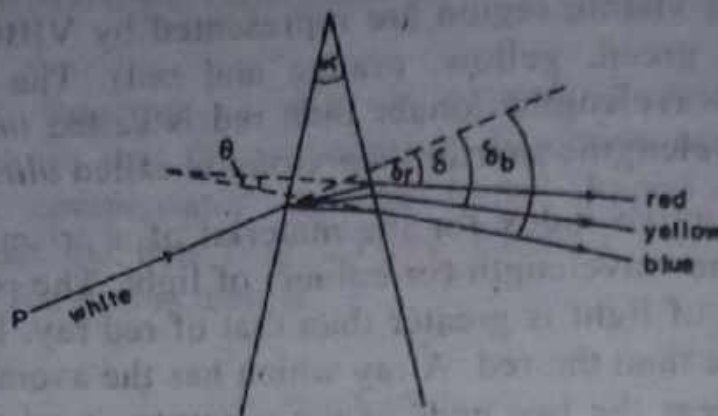


Fig. 25.16

From eqn. (25.15)

$$\delta_b = (\mu_b - 1) \alpha ; \delta_r = (\mu_r - 1) \alpha$$

$$\text{and } \delta = (\mu - 1) \alpha$$

where  $\alpha$  is the angle of the prism and  $\mu_b$ ,  $\mu_r$  and  $\mu$  are the respective refractive indices of the blue, red and yellow rays for the material of the prism. The difference in deviation between any two colours is called the *angular dispersion* for those two colours. The angular dispersion between the blue and the red ray, is therefore,

$$\begin{aligned} \delta_b - \delta_r &= (\mu_b - 1) \alpha - (\mu_r - 1) \alpha \\ &= (\mu_b - \mu_r) \alpha \end{aligned} \quad (25.10)$$

Dividing equation (25.10) by  $\delta$ , the deviation for the mean ray, we have

$$\frac{\delta_b - \delta_r}{\delta} = \frac{(\mu_b - \mu) \alpha}{(\mu - 1) \alpha} = \frac{\mu_b - \mu_r}{\mu - 1} = \frac{d\mu}{\mu - 1} \quad (25.11)$$

where  $d\mu$  is the change in refractive index of the material of the prism as the colour changes from red to blue. But  $(\delta_b - \delta_r)/\delta$  by definition is the dispersive power of the optical medium (material of the prism) with respect to the blue and red colours. Hence, the dispersive power of an optical medium, usually denoted by  $\omega$ , is given by



$$\omega = \frac{\mu_b - \mu_r}{\mu - 1} \quad (25.12)$$

The dispersive power of an optical medium is constant for two colours considered. The reciprocal of the dispersive power is called the *constringence*.

**Note :** If the dispersion through a prism does not follow the order given by *vibgyor*, it is said to be *anomalous dispersion*.

### 25.13 Chromatic aberration in a lens

Every refracting medium has a different refractive index for different wavelengths of light. The focal length of a thin lens is related to the refractive index of the optical material by the relation,

$$\frac{1}{f} = (\mu - 1) \left( \frac{1}{R_1} - \frac{1}{R_2} \right)$$

The focal length of a lens, therefore, depends upon the wavelength of light and since  $\mu_v > \mu_r$ , the focal length  $f_v$  for the violet ray is less than the focal length  $f_r$  of the red ray. Thus the focal length decreases as we pass from red to the violet end of the spectrum. A single lens, therefore, forms not merely one image of an object point but a series of coloured images at varying distances from the lens, one for each of the colours constituting the incident beam. Thus, even if the lens or the lens system is somehow corrected simultaneously for the monochromatic aberrations (spherical aberration, coma, curvature, astigmatism and distortion), which is practically impossible, there will be present another aberration called *chromatic aberration* in the images formed by lenses, when instead of being monochromatic, the light diverging from the object is polychromatic. A beam of white light parallel to the principal axis of a convex lens will converge to different foci at various distances from the lens. The point  $F_v$  (Fig. 25.17) where the violet rays converge is known as the *violet focus*, the point  $F_r$  where the red rays meet as the *red focus* and so on. The distances  $f_v, f_r$ , etc. from the optical centre of the lens to the different foci are known as the violet focal length, red focal length, etc. If a screen is placed at  $F_v$  perpendicular to the axis, the image would be a violet point surrounded by annular coloured patches. If the screen is placed at  $F_r$ , the image will be a red point surrounded by

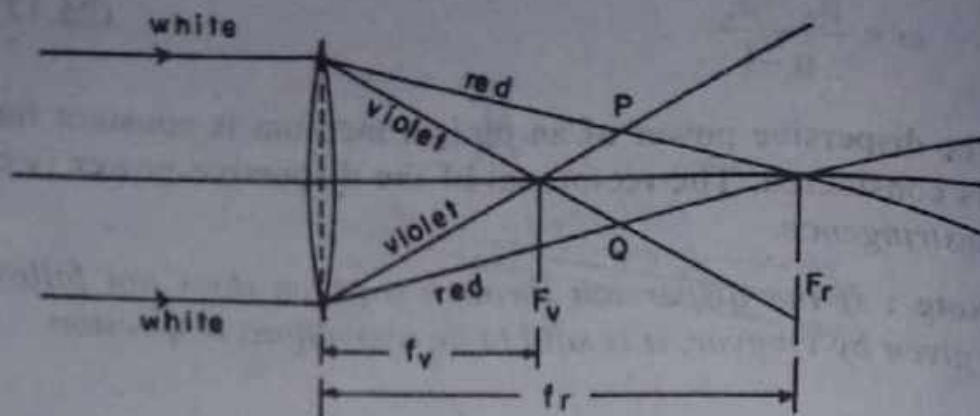


Fig. 25.17

annular coloured patches. But at PQ the image will be an approximately white circular patch. This is called the *circle of least confusion* and is the nearest approach to the image.

Figs. 25.18 (a) & (b) show the image of an extended white object formed by a convex and a concave lens respectively. Since  $f_v < f_r$  the violet image is formed nearest to the lens and is also the smallest, while the red image is farthest away and also the largest. The image of other colours (not shown in the figure) are formed at the intermediate positions and are of intermediate sizes. Evidently there is no one plane where all the images are simultaneously in good focus. Therefore, on a screen, we shall get a blurred image due to superposition of numerous coloured images out of focus, the best possible image being obtained at near about the yellow image.

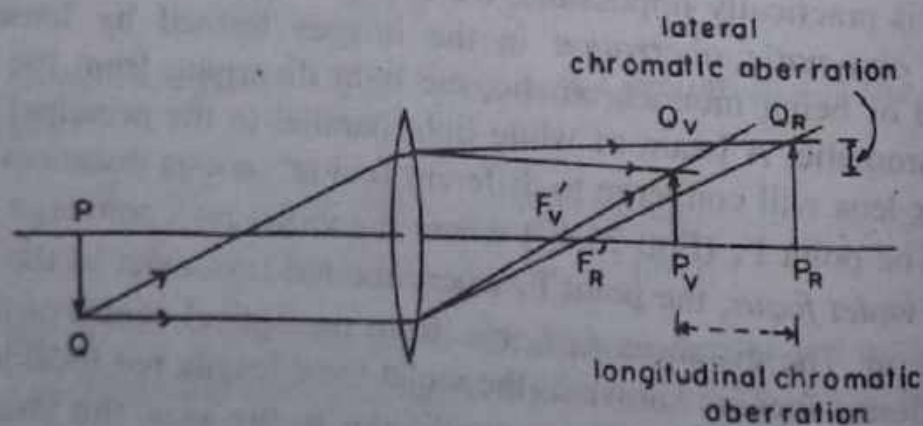


Fig. 25.18 (a)



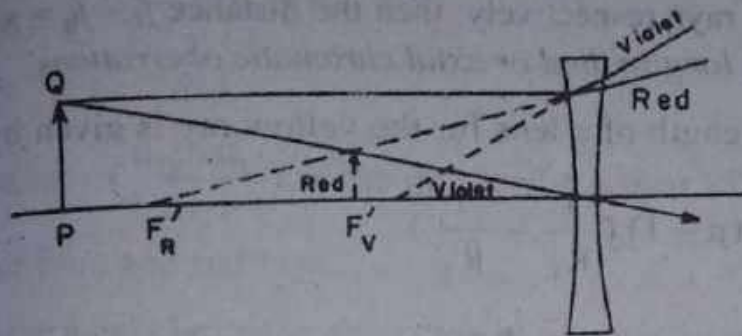


Fig. 25.18 (b)

The variation of the image distance along the axis of the lens due to change in wavelength,  $\frac{dv}{d\lambda}$ , is called the *axial or longitudinal chromatic aberration*. It is measured by the linear distance along the axis between the extreme images and is said to be *positive* when the violet image is situated to the left of the red image. The variation in the image size with wavelength,  $\frac{dy}{d\lambda}$ , is called the *lateral chromatic aberration* and is measured by the difference in the lateral sizes of the extreme images. It is said to be *positive* when the red image is more magnified than the violet.

Convergent lens produces positive longitudinal and lateral chromatic aberrations while divergent lens produces negative longitudinal chromatic aberration but positive lateral chromatic aberration.

#### 25.14 Axial chromatic aberration in a lens

When a parallel beam of white light is passed through a lens, the beam gets refracted as well as dispersed and rays of light of different colours come to focus at different points along the axis. The violet light is closest to the lens while the red light is the farthest away from the lens (Fig. 25.17). The refractive indices of glass are usually given for Fraunhofer lines viz. C-Red ( $\lambda_C = 6563$  A.U.), D-yellow ( $\lambda_D = 5893$  A.U.) and F-Blue ( $\lambda_F = 4862$  A.U.). The wavelength of the yellow light may, therefore, be regarded as the approximate average of the blue and red rays. Hence in the consideration of the chromatic aberration in a lens,



blue and red rays are taken as the two extreme rays instead of the violet and red rays. If  $f_r$ ,  $f_b$  and  $f$  are the focal lengths of the lens for the red, blue and yellow rays respectively, then the distance  $f_r - f_b = x$  along the axis is called the *longitudinal or axial chromatic aberrations*.

The focal length of a lens for the yellow ray is given by

$$\frac{1}{f} = (\mu - 1) \left( \frac{1}{R_1} - \frac{1}{R_2} \right)$$

$$\text{or, } \left( \frac{1}{R_1} - \frac{1}{R_2} \right) = \frac{1}{f(\mu - 1)}$$

The focal length of the lens for the blue ray is given by

$$\frac{1}{f_b} = (\mu_b - 1) \left( \frac{1}{R_1} - \frac{1}{R_2} \right) = \frac{(\mu_b - 1)}{f(\mu - 1)} \quad (25.13)$$

Similarly, the focal length for the red ray is

$$\frac{1}{f_r} = \frac{(\mu_r - 1)}{f(\mu - 1)} \quad (25.14)$$

From equations (25.13) and (25.14)

$$\frac{1}{f_b} - \frac{1}{f_r} = \frac{(\mu_b - 1)}{f(\mu - 1)} - \frac{(\mu_r - 1)}{f(\mu - 1)}$$

$$\begin{aligned} \text{or, } \frac{f_r - f_b}{f_r f_b} &= \frac{1}{f(\mu - 1)} (\mu_b - 1 - \mu_r + 1) \\ &= \frac{(\mu_b - \mu_r)}{f(\mu - 1)} \end{aligned} \quad (25.15)$$

Since,  $f$  is the focal length of the mean ray, it is possible to write  $f_r f_b \simeq f^2$ . Eqn. 25.15 may, therefore, be written as

$$\frac{f_r - f_b}{f_r f_b} = \frac{(\mu_b - \mu_r)}{f(\mu - 1)}$$

$$\text{or, } f_r - f_b = \frac{(\mu_b - \mu_r) f^2}{f(\mu - 1)}$$

$$\begin{aligned}
 &= \frac{(\mu_b - \mu_r)}{(\mu - 1)} \cdot f \\
 &= \omega \cdot f.
 \end{aligned}
 \tag{25.16}$$

where  $\omega = \left( \frac{\mu_b - \mu_r}{\mu - 1} \right)$  is the dispersive power of the material of the lens for blue and red rays.

Thus the axial chromatic aberration of a lens is given by the product of the dispersive power of the material of the lens for the two extreme colours and their mean focal length. From this it is clear that *a single lens cannot form an image free from chromatic aberration.*

### 25.15 Achromatism

An optical instrument free from chromatic aberrations is called an *achromatic* instrument. A perfectly achromatic system is that in which all the coloured images should be formed at the same place and the heights of all these images should also be same. This complete achromatism demands that not only the focal points must be made the same but also the principal planes must be made to coincide for different wavelengths *i.e.*, both axial chromatism ( $dv/d\lambda$ ) and lateral chromatism ( $dy/d\lambda$ ) should be zero. These conditions should be fulfilled not for one position of the object but for its every position. However, this ideal achromatism is extremely difficult to accomplish in the case of any actual system. In practice, only partial achromatism is achieved by removing either axial or lateral chromatic aberration but not both simultaneously. In making the lens system partially achromatic, it is essential to keep in mind the aperture and the field of view of the optical system in deciding the type of chromatic aberration to be eliminated because the type of achromatism which is disadvantageous in one type of instrument may be entirely advantageous in the other. For example, if the optical system has a large field of view but small aperture, lateral chromatic aberration should be eliminated so as to provide uniform magnification for all wavelengths although the images would be formed at different planes. On the other hand, if the optical system has a small field of view but large aperture then it should be corrected for axial chromatic aberration. In that case rays of different wavelengths are brought to focus at the same plane but since the principal points of the



system are different for different wavelengths, the focal lengths would not be equal with the result that magnification would be different for different wavelengths. Attainment of even this partial achromatism is possible for two colours and their mean colour but not for all colours.

### 25.16 Condition of achromatism for two lenses in contact

Chromatic aberration is ordinarily corrected by suitably combining two lenses such that the combination is free from axial chromatic aberration, for the two given colours (say, blue and red), the combination being itself considered as a thin lens. If two lenses of focal length  $f_1$  and  $f_2$  are placed in contact then the equivalent focal length  $f$  is given by

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} \quad (25.17)$$

where  $f_1$ ,  $f_2$  and  $f$  refer to the respective focal lengths for the mean ray (yellow) of the colours considered.

Now

$$\begin{aligned} \frac{1}{f_1} &= (\mu_1 - 1) \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \\ &= k_1 (\mu_1 - 1) \end{aligned} \quad (25.18)$$

$$\begin{aligned} \text{and } \frac{1}{f_2} &= (\mu_2 - 1) \left( \frac{1}{R'_1} - \frac{1}{R'_2} \right) \\ &= k_2 (\mu_2 - 1) \end{aligned} \quad (25.19)$$

where  $\mu_1$  and  $\mu_2$  are the refractive indices of the mean ray for the materials of the first and second lenses respectively.

Eqn. 25.17 then reduces to

$$\frac{1}{f} = k_1 (\mu_1 - 1) + k_2 (\mu_2 - 1) \quad (25.20)$$

Eqn. (25.20) shows that the equivalent focal length  $f$  is a function of wavelength ( $\lambda$ ), since both  $\mu_1$  and  $\mu_2$  are functions of wavelength. Differentiating eqn. (25.20) with respect to  $\lambda$ , we get

$$\frac{d}{d\lambda} \left( \frac{1}{f} \right) = k_1 \frac{d\mu_1}{d\lambda} + k_2 \frac{d\mu_2}{d\lambda} \quad (25.21)$$



Achromatism of the combination is attained when the focal length  $f$  or,  $\frac{1}{f}$  does not change with colour, *i.e.*, wavelength. Hence

for achromatism,  $\frac{d}{d\lambda} \left( \frac{1}{f} \right)$  should be zero. Thus the condition of achromatism is given by

$$\frac{d}{d\lambda} \left( \frac{1}{f} \right) = k_1 \frac{d\mu_1}{d\lambda} + k_2 \frac{d\mu_2}{d\lambda} = 0 \quad (25.22)$$

From eqns. 25.18 and 25.19, we have

$$k_1 = \frac{1}{f_1 (\mu_1 - 1)} \quad \text{and} \quad k_2 = \frac{1}{f_2 (\mu_2 - 1)}$$

Eqn. (25.22), therefore, becomes

$$\frac{1}{f_1 (\mu_1 - 1)} \cdot \frac{d\mu_1}{d\lambda} + \frac{1}{f_2 (\mu_2 - 1)} \cdot \frac{d\mu_2}{d\lambda} = 0 \quad (25.23)$$

If we deal with a finite change in refractive index ( $= d\mu$ ) for a finite change in wavelength ( $= d\lambda$ ) of light between the two given colours (say, blue and red), then we may write,

$$\frac{1}{f_1} \cdot \frac{d\mu_1}{(\mu_1 - 1)} + \frac{1}{f_2} \cdot \frac{d\mu_2}{(\mu_2 - 1)} = 0 \times d\lambda = 0 \quad (25.24)$$

$$\text{But } \frac{d\mu_1}{\mu_1 - 1} = \omega_1 \quad \text{and} \quad \frac{d\mu_2}{\mu_2 - 1} = \omega_2$$

where  $\omega_1$  and  $\omega_2$  are the dispersive powers of the materials of the two lenses for the blue and red rays of light. Eqn. (25.24), therefore, becomes

$$\frac{\omega_1}{f_1} + \frac{\omega_2}{f_2} = 0 \quad (25.25)$$

Eqn. 25.25 is the required condition of achromatism for two lenses in contact.

From eqn. 25.25 it follows that

- (i) *One of the lens is convergent and the other divergent.*

Since  $\mu_b > \mu_r$ ,  $d\mu$  is positive and since  $\mu > 1$ , the dispersive powers  $\omega_1$  and  $\omega_2$  are always positive. Hence, in order that the eqn. (25.25) be satisfied  $f_1$  and  $f_2$  should be of opposite signs.

(ii) *The lenses should be made of different materials* : If we take two lenses of the same material, then  $\omega_1 = \omega_2 = \omega$  (say). Then the condition (25.25) reduces to

$$\omega \left( \frac{1}{f_1} + \frac{1}{f_2} \right) = 0 \quad \text{or, } \omega \cdot \frac{1}{f} = 0.$$

But since  $\omega$  cannot be zero,  $\frac{1}{f} = 0$  or,  $f = \infty$ . The combination will behave like a parallel plate of glass and will not act like a lens. Thus achromatism cannot be achieved by taking two lenses of the same material in contact.

(iii) *The choice of the focal length and dispersive power of the lens is governed by eqn. (25.25).*

In practice a convergent lens of crown glass and a divergent lens of flint glass are used. The convergent lens is more strongly convergent for the blue rays than for the red rays while the divergent lens is more strongly divergent for the blue rays than for the red rays. Hence if a combination is made of these lenses satisfying eqn. (25.25), the coloured images formed by the combination for different wavelengths will fold on one another and the effects of chromatic aberration will be absent. Since the relation (25.25) is independent of the position of the object, achromatism will be obtained for all object distances.

### 25.17 Achromatic doublet

Since the removal of chromatic aberration only restricts the choice of focal lengths and not the radii of curvature of the lenses (eqn. 25.25), the latter may be chosen to make spherical aberration minimum as well. In practice a double convex lens of crown glass is combined with a plano-concave lens of flint glass. The double convex lens is a crossed lens, its radii of curvature  $R_1$  and  $R_2$  being in the ratio of 1:6. The radius of curvature of the concave surface of the plano-concave lens is made equal to the surface of the convex lens with the larger radius. The two equal surfaces are then put in



contact to form what is known as *achromatic doublet* (Fig. 25.19). To avoid any loss of light the surface in contact are cemented together by *Canada Balsam*. To reduce spherical aberration the curved

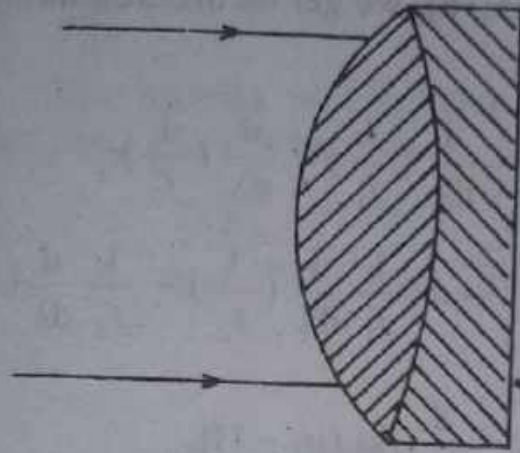


Fig. 25.19

surface of the convex lens with the smaller radius is made to face the beam of light. The focal lengths,  $f_1$  and  $f_2$  are chosen to satisfy the relation  $\frac{\omega_1}{f_1} + \frac{\omega_2}{f_2} = 0$ ,  $\omega_1$  and  $\omega_2$  being the dispersive powers for the materials of the two lenses. The achromatic doublet is commonly used as an *objective* in telescopes.

In the above cases achromatism has been achieved only in a restricted sense in that the combination has been corrected only for two extreme colours. But if the two colours are suitably chosen the combination will be practically achromatic for the other colours also.

#### 25.18 Condition of achromatism of two lenses separated by a distance (separated doublet)

Let two lenses of focal lengths  $f_1$  and  $f_2$  (for the mean rays) and of dispersive powers  $\omega_1$  and  $\omega_2$  respectively, are separated by a distance  $d$ . Then the equivalent focal length  $f$  of the combination is

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2} \quad (25.26)$$

As in Art. 25.16 the focal lengths of the individual lenses are given by



$$\frac{1}{f_1} = k_1 (\mu_1 - 1) \text{ and } \frac{1}{f_2} = k_2 (\mu_2 - 1)$$

Since the focal lengths are functions of the wavelengths  $\lambda$  (already discussed in Art. (25.16), we get on differentiating eqn. (25.26) with respect to  $\lambda$

$$\begin{aligned} \frac{d}{d\lambda} \left( \frac{1}{f} \right) &= \frac{d}{d\lambda} \left( \frac{1}{f_1} \right) + \frac{d}{d\lambda} \left( \frac{1}{f_2} \right) \\ &= \frac{d}{d\lambda} \left[ \frac{1}{f_1} (\mu_1 - 1) + \frac{1}{f_2} (\mu_2 - 1) \right] \quad (25.27) \end{aligned}$$

$$\begin{aligned} \text{But } \frac{d}{d\lambda} \left( \frac{1}{f_1} \right) &= \frac{d}{d\lambda} [k_1 (\mu_1 - 1)] \\ &= k_1 \frac{d\mu_1}{d\lambda} = \frac{1}{f_1 (\mu_1 - 1)} \frac{d\mu_1}{d\lambda} \end{aligned}$$

Similarly

$$\frac{d}{d\lambda} \left( \frac{1}{f_2} \right) = \frac{1}{f_2 (\mu_2 - 1)} \frac{d\mu_2}{d\lambda}$$

Eqn. (25.27), therefore, may be written as

$$\begin{aligned} \frac{d}{d\lambda} \left( \frac{1}{f} \right) &= \frac{1}{f_1 (\mu_1 - 1)} \frac{d\mu_1}{d\lambda} + \frac{1}{f_2 (\mu_2 - 1)} \frac{d\mu_2}{d\lambda} \\ &= \frac{d}{d\lambda} \left[ \frac{1}{f_1} \cdot \frac{1}{f_2 (\mu_2 - 1)} \frac{d\mu_2}{d\lambda} + \frac{1}{f_2} \cdot \frac{1}{f_1 (\mu_1 - 1)} \frac{d\mu_1}{d\lambda} \right] \quad (25.28) \end{aligned}$$

For the combination to be achromatic,

$$\frac{d}{d\lambda} \left( \frac{1}{f} \right) = 0$$

Hence

$$\frac{1}{f_1(\mu_1 - 1)} \cdot \frac{d\mu_1}{d\lambda} + \frac{1}{f_2(\mu_2 - 1)} \cdot \frac{d\mu_2}{d\lambda} - \frac{d}{f_1 f_2} \left[ \frac{1}{(\mu_1 - 1)} \cdot \frac{d\mu_1}{d\lambda} + \frac{1}{(\mu_2 - 1)} \cdot \frac{d\mu_2}{d\lambda} \right] = 0 \quad (25.29)$$

Considering a finite change in refractive index for a finite change in wavelength, eqn. (25.29) becomes

$$\frac{1}{f_1} \cdot \frac{d\mu_1}{(\mu_1 - 1)} + \frac{1}{f_2} \cdot \frac{d\mu_2}{(\mu_2 - 1)} - \frac{d}{f_1 f_2} \left[ \frac{d\mu_1}{(\mu_1 - 1)} + \frac{d\mu_2}{(\mu_2 - 1)} \right] = 0 \times d\lambda = 0$$

$$\text{or, } \frac{\omega_1}{f_1} + \frac{\omega_2}{f_2} - \frac{d}{f_1 f_2} (\omega_1 + \omega_2) = 0$$

$$\text{or, } \frac{d}{f_1 f_2} (\omega_1 + \omega_2) = \frac{\omega_1}{f_1} + \frac{\omega_2}{f_2} = \frac{\omega_1 f_2 + \omega_2 f_1}{f_1 f_2}$$

$$\therefore \text{ or, } d = + \frac{\omega_1 f_2 + \omega_2 f_1}{\omega_1 + \omega_2} \quad (25.30)$$

Eqn. (25.30) gives the condition of achromatism of a *separated doublet* when the lenses are made of different material. If the lenses are of the same material  $\omega_1 = \omega_2 = \omega$  (say), eqn. (25.30) then becomes

$$d = + \frac{1}{2} (f_1 + f_2) \quad (25.31)$$

Thus if the lenses are of the same material, chromatic aberration can be eliminated by keeping them separated by half the sum of their focal lengths. As  $d$  is positive ( $f_1 + f_2$ ) should be positive. Hence *both the lenses or the lens with the longer focal length must be convergent*. The combination will then be achromatic for all the colours near those for which the mean focal lengths  $f_1$  and  $f_2$  have been calculated. This principle is utilised in the construction of *eye-pieces* for optical instruments.

**Example 25.1.** The focal lengths of a convex lens are 100 cm and 98 cm for red and blue rays respectively. Calculate the dispersive power of the material of the lens.

**Soln.**

In terms of the principal focal lengths the dispersive power may be written as

$$\omega = \frac{f_r - f_b}{f} \quad (\text{eqn. 25.16})$$

where  $f$  is the focal length of the mean ray. Since  $f$  is intermediate in value between  $f_b$  and  $f_r$

$$f = \sqrt{f_r \times f_b}$$

Hence

$$\omega = \frac{f_r - f_b}{\sqrt{f_r \times f_b}} = \frac{100 - 98}{\sqrt{100 \times 98}} = \frac{2}{99} = .02$$

**Example 25.2.** An achromatic objective of focal length 50 cm is to be made of different kinds of glass shown below. Find the focal length of each lens, stating whether it is convergent or divergent.

	Glass A	Glass B
$\mu_{\text{red}}$	1.51	1.64
$\mu_{\text{blue}}$	1.52	1.66

**Soln.**

The refractive index of the mean ray in glass A,

$$\mu_A = \frac{1.51 + 1.52}{2} = 1.515$$

Hence the dispersive power of glass A,

$$\omega_A = \frac{\mu_b - \mu_r}{\mu_A - 1} = \frac{1.52 - 1.51}{1.515 - 1} = 0.0194$$

The refractive index of the mean ray in glass B,



$$\mu_B = \frac{1.64 + 1.66}{2} = 1.65$$

Hence the dispersive power of the glass B,

$$\omega_B = \frac{1.66 - 1.64}{1.65 - 1} = 0.0307$$

Now, the condition for lenses in contact to be achromatic for red and blue light is

$$\frac{\omega_A}{f_A} + \frac{\omega_B}{f_B} = 0$$

$$\text{or, } \frac{f_B}{f_A} = - \frac{\omega_B}{\omega_A} = - \frac{0.0307}{0.014} = - 1.582$$

$$\text{or, } f_B = - 1.582 f_A$$

Again from the condition of equivalent focal length of two lenses in contact,

$$\frac{1}{f} = \frac{1}{f_A} + \frac{1}{f_B}; \quad \text{or, } \frac{1}{50} = \frac{1}{f_A} - \frac{1}{1.582 f_A}$$

$$\text{or, } 1.582 f_A = 29.1; \quad \text{or, } f_A = + 18.4 \text{ cm}$$

focal length is +ve, hence the lens is convergent.

$$f_B = - 1.582 f_A = - 1.582 \times 18.4 = - 19.1 \text{ cm}$$

focal length is - ve, hence the lens is divergent.

**Example 25.3.** A converging lens of focal length 35.0 cm and achromatic for the C and F lines is to be made with two lenses, one of hard crown and the other of dense flint glass, placed in contact. Find the focal length of these lenses.

$$\text{Hard crown : } \mu_D = 1.5175,$$

$$\mu_F - \mu_C = 0.00856$$

$$\text{Dense flint : } \mu_D = 1.6264,$$

$$\mu_F - \mu_C = 0.01722$$

$\mu_D$  is the refractive index for mean ray.

**Soln.**

Let the focal length and dispersive power of the hard crown glass lens be  $f_1$  and  $w_1$  respectively, the corresponding quantities for the dense flint glass lens being  $f_2$  and  $w_2$ .

$$w_1 = \frac{d\mu}{\mu - 1} = \frac{0.00856}{1.5175 - 1} = 0.0165$$

$$w_2 = \frac{d\mu}{\mu - 1} = \frac{0.01722}{1.6264 - 1} = 0.0275$$

From the condition of achromatism for two lenses in contact

$$\frac{w_1}{f_1} + \frac{w_2}{f_2} = 0;$$

$$\text{or, } \frac{f_2}{f_1} = - \frac{w_2}{w_1} = - \frac{0.0275}{0.0165}$$

$$\text{or, } f_2 = -1.666 f_1$$

The equivalent focal length of the combination

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}$$

$$\text{or, } \frac{1}{35} = \frac{1}{f_1} - \frac{1}{1.666 f_1} = \frac{0.666}{1.666 f_1}$$

$$\text{or, } 1.666 f_1 = 23.3$$

$$\text{or, } f_1 = +13.98 \text{ cm}$$

Hard crown glass lens is convex; focal length +13.98 cm

$$f_2 = -1.666 f_1$$

$$= -1.666 \times 13.98 \text{ cm} = -23.3 \text{ cm.}$$

Dense flint glass lens is concave, focal length -23.3 cm

**Example 25.4.** The dispersive powers for crown and flint glasses are in the ratio of 1:2. Calculate the focal lengths of the lenses made of crown and flint glass which form an achromatic combination of focal length 20 cm when placed in contact.

**Soln.**

Let  $f_1$  and  $f_2$  be the focal lengths of the crown and flint glass lens respectively, the corresponding dispersive powers being  $\omega_1$  and  $\omega_2$ .

From condition of achromatism

$$\frac{\omega_1}{f_1} + \frac{\omega_2}{f_2} = 0;$$

$$\text{or, } \frac{f_2}{f_1} = - \frac{\omega_2}{\omega_1} = - \frac{2}{1}$$

$$\text{or, } f_2 = -2f_1$$

Equivalent focal length,

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}; \quad \text{or, } \frac{1}{20} = \frac{1}{f_1} - \frac{1}{2f_1} = \frac{1}{2f_1}$$

$$\therefore f_1 = +10 \text{ cm}$$

$$\text{and } f_2 = -2f_1 = -20 \text{ cm}$$

**Example 25.5.** The dispersive power of crown and flint glasses are 0.01 and 0.02 and their refractive indices are  $3/2$  and  $5/2$  respectively. Calculate the focal lengths and radii of curvature of the lenses required to design an achromatic doublet of focal length 100 cm with these glasses.

**Soln.**

Let  $f_1$  and  $f_2$  be the focal lengths of the crown and flint glass respectively; the corresponding dispersive powers being  $\omega_1$  and  $\omega_2$ .

From condition of achromatism,

$$\frac{\omega_1}{f_1} + \frac{\omega_2}{f_2} = 0; \quad \frac{f_2}{f_1} = - \frac{\omega_2}{\omega_1} = - \frac{0.02}{0.01}$$

$$\text{or, } f_2 = -2f_1$$

Equivalent focal length,

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}; \quad \text{or, } \frac{1}{100} = \frac{1}{f_1} - \frac{1}{2f_1} = \frac{1}{2f_1}$$

$$\text{or, } 2f_1 = 100; f_1 = +50 \text{ cm}$$



$$\therefore f_2 = -2f_1 = -100 \text{ cm.}$$

Achromatic doublet means that the crown glass lens is double convex and the flint glass lens is plano-concave. Further the second surface of the convex lens and the concave surface of the plano-concave lens are in contact with each other and the radii of curvature of these surfaces are same.

For the flint glass lens we have  $R_2' = \infty$ ,

$$\mu_2 = \frac{5}{3}, \quad f_2 = -100 \text{ cm.}$$

If  $R_1'$  is the radius of curvature of the first surface of the lens, then

$$\frac{1}{f_2} = (\mu_2 - 1) \left( \frac{1}{R_1'} - \frac{1}{R_2'} \right)$$

$$\text{or, } -\frac{1}{100} = \left( \frac{5}{3} - 1 \right) \frac{1}{R_1'} = \frac{2}{3R_1'}$$

$$\text{or, } R_1' = -66.66$$

Now for the crown glass lens,

$$R_2 = R_1' = -\frac{200}{3} \text{ cm, } \mu_1 = \frac{3}{2}$$

$$f_1 = +50$$

$$\frac{1}{f_1} = (\mu_1 - 1) \left( \frac{1}{R_1} - \frac{1}{R_2} \right)$$

$$\text{or, } \frac{1}{50} = \left( \frac{3}{2} - 1 \right) \left( \frac{1}{R_1} + \frac{3}{200} \right)$$

$$= \frac{1}{2R_1} + \frac{3}{400}$$

whence  $R_1 = +40 \text{ cm.}$

Thus the required condition for the design of an achromatic doublet is

crown glass lens :  $f_1 = + 50$  cm,

$R_1 = + 40$  cm,  $R_2 = - 66.66$  cm.

flint glass lens :  $f_2 = - 100$  cm,

$R_1' = - 66.66$  cm,  $R_2' = \infty$ .

**Example 25.6.** A converging achromat of 60 cm focal length is to be constructed out of a thin crown glass lens, the surfaces in contact having a common radius of curvature of 40 cm. If the refractive indices and dispersive powers are 1.5 and 0.018 for crown glass, and 1.6 and 0.036 for flint glass, what will be the radius of curvature of the other surface of each lens?

**Soln.**

For crown glass lens :  $\omega_1 = 0.018$ ,  $\mu_1 = 1.5$ ,  $f_1 = ?$   $R_2 = - 40$  cm.

For flint glass lens :  $\omega_2 = 0.036$ ,  $\mu_2 = 1.6$ ,  $f_2 = ?$   $R_1' = - 40$  cm.

Now

$$\frac{\omega_1}{f_1} + \frac{\omega_2}{f_2} = 0;$$

$$\text{or, } \frac{f_2}{f_1} = - \frac{\omega_2}{\omega_1} = - \frac{0.036}{0.018} = - 2$$

$$\text{or, } f_2 = - 2f_1$$

Again

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}; \quad \text{or, } \frac{1}{6} = \frac{1}{f_1} - \frac{1}{2f_1} = \frac{1}{2f_1}$$

$$\text{or, } 2f_1 = 60 \text{ cm; or, } f_1 = + 30 \text{ cm}$$

$$f_2 = - 2f_1 = - 60 \text{ cm}$$

From

$$\frac{1}{f_1} = (\mu_1 - 1) \left( \frac{1}{R_1} - \frac{1}{R_2} \right)$$

$$\frac{1}{30} = (1.5 - 1) \left( \frac{1}{R_1} + \frac{1}{40} \right) = \frac{1}{2R_1} + \frac{1}{80}$$

$$\text{or, } R_1 = + 24 \text{ cm}$$

$$\text{and } \frac{1}{f_2} = (\mu_2 - 1) \left( \frac{1}{R'_1} - \frac{1}{R'_2} \right)$$

$$\begin{aligned} \text{or, } -\frac{1}{60} &= (1.6 - 1) \left( -\frac{1}{40} - \frac{1}{R'_2} \right) \\ &= - \left( \frac{6}{400} + \frac{6}{10R'_2} \right) \end{aligned}$$

$$\text{or, } R'_2 = + 360 \text{ cm}$$

$$R_1 = + 24 \text{ cm, } R'_2 = + 360 \text{ cm.}$$

**Example 25.7.** Two convex lenses of focal length 12 cm and 4 cm and of the same material are placed at a certain distance apart so as to satisfy the condition for minimum spherical aberration. Is the combination achromatic?

**Soln.**

Let the distance of separation be  $d$ . For minimum spherical aberration,  $d = f_1 - f_2 = 12 - 4 = 8 \text{ cm}$

Since the lenses are of the same material, for the combination to be achromatic,

$$d = \frac{1}{2} (f_1 + f_2) = \frac{1}{2} (12 + 4) = 8 \text{ cm}$$

Hence the combination is achromatic.

**Example 25.8.** Two thin lenses of focal lengths  $f_1$  and  $f_2$  separated by a distance  $d$  have an equivalent focal length of 50 cm. The combination satisfies the conditions for minimum spherical



aberration and is also achromatic. Find the values of  $f_1$ ,  $f_2$  and  $d$ . Assume that both the lenses are of the same material.

**Soln.**

The equivalent focal length of the combination

$$f = \frac{f_1 f_2}{f_1 + f_2 - d}; \quad \text{or, } 50 = \frac{f_1 f_2}{f_1 + f_2 - d} \quad (i)$$

$$\text{For achromatism, } d = \frac{f_1 + f_2}{2} \quad (ii)$$

For minimum spherical aberration,

$$d = f_1 - f_2 \quad (iii)$$

From (ii) and (iii)

$$f_1 = \frac{3d}{2} \quad \text{and} \quad f_2 = \frac{d}{2}$$

Substituting these values of  $f_1$  and  $f_2$  in equation (i).

$$50 = \frac{\frac{3d}{2} \times \frac{d}{2}}{\frac{3d}{2} + \frac{d}{2} - d} = \frac{3d}{4}$$

$$\text{or, } d = \frac{200}{3} = 66.67 \text{ cm}$$

$$\therefore f_1 = \frac{3d}{2} = 100 \text{ cm and } f_2 = \frac{d}{2} = 33.33 \text{ cm.}$$

**Example 25.9.** Find the ratio of two radii of curvature of a crossed lens to exhibit minimum spherical aberration. Given  $\mu = 1.6$ .

**Soln.**

The ratio  $R_1/R_2$  for spherical aberration to be minimum is given by the relation

$$k = \frac{R_1}{R_2} = \frac{2\mu^2 - \mu - 4}{\mu(2\mu + 1)}$$

$$\begin{aligned}
 &= \frac{2 \times (1.6)^2 - 1.6 - 4}{1.6 (2 \times 1.6 + 1)} \\
 &= \frac{1.6 (3.2 - 1) - 4}{1.6 \times 4.2} \\
 &= \frac{1.6 \times 2.2 - 4}{6.72} \\
 &= \frac{3.52 - 4}{6.72} \\
 &= -\frac{0.48}{6.72} = -\frac{1}{16}
 \end{aligned}$$

Hence  $k = -1/16$ .

**Example 25.10.** If two convex lenses made of same glass and of focal lengths 32 cm and 20 cm are to be used to exhibit minimum spherical aberration ; find the distance between the two. If  $\mu = 1.5$ , find the radii of curvature of the lens surfaces. One surface of each lens is plane.

**Soln.**

For minimum spherical aberration, we have

$$\begin{aligned}
 d &= f_1 - f_2 \\
 &= 32 - 20 = 12 \text{ cm.}
 \end{aligned}$$

From the relation

$$\frac{1}{f} = (\mu - 1) \left( \frac{1}{R_1} - \frac{1}{R_2} \right)$$

we get, for a plano-convex lens

$$\frac{1}{32} = (1.5 - 1) \frac{1}{R_1}$$

( $R_2 = \infty$  for a plano-convex lens)

$$= \frac{1}{2R_1}$$

or,  $R_1 = 16 \text{ cm.}$

Similarly for the second lens,

$$\begin{aligned}\frac{1}{20} &= (1.5 - 1) \frac{1}{R'_1} \\ &= \frac{1}{2R'_1}\end{aligned}$$

or,  $R'_1 = 10 \text{ cm.}$

**Example 25.11.** Find the focal length of an equi-curved lens to reduce curvature of field to zero. The refractive indices of the two lenses are 1.5 and 1.65 respectively and the focal length of the first lens is 33 cm.

**Soln.**

Petzval condition for zero curvature of field is given by

$$\frac{f_1}{f_2} = - \frac{\mu_2}{\mu_1}$$

Hence  $\mu_1 = 1.5$ ,  $\mu_2 = 1.65$ ,  $f_1 = 33 \text{ cm}$

$$\therefore f_2 = - \frac{\mu_1}{\mu_2} \cdot f_1 = - \frac{1.5 \times 33}{1.65}$$

$$\text{or, } f_2 = - 1.5 \times 20 = - 30 \text{ cm.}$$

Thus a combination of convex lens of focal length 33 cm has to be made with a concave lens of 30 cm focal length to reduce curvature of field to zero. Evidently such a combination shall be a divergent combination.

### EXERCISES

- [1] Explain with suitable diagrams what you mean by spherical aberration. Define the terms longitudinal and lateral spherical aberrations and circle of least confusion.
- [2] Explain the causes of spherical aberration fully and the methods to minimize it.



- [3] Outline the simple methods used for correction of spherical aberration in the case of an ordinary lens.
- [4] What is spherical aberration? How is it minimized when two thin lenses are placed at a distance from each other?
- [5] Describe astigmatism coma, curvature and distortion. How they may be reduced to a minimum?
- [6] What is an aplanatic lens? Outline the property of an aplanatic lens and the condition that must be satisfied for the lens to be aplanatic.
- [7] Explain what is dispersive power of a substance.

Deduce the relation,  $\omega = \frac{d\mu}{\mu - 1}$

- [8] Explain what is meant by chromatic aberration in lenses. Describe, with suitable diagrams, the lateral and longitudinal chromatic aberration. Derive an expression for the axial chromatic aberration for a thin lens.
- [9] What do you understand by the term achromatism?  
Derive and discuss the condition of achromatism for two thin lenses of focal lengths  $f_1$  and  $f_2$  (a) when they are made of different materials but placed in contact and (b) when they are made of same material but separated by a distance  $d$ .
- [10] What do you understand by the term achromatism? Derive and discuss the condition of achromatism for two thin lenses in contact.
- [11] Describe an achromatic doublet and explain its principle.
- [12] Describe and explain chromatic aberration. Deduce a condition for achromatism of two lenses separated by a distance.  
Explain what is meant by an achromatic system.
- [13] Write a brief essay on aberrations of optical image.
- [14] What are the two main defects of optical images as formed by ordinary lenses? How will you achromatise a system of two thin lenses (a) of different materials and (b) of the same materials. Is such a combination of lenses truly achromatic?
- [15] Derive a condition for minimum spherical aberration and for achromatism of two thin co-axial lenses placed in air a certain distance apart.

- [16] The focal lengths of a thin convex lens are 100 cm and 96.8 cm for red and blue rays respectively. Calculate the dispersive power of the material of the lens. (0.0325)

- [17] Calculate the dispersive powers for crown and flint glass from the following data :

	Red	Yellow	Blue
crown	1.5145	1.5170	1.55230
flint	1.6444	1.6520	1.6637

(Crown : 0.01644; flint : 0.02961)

- [18] The refractive indices of two kinds of glass for violet and red rays are as follows :

For glass A

$$\mu_v = 1.523$$

$$\mu_r = 1.517$$

For glass B

$$\mu_v' = 1.664$$

$$\mu_r' = 1.650$$

What should be the focal length of two lenses made of these glasses which will form an achromatic doublet of focal length 50 cm?

(glass A = + 22.93 cm, glass B = - 42.93 cm)

- [19] An achromatic converging combination of focal length 60 cm is formed with a convex lens of crown glass and a concave lens of flint glass placed in contact. Calculate their focal lengths if the dispersive power of crown glass is 0.03 and that of flint glass is 0.05. (crown glass : + 24 cm; flint glass = - 40 cm).

- [20] It is desired to make a converging achromatic lens of mean focal length 20 cm by using two lenses of materials A and B. If the dispersive powers of A and B are in the ratio of 2:3, find the focal length of each lens. (A = + 6.67 cm, B = - 10 cm).

- [21] The object glass of a telescope is made of a convex lens of crown glass (dispersive power 0.012) and a concave lens of flint glass (dispersive power 0.020). If the focal length of the object glass is 30 cm, calculate the focal lengths of the component lenses. (crown glass = + 12 cm, flint glass = - 20 cm).

- [22] A thin crown glass lens is in contact with a thin flint glass lens, the radius of curvature of the common surface being 25 cm. The combination forms an achromatic combination of 40 cm focal length, find the radii of curvature of the other face of the two lenses.



$\mu$  for crown glass = 1.50;  $\mu$  for flint glass = 1.60

dispersive power of crown glass = 0.021,

dispersive power of flint glass = 0.045,

(crown glass :  $f_1 = + 21.33$ ,  $R_1 = +18.5$  cm,  $R_2 = - 25$  cm,  
flint glass :  $f_2 = - 45.7$  cm,  $R_1' = - 25$  cm,  $R_2' = 286$  cm.)

- [23] A convex lens of crown glass and a concave lens of flint glass are combined to form an achromatic doublet of focal length 80 cm. The dispersive powers of crown and flint glass are 0.015 and 0.030 respectively, their respective refractive indices being 1.52 and 1.65. Calculate the focal lengths and the radii of curvature of both the surfaces of the lens. (crown glass :  $f_1 = + 40$  cm,  $R_1 = + 34.67$  cm,  $R_2 = - 52$  cm, flint glass :  $f_2 = - 80$  cm,  $R_1' = - 52$  cm,  $R_2' = \infty$ ).
- [24] A converging achromat of 40 cm focal length is to be constructed out of a thin crown glass lens and a thin flint glass lens, the surfaces in contact having a common radius of curvature of 25 cm. Calculate the radius of curvature of the other surface of each lens, given that the values of the dispersive powers and refractive indices are 0.017 and 1.5 for crown glass, and 0.034 and 1.7 for flint glass. (crown glass :  $f_1 = + 20$  cm,  $R_1 = + 16.67$  cm,  $R_2 = - 25$  cm, flint glass :  $f_2 = - 40$  cm,  $R_1' = - 25$  cm,  $R_2' = - 233.33$  cm).
- [25] At what distance apart should two lenses of focal lengths 24 cm and 8 cm be placed so as to form an achromatic combination? Assume the lenses to be of the same material. Will the combination also satisfy the condition for minimum spherical aberration? (16 cm; yes, the combination satisfies the condition of minimum spherical aberration).
- [26] Find the value of shape factor to reduce spherical aberration to minimum for  $\mu = 1.686$ ,  $\mu = 1.66$  and  $\mu = 1.5$ . [0, -1/6.5, -1/6]
- [27] The focal length of a lens of refractive index  $\mu_1 = 1.70$  is 30 cm. Find the focal length of another lens of refractive index  $\mu_2 = 1.5$  to form a doublet to reduce curvature of field. [ $f_2 = - 34$  cm].
- [28] A convergent doublet of separated lenses, corrected for spherical aberration, has an equivalent focal length of 10 cm. The lenses of the doublet are separated by 2 cm. What are the focal lengths of the component lenses ? [18 cm, 20 cm].



## CHAPTER XXVI

## OPTICAL INSTRUMENTS

Photographic camera - Visual angle - Simple microscope - Compound microscope - Entrance and exit pupil of a compound microscope - Microscope objective - Eye-pieces or oculars - Ramsden eye-piece - Huygens' eye-piece - Comparison of Ramsden eye-piece and Huygens' eye-piece - Binocular microscope - Telescope - Astronomical telescope - Terrestrial telescope - Reflecting telescope - The prism binocular - The spectrometer - Solved problems - Exercises.

## 26.1 Photographic camera

The term camera comes from the latin phrase *camera obscura* or dark chamber, for all picture taking instruments have a dark chamber to protect the sensitive film from light.

A photographic camera essentially consists of the following parts (Fig. 26.1)

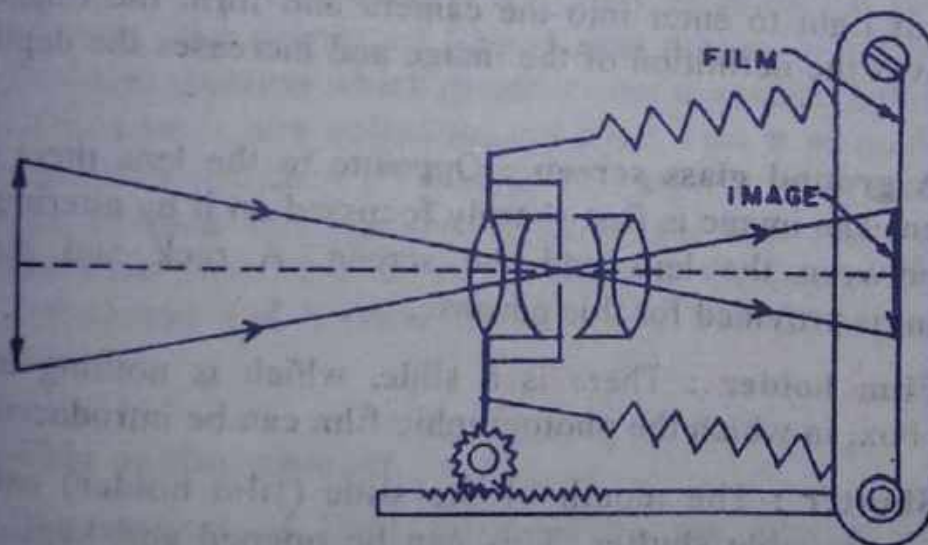


Fig. 26.1

(i) **A light-tight box :** The inside of this box is painted black to absorb the light which falls on it. The sides of the box are made

of folded leather so that the distance between the lens and the screen can be altered.

(ii) **The camera objective :** In front of the box, there is a lens known as the objective. Usually the objective consists of two achromatic doublets separated by a distance with a diaphragm in between them. The combination is equivalent to a single converging lens. For the sake of clear and well-defined image the objective is called upon to cover a field of more than  $50^\circ$ . Furthermore, the relative aperture of the lens must be large in order that it may collect sufficient light to permit short exposure. The problem of correcting a photographic lens for different aberrations becomes a difficult one because of this combination of wide field and large apertures. Never the less, even the simplest photographic lens are free from chromatic aberration and curvature. More sophisticated lenses, such as the *astigmatic lens*, are corrected for spherical and chromatic aberrations, coma, astigmatism, curvature and distortion.

(iii) **Diaphragm :** The diaphragm is an adjustable aperture, that regulates the intensity of the image by permitting different quantities of light to enter into the camera and form the image. It also improves the definition of the image and increases the depth of the field.

(iv) **A ground glass screen :** Opposite to the lens there is a glass screen. The image is first sharply focussed on it by altering the distance between the lens and the screen. A rack and pinion arrangement is provided for this purpose.

(v) **Film holder :** There is a slide, which is nothing but a light-tight box, in which the photographic film can be introduced.

(vi) **Shutter :** The mouth of the slide (film holder) can be closed by a movable shutter. This can be opened and kept open for any interval to allow light from the subject to enter the camera and strike the film for a given time. Lens shutters are usually placed between or just behind the elements of the lens and usually have a set of leaves that snap open for a desired time and then snap shut.



## Action

The camera is loaded with the sensitive film with the shutter closed. The sensitive film consists of a film of celluloid coated with silver halide (bromide) not exposed to light. To take a photograph the object is first sharply focussed on the film by adjusting the lens-film distance. An appropriate exposure is then given to the film by keeping the shutter open for the desired length of time. A mechanism opens the shutter and closes it automatically after the desired length of time. This time depends upon the brightness of the object, the sensitivity of the photographic emulsion, the focal length of the lens and the size of the stop. Typical values of this time are from 10 seconds to  $1/1000$  of a second, so that (a) the film receives just the amount of light required to produce a satisfactory image, and (b) moving objects produce reasonably sharp images. Some cameras take pictures at a millionth of a second enabling man to see the unseeable.

After the exposure, the exposed film is taken out of the camera and dipped in a *developer* (a solution of potassium ferrous oxalate) in a dark room whereby the silver halide is reduced to metallic silver. The degree of reduction is exactly proportional to the degree of exposure. The developed film is now dipped in a *hypo* (sodium thiosulphate) solution which dissolves out unreacted halide from the film. This film is now called the *negative*. This is washed with clear running water and then allowed to dry. By allowing a strong beam of light to pass through the negative and fall upon a sensitive photographic paper, a *print* is obtained on the paper. The paper is also developed and washed as the film. The photograph is now prepared.

## *f*-number or stop number

The amount of light falling upon the film is proportional directly to the area or square of the lens aperture and inversely to the square of the object distance. The objects to be photographed are typically at large distances compared to the focal length of the lens. The images are, therefore, formed always close to the focal point. Thus, for large distances used in photography, the object distances is practically proportional to focal length  $f$ . The quantity of light reaching the film is controlled by adjustment of the diaphragm or



stop, which is used to mask off outer portions of the lens. Hence, if  $d$  is the lens aperture i.e., the diameter of the effective opening of the lens, then the quantity of light falling on the film per unit area per second i.e., the *illuminance or brightness* of the image formed is proportional to  $d^2/f^2$ . The time of exposure should be inversely proportional to this. Hence time of exposure is proportional to  $f^2/d^2$ .

The diameter  $d$  of the lens (or of the stop) is usually expressed as a fraction of the focal length  $f$  of the lens system and the ratio  $f/d$ , i.e.,

$$\frac{\text{focal length}}{\text{diameter of the stop}}$$
 is called the *f-ratio* or *f-number* or *stop*

*number (n)*. The ratio  $d/f$  is called the *relative aperture* of the lens. The aperture is usually expressed by its *f-number*. If the aperture is *f-8*, this means that the diameter  $d$  of the aperture is  $f/8$  where  $f$  is the focal length of the lens. Alternately if the focal length of the lens is 11 cm and the diameter of the stop is 1 cm, then the *f-number* or stop number is 11 and  $d = \frac{f}{11}$ .

The illuminance of the image is, therefore,  $1/(\text{f-number})^2$ . Since this result is independent of the focal length of the lens, a photographer can properly adjust the quantity of light striking the film by always using the same *f-number* for a given light level on the object, regardless of which particular camera he is using. The *f-number* or stop number of a camera lens is controlled by adjusting the diaphragm or the stop, which limits the active area of the lens. This adjustment is usually calibrated in terms of *f-numbers*. Each step changes the *f-number* by a factor of about  $\sqrt{2}$ , so that it changes the illuminance of the film by a factor 2. The exposure time is, therefore, doubled between each position e.g.,

$$\frac{f}{2}, \frac{f}{2.8}, \frac{f}{4}, \frac{f}{5.6}, \frac{f}{8}, \frac{f}{11}, \frac{f}{16}, \frac{f}{22}$$

Thus the *f-numbers* are 2, 2.8, 4, 5.6, 8, 11, 16, 22 etc. and their squares are in the ratio of 1 : 2 : 4 : 8 : 16. Hence, if the correct exposure time for  $\frac{f}{5.6}$  is one second, then for  $\frac{f}{8}$  it is 2 seconds, for

$\frac{f}{11}$ , 4 seconds and so on. The actual  $f$ -number marking on a camera lens are rounded off to two figures. As a result, the ratio of successive  $f$ -numbers are not exactly equal to the square root of 2.

### Typical $f$ -number markings on a camera

increasing illuminance	$f$ -numbers	of $f$ -number (ideally $\sqrt{2} = 1.414$ )
	$f-16$	$16/11 = 1.45$
	$f-11$	$11/8 = 1.38$
	$f-8$	$8/5.6 = 1.43$
	$f-5.6$	$5.6/4 = 1.40$
	$f-4.0$	$4/2.8 = 1.43$
	$f-2.8$	$2.8/2.0 = 1.40$
	$f-2.0$	

### Depth of focus

When a lens is focussed for a certain distance, object points at that distance only are imaged with maximum sharpness. Image of points at other distances from the lens appears as blurred circles, termed *circles of confusion*. The greater is the distance of the point from the plane focussed on, the greater will be the size of the circle of confusion. If the circle of confusion is below a certain size, it appears to the eye as a point and the image appears sharp. Thus there is a range of distance on the rear and far sides of the plane focussed upon, within which the details are imaged with acceptable sharpness i.e., the circles of confusion of points within this range are not so large for the image to be considered unsatisfactory. This range is known as the *depth of focus* and is of particular importance in photography, since a camera is sometimes required to take a photograph of an object extending over a distance from the camera.

Let  $r$  be the radius of the exit pupil of the coaxial optical system (camera lens system) and  $P$  the point image at a distance  $x$  from the lens system (Fig. 26.2)



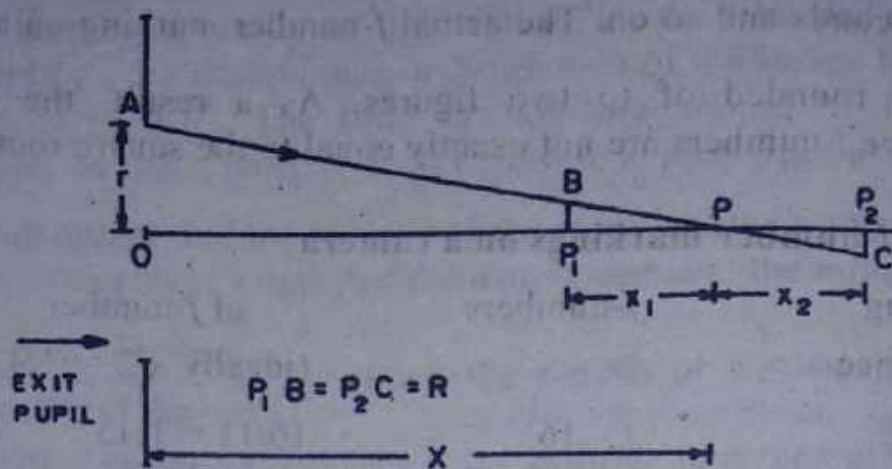


Fig. 26.2

When the film is moved forward by a distance  $x_1$  without causing any blurring of image, the ray passing through the point A on the circumference of the exit pupil cuts the plane passing through  $P_1$  and normal to the axis at the point B. When the film is moved backward by a distance  $x_2$  again without causing any blurring of the image, the ray passing through the point A cuts the plane passing through  $P_2$  and normal to the axis at the point C. The location of the point  $P_2$  is such that  $x_1 = x_2$ .

Let  $P_1B = P_2C = R$  (say),

Triangles  $PP_1B$  and  $POA$  are similar

$$\frac{P_1B}{OA} = \frac{PP_1}{OP} \quad \text{or,} \quad \frac{R}{r} = \frac{x_1}{x}$$

$$\therefore x_1 = \frac{R \cdot x}{r} \quad \text{but,} \quad x_2 = \frac{R \cdot x}{r} = x_1$$

Hence, depth of focus,

$$P_1P_2 = x_1 + x_2 = \frac{2Rx}{r}$$

Determination of the depth of focus in a camera is of great importance. It helps in estimating the amount by which the film in the camera may buckle without disturbing the image beyond the tolerable limit. Of course, the increased depth of field is gained at a



cost of decreased illumination of the image on the film. The exposure time must be increased.

### Telephoto lens

Small object, such as animals and birds, may have to be photographed sometimes from a large distance, since they are likely to be disturbed if a cameraman approaches them too closely. On the other hand, if such objects are not photographed from a close range, their photo-images, as obtained with a normal camera lens, are very small with likely loss of details.

Magnification by a lens is given by the relation  $m = f/(u + f)$ . Thus, the image size of a distant object is directly proportional to this focal length of the converging camera lens. Obviously, to obtain a large photo image size of a remote object, a convex lens of large focal length should be used, i.e., the distance between the lens and the film should be large. Greater the lens focal length, larger will the image size be. Unfortunately, large focal length will require a very large size of the camera box which may be unmanageable and inconvenient and very few cameras have bellows capable of large extension. To avoid this difficulty, Taylor suggested the use of a *telephoto lens*, which increases the effective focal length without increasing the size of the box.

The telephoto lens consists of two achromatic lenses – one convex and one concave, mounted in a tube (Fig. 26.3). The tube is fixed in the camera in such a manner that the concave lens  $L_2$  occupies the

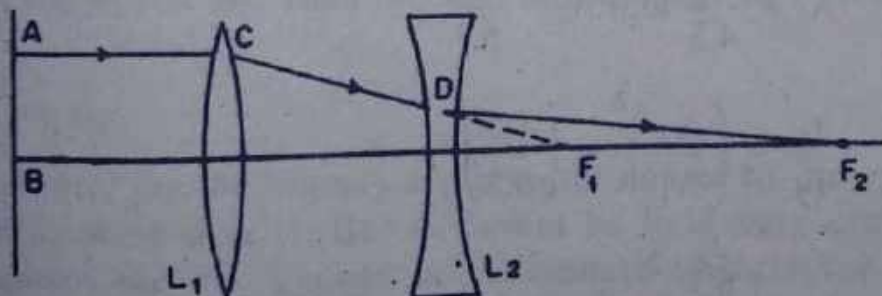


Fig. 26.3

original position of the lens in the camera, whereas the convex lens  $L_1$  is placed in front of the lens  $L_2$ . With camera lens  $L_1$  alone the rays from a distance object (parallel rays) would have come to focus at  $F_1$  – the (second) principal focus of  $L_1$  where photo film would have been placed. However,  $L_2$  is so placed that  $F_1$  acts as a virtual object for it. The rays are then brought to focus at a farther point  $F_2$ . The whole combination then becomes equivalent to a single lens placed along the plane  $AB$  where the incident and final convergent rays meet each other.

Its focal length  $BF_2$  is much larger than the focal length of the original camera lens  $L_1$ . If  $d$  is the distance between the two lenses  $L_1$  and  $L_2$  then the equivalent focal length  $f(BF_2)$  is given by

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}$$

$f_1$  and  $f_2$  being the respective focal lengths of  $L_1$  and  $L_2$ . Due allowance of sign should be made while calculating  $f$ .

**Example 26.1.** Under certain conditions the correct film exposure time is  $\frac{1}{20}$  sec with a lens speed of  $f/4.5$ . What is the correct exposure time when the lens is diaphragmed to  $f/6.3$ ?

**Soln.**

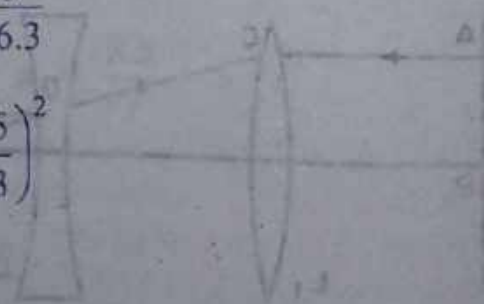
The apertures are

$$A_1 = \frac{f}{4.5} \text{ and } A_2 = \frac{f}{6.3}$$

$$\frac{t_1}{t_2} = \left( \frac{A_2}{A_1} \right)^2 = \left( \frac{4.5}{6.3} \right)^2$$

$$\therefore t_2 = \left( \frac{6.3}{4.5} \right)^2 \times t_1$$

$$\left( \frac{6.3}{4.5} \right)^2 \times \frac{1}{20} = \frac{1}{10} \text{ sec.}$$





## 26.2 Visual angle

The angle subtended by the object at the human eye is called the *visual angle*. Let two objects  $A_1$  and  $A_2$  of different sizes be placed in front of the eye (Fig. 26.4). Both subtend equal angles at the

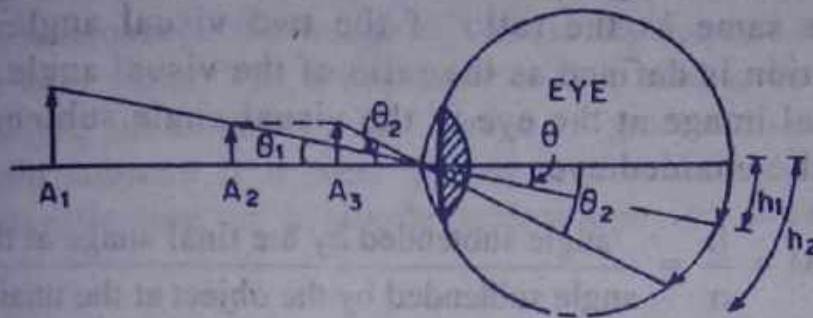


Fig. 26.4

eye and hence their images formed on the retina of the eye are of the same size, although they themselves are of different sizes. If  $k$  is the distance between the retina and the lens, then the size  $h_1$  of the image is given by

$$\frac{h_1}{k} = \theta_1 ; \quad \text{or, } h_1 = k\theta_1$$

$$\text{or, } h_1 \propto \theta_1$$

Thus the size of the image formed on the retina is proportional to  $\theta$ , the angle subtended by the object at the eye. Suppose the object  $A_1$  is brought nearer to the eye to a position  $A_3$  and let the angle subtended at the eye be  $\theta_2$ . Then the size of the image  $h_2$  formed on the retina is

$$h_2 = k \theta_2$$

Consequently, as the objects are brought nearer to the eye, it can be seen in more details. But it cannot be held very close to the eye because the eye cannot accommodate objects for very short distance. This difficulty is overcome by using *telescopes* and *microscopes* which project larger images on the retina than would be seen by the unaided eye. These instruments are designed to increase the visual angle so that the object viewed can be made to appear large with their aid. However, as the



telescope image is usually far smaller than the actual object being observed, linear magnification is not a good index of the value of the telescope. Rather, the comparison to be made is between the size of the image formed at the eye with the telescope and that formed at the unaided eye *i.e.*, without the telescope. The ratio of the two retinal image sizes is called the *angular magnification* and is the same as the ratio of the two visual angles. Angular magnification is defined as the ratio of the visual angle subtended by the final image at the eye to the visual angle subtended by the object at the unaided eye.

$$M = \frac{\beta}{\alpha} = \frac{\text{angle subtended by the final image at the eye}}{\text{angle subtended by the object at the unaided eye}}$$

*It is to be remembered that the visual angles subtended by the object and the image are more important in the theory of optical instruments. The sizes of the object and the image are of less significance.*

### 26.3 Simple microscope

A simple microscope [Fig. 26.5 (i)] is simply a convex lens mounted upon a frame with a handle. This is more popularly referred to as a *magnifying glass*. The object *AB* to be viewed is placed

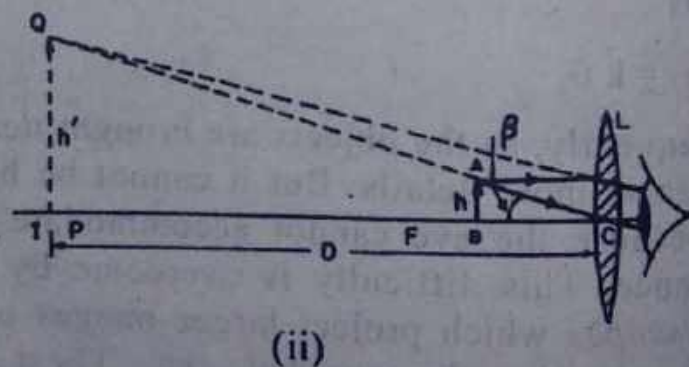
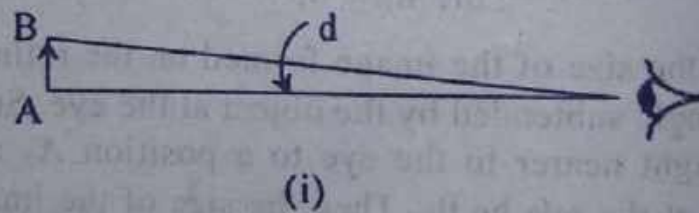


Fig. 26.5

within the focal length of the lens. Then a virtual erect and magnified image  $PQ$  of the object can be seen by placing the eye nearer the lens on the other side. The apparent size of an object depends upon the angle subtended by it at the eye. So the nearer is an object, the larger is its apparent size. But the object cannot be seen clearly if it lies nearer than the *least distance of distinct vision*. In a simple microscope the image  $PQ$  is usually formed at this least distance or *near point*. The microscope is then said to be in *normal use*. The angle subtended by the image is greater than what the object would subtend if it were placed at this distance. Thus the angular magnification of a simple microscope in normal use is given by

$$M = \frac{\text{angle subtended at the eye by the image}}{\text{angle subtended at the unaided eye by the object placed at the near point}} \\ = \beta/\alpha$$

Suppose the object  $AB$  of length  $h$  is placed at the near point  $A$  and viewed by the eye [Fig. 26.5(ii)]. Then the visual angle is  $\alpha$  and is given by

$$\tan \alpha = \frac{h}{D}$$

where  $D$  is the distance of distinct vision. If the object be now placed between the optical centre  $O$  and principal focus  $F$  of a lens  $L$  of focal length  $f$ , then a magnified, erect and virtual image  $PQ$  will be formed. If the observer's eye is placed close to the lens on the other side, then the distance  $OP$  can be taken as equal to the least distance of distinct vision. Hence the visual angle  $\beta$  subtended by the eye is given by

$$\tan \beta = \frac{PQ}{D}$$

For small angles  $\tan \alpha = \alpha$  and  $\tan \beta = \beta$ . Hence the magnification is given by

$$M = \frac{\beta}{\alpha} = \frac{PQ}{h}$$



But  $PQ$  and  $h$  are the heights of the image and the object respectively. Thus

$$M = \frac{PQ}{h} = \frac{v}{u} = \frac{D}{u}$$

From the lens equation, we have

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

But  $v = -D$  and  $u$  is also  $-ve$ ,

$$\therefore -\frac{1}{D} + \frac{1}{u} = \frac{1}{f}$$

$$\text{or, } -1 + \frac{D}{u} = \frac{D}{f}$$

$$\text{or, } \frac{D}{u} = 1 + \frac{D}{f}$$

$$\text{or, } M = 1 + \frac{D}{f} \quad (26.1)$$

#### 26.4 Compound microscope

The magnifying power of a simple microscope is given by

$$M = 1 + \frac{D}{f}. \text{ Hence it can be increased by decreasing the focal}$$

length. However, due to constructional difficulties, the focal length of a lens cannot be decreased beyond a certain limit. Moreover, a lens of small focal length has also a small aperture (diameter), since the curvature of the surface is large. This results in a small field of view.

In the compound microscope, invented by Galileo in 1610, the magnifying power is increased by using two separate lenses instead of a single lens. The lens near the object is called the *objective* and the lens nearer the eye is known as the *eye-piece*. Fig. 26.6 illustrates the optical parts of a compound microscope. The focal length of the objective is very small while that of the eye-piece is some-what large. In the figure, the objective as well as the eye-piece



are shown as single lens but in reality they are highly corrected compound lenses; the former is specially corrected for spherical aberration, coma and chromatic aberration for object positions at a very close distance. The actual microscope objective and eye-piece are described in detail in Arts. 26.6 and 26.7 respectively.

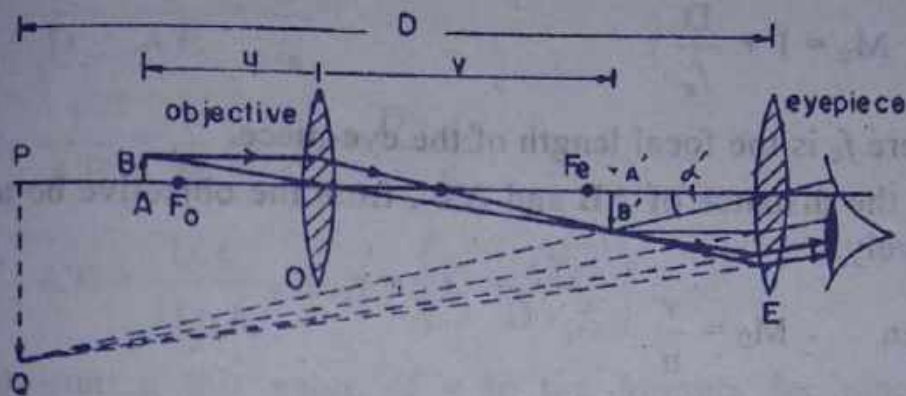


Fig. 26.6

In the actual arrangement, a small object  $AB$  is placed just beyond the first principal focus  $F_0$  of the objective. A real, inverted and magnified image is formed at  $A'$ . The eye-piece is then adjusted so that the distance of  $A'$  from it is less than its focal length. The eye-piece is, therefore, positioned as a simple magnifier with respect to the image  $A'B'$ . As a consequence, the eye-piece forms a large virtual image  $PQ$ . For an eye placed close to and on the other side of the eye-piece, *i.e.*, when the microscope is in normal use, the distance of this final image from the eye-piece can be regarded as the least distance of distinct vision,  $D$ .

The magnifying power of a compound microscope in normal use, like the magnifying power of a simple microscope, is defined as

$$M = \frac{\text{angle subtended at the eye by the final image at } D}{\text{angle subtended at the eye by the object at the same distance } D}$$

where  $D$  is the least distance of distinct vision.

$$\text{or, } M = \frac{PQ/D}{AB/D} = PQ / AB$$

$$\text{or, } M = \frac{PQ}{A'B'} \times \frac{A'B'}{AB}$$

But  $PQ/A'B' = M_e = \text{Magnification of the eye-piece}$   
 and  $A'B'/AB = M_o = \text{Lateral magnification of the objective}$

$$\therefore M = M_e \times M_o$$

Since the eye-piece acts as a simple magnifier,

$$M_e = 1 + \frac{D}{f_e}$$

where  $f_e$  is the focal length of the eye-piece.

Let the distance of  $AB$  and  $A'B'$ , from the objective be  $u$  and  $v$  respectively.

$$\text{Then } M_o = \frac{v}{u}$$

Again from the lens equation we get

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f_o}$$

where  $f_o$  is the focal length of the objective.

Applying sign convention  $u$  is -ve,

$$\therefore \frac{1}{v} + \frac{1}{u} = \frac{1}{f_o}; \quad \text{or, } \frac{v}{u} = \frac{v}{f_o} - 1$$

$$\text{or, } M_o = \frac{v}{u} = \left( \frac{v}{f_o} - 1 \right)$$

$$\therefore M = M_o \times M_e = \frac{v}{u} \left( 1 + \frac{D}{f_e} \right)$$

$$= \left( \frac{v}{f_o} - 1 \right) \left( 1 + \frac{D}{f_e} \right)$$

In the case of most microscopes the objective and the eye-piece are fixed at the two ends of the tube i.e., the distance between them is fixed. The microscope is then focussed by moving the tube bodily either towards or away from the object with the help of a rack and pinion arrangement. Therefore, in the relation for the magnifying power,

$M = (v/f_o - 1) (1 + D/f_e)$  it is necessary to find  $v$  in terms of  $f_o, f_e$  and distance between the objective and the eye-piece.



Suppose, the distance between the objective and the eye-piece =  $OE = L$  and  $v = OA' = OE - A'E = L - A'E$

As  $A'$  acts as a point object for the eye-piece whose image is formed at  $P$ , we have, from the lens equation,

$$\frac{1}{D} - \frac{1}{A'E} = \frac{1}{f_e}$$

$$\text{or, } \frac{1}{A'E} = \frac{1}{D} + \frac{1}{f_e} = \frac{D + f_e}{D \cdot f_e}$$

$$\therefore A'E = \frac{D \cdot f_e}{D + f_e}, \text{ or, } v = \left( L - \frac{D f_e}{D + f_e} \right)$$

Substituting this value of  $v$  in the formula for magnifying power,

$$\begin{aligned} M &= \left( 1 + \frac{D}{f_e} \right) \left( \frac{L - \frac{D f_e}{D + f_e}}{f_o} - 1 \right) \\ &= \frac{(D + f_e) \left( L - f_o - \frac{D f_e}{D + f_e} \right)}{f_o f_e} \\ &= \frac{(DL - D f_o + L f_e - f_e f_o - D f_e)}{f_o f_e} \\ &= \frac{D(L - f_o - f_e) + f_e(L - f_o)}{f_o f_e} \end{aligned} \quad (26.2)$$

From equation (26.2), knowing  $f_e$ ,  $f_o$ ,  $D$  and  $L$ , the magnifying power of the microscope can be calculated. It can further be seen from the equation that to obtain high magnification, the focal lengths of the eye-piece and the object glass should both be small.

**Note :** However, it is better to remember that the total magnification  $M = m_1 \times m_2 = \frac{v}{u} \times \frac{v'}{u'}$ . If  $l$  is the length of the object, then the length of the final image is  $l \times \frac{v}{u} \times \frac{v'}{u'}$



### 26.5 Entrance and exit pupil of a compound microscope

If a diaphragm with a small hole in it is placed over the objective, it will be observed that the image becomes less bright than before but retains the same size. Thus it is the periphery of the objective, which limits the cone of rays entering the microscope and determines the brightness of the image. However, it does not affect the size of the image since the size depends upon the focal length. The lens thus acts as a *stop* to the light from the object. The *clear area* of the objective is called the *entrance pupil*. On the other hand to have a full field of view, all the rays emerging from the microscope must enter the pupil of the eye. If the rays leaving all the points of an object  $AB$  are traced, it is found that all rays after passing through the eye-piece pass through a small circular area. This is called the *exit pupil* or *eye ring* (Fig. 26.7). It is evident from Fig. 26.7, that the eye must be placed at the position of the exit pupil so as to collect as much light that passes through the objective as possible. Actually  $R$  is the image of the objective through the eye-piece and all the emergent rays pass through  $R$ . Moreover, the diameter of the exit pupil should be equal to that of the pupil of the eye.

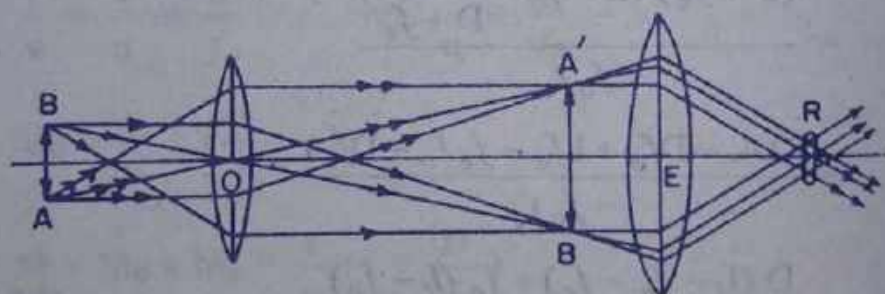


Fig. 26.7

Under this condition, when the eye of the observer coincides with the ring, the whole of the image will be seen clearly and the magnifying power of the microscope is called *normal magnifying power*.

While constructing microscopes, a circular hole slightly greater than the exit pupil is placed behind the eye-piece so that the two coincide. The eye is placed behind the hole to view the image.





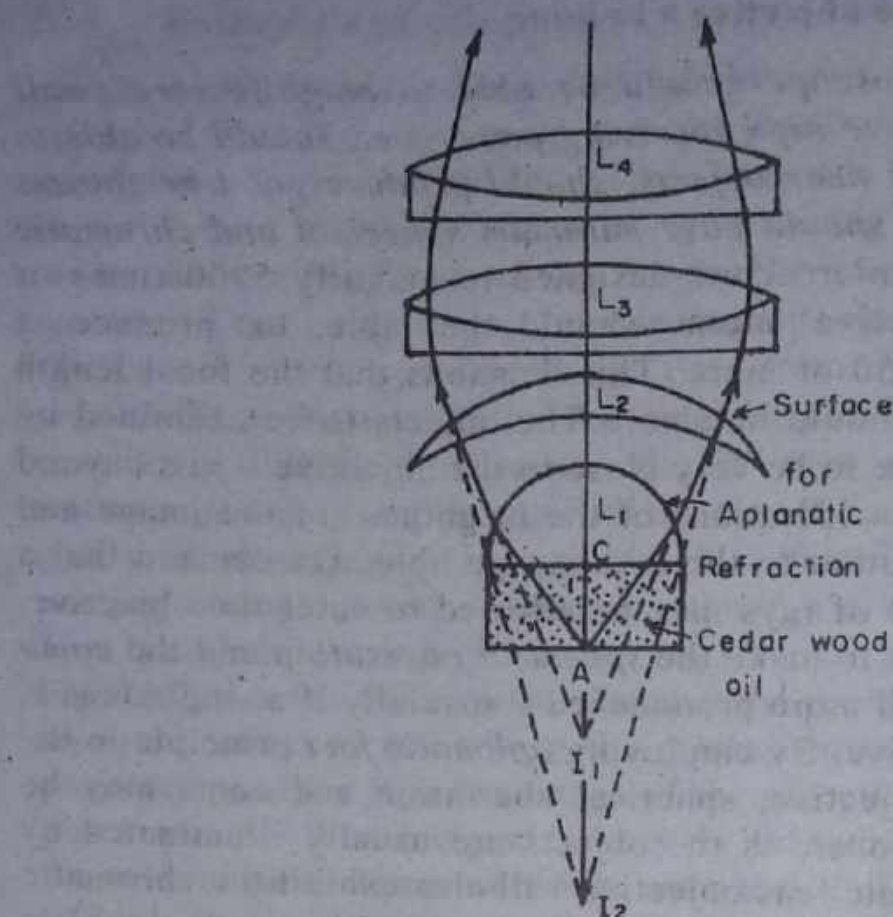


Fig. 26.8

the inclination of the incident rays to the axis, appear to come from the conjugate second aplanatic point  $I_1$  at a distance  $\mu R$  from  $C$ . The image will therefore be free from spherical aberration and the magnification produced is  $\frac{CI_1}{AC} = \frac{\mu R}{R/\mu} = \mu^2$ . The second lens  $L_2$  in the objective is a meniscus lens, the centre of curvature of its first concave surface is  $I_1$ . Therefore, the rays emerging from the hemispherical lens  $L_1$  will fall on this lower concave surface normally and will pass undeviated and fall on the convex surface. The radius of curvature of the convex surface is so selected that  $I_1$  becomes the first aplanatic point for it. The rays after refraction through this second surface, therefore, appear to come from the conjugate aplanatic point  $I_2$ . This gives added magnification without introducing spherical aberration. This property of the lens  $L_2$  holds true only for rays from the point  $A$  and not for points adjacent to it. The rays emerging from the lens  $L_2$  are rendered still less divergent and finally convergent by the other achromatic lenses  $L_3$  and  $L_4$  which also serve to



eliminate the chromatic aberration introduced by the first two lenses. This type of objective is known as *Abbe's homogeneous oil immersion objective*.

The *numerical aperture* of the objective is defined as  $\mu \sin \alpha$  where  $\mu$  is the refractive index of the object space and  $\alpha$  is the semi-vertical angle of the cone of rays entering the objective. Compared with objectives having air in the object space, oil immersion type objectives have, therefore, a large numerical aperture. As a consequence, the *resolving power* of microscope with oil immersion type objective is much better. The resolving power of an optical instrument is its ability to separate two very near objects. A microscope with high resolving power will make the details of an object visible. The minimum distance  $s$  between two elements of an object which can be just separately distinguished, also known as the resolution, is given by

$$s = \frac{1.22 \lambda}{2\mu \sin \alpha} = \frac{1.22 \lambda}{2 \times \text{numerical aperture}}$$

## 26.7 Eye-pieces or oculars

An *eye-piece* or *ocular* is employed to magnify the image already formed and magnified by the lens or lenses (objective) preceding it in a microscope or telescope. The final magnified image, as seen by the eye placed behind the eye-piece must be free from the various aberrations. This is not possible when a single lens is used as the eye-piece. To minimise aberrations and also to increase the field of view, it is convenient to use two lenses separated by a given distance instead of a single lens as an eye-piece. *By field of view we mean the total range of an extended object from which rays of light after refraction through the instrument enter the pupil of the eye.* The most popular eye-pieces consist of two convergent lenses, constructed from the same glass and separated by a distance so chosen that the spherical aberration and chromatic aberration are minimised. The spherical aberration can also be further reduced by adjusting the four radii of curvature of the surfaces of the lenses constituting the eye-piece.

In the eye-piece, the lens which faces the objective is called the *field lens* and its principal function is to enlarge the field of view. This is achieved by simply deviating the cone of rays from each point of the real image, formed by the objective, towards the axis so that these cones pass

through the central region of the *eye-lens*, the lens nearest to the eye. Thus, pupil of the eye placed close to the eye-lens receives every cone of rays simultaneously, thereby making the whole field of view visible simultaneously. The principal function of the eye-lens is to magnify the image formed by the objective.

Two types of eye-pieces commonly used are described below.

### 26.8 Ramsden eye-piece

Ramsden eye-piece consists of two plano-convex lens of the same materials and the same focal length  $f$  placed coaxially at a distance equal to two-thirds of the common focal length (Fig. 26.9).

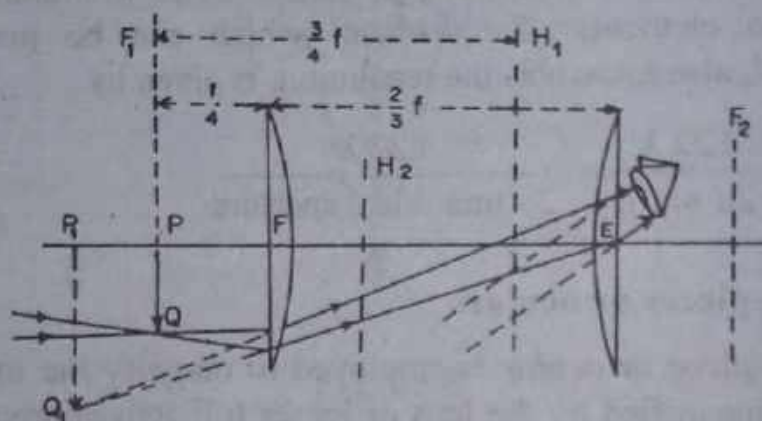


Fig. 26.9

The curved surfaces of the lenses face each other and the eye-piece is placed beyond the image formed by the objective. In this eye-piece cross-wires are provided and it is used in instruments where accurate quantitative measurements are made.

#### Action

The real image PQ of a distant object, formed by the objective (not shown in the figure), serves as a real object for the eye-piece. For normal vision, this image PQ should be located at the first focal plane  $F_1$  of the eye-piece so that the final image formed by the eye-piece may be at infinity. In reality the final image is formed in two steps. In the first step, PQ serves as a real object to the field lens F which forms its virtual image  $P_1Q_1$  at the first plane of the eye-lens E.  $P_1Q_1$  now serves as an object for the eye-lens E and the final image is formed at infinity.



For distinct vision, the objective must form an image PQ within the first focal length  $H_1F_1$  of the equivalent lens  $H_1$  so that the final image may be formed at the least distance of distinct vision. In other words, the image  $P_1Q_1$  formed by the field lens should be situated within the first focal plane of the eye-lens. If any cross-wire or micrometer scale is to be attached for measurement, it should be placed at the focal plane of the objective where the real image of the object is formed.

### Cardinal points of a Ramsden's eye-piece

$$\text{Since } f_1 = f_2 = f \quad \text{and } d = \frac{2}{3} \cdot f.$$

the equivalent focal length  $F$  of the eye-piece is given by

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2} = \frac{1}{f} + \frac{1}{f} - \frac{2/3f}{f^2}$$

$$\text{or, } F = + \frac{3}{4}f$$

The position of the first principal plane of the eye-piece is given by

$$\alpha = + \frac{F.d}{f_2} = + \frac{\frac{3}{4}f \cdot \frac{2}{3}f}{f} = \frac{1}{2}f.$$

Thus, for the final image to be formed at infinity, the position of the equivalent lens must be at a distance of  $\frac{f}{2}$  to the right of (or behind) the field lens. Since the equivalent focal length of the eye-piece is  $\frac{3}{4}f$ , the position of the principal focus of the eye-piece is, therefore, at a distance

$$\frac{3}{4}f - \frac{1}{2}f = \frac{1}{4}f$$

in front of (or left of) the field lens. This is where the objective-image must be formed for the final image to be formed at infinity. As the image formed by the objective lies in front of the field lens. Ramsden eye-piece is often referred to as the *positive eye-piece*.



The second principal point is at a distance  $\beta$  from the eye-lens.

$$\beta = -\frac{F.d}{f_1} = -\frac{\frac{3}{4}f \cdot \frac{2}{3}f}{f} = -\frac{1}{2}f.$$

Therefore, the second principal point  $H_1$  is at a distance of  $\frac{f}{2}$  to the left of the eye-lens. Since the system is in air, the nodal points  $N_1$  and  $N_2$  coincide with the principal points. And the second principal focus  $F_2$  is located at a distance of  $(\frac{3}{4}f - \frac{1}{2}f) = \frac{1}{4}f$  to the right of the eye-lens.

### Advantages

As explained above, the first principal focal plane lies in front of the field lens. This makes it more convenient to introduce a micrometer screw or cross-wire, movable by a fine calibrated screw in the focal plane of the objective where the real image PQ is formed. As the final image of the object as well as that of the micrometer scale or cross-wire are formed by both the lenses of the eye-piece, the two images are equally magnified and hence faithful measurement is possible. This is the main reason which lead to the designing of Ramsden's eye-piece.

### Disadvantages

The conditions for minimum spherical and chromatic aberrations of two lenses of the same material of focal lengths  $f_1$  and  $f_2$  at a distance  $d$  apart are respectively given by

$$d = f_1 - f_2 \quad \text{and} \quad d = \frac{1}{2}(f_1 + f_2)$$

In Ramsden eye-piece  $f_1 = f_2 = f$  and  $d = \frac{2}{3}f$  and hence none of the above two conditions are satisfied. Consequently the eye-piece neither reduces the spherical aberration to a minimum nor it is achromatic. Since the final image is seen by refraction at four surfaces the *spherical aberration* and coma is, however, made small by distributing, as far as possible, equally the total deviation over the four refracting surfaces of the two lenses. It is further reduced by

employing plano-convex lenses as the field lens and the eye-lens. *Chromatic aberration* can be almost eliminated by making the eye-lens an achromatic doublet whose plane surface should be directed towards the eye. Although achromatism is sacrificed a little no

doubt by making  $d = \frac{2}{3}f$ , but since the field lens is not in the place

of the real image formed by the objective, the undesirable feature of bringing into focus dust particles and scratches, if any, on the field lens is avoided.

### 26.9 Huygens' eye-piece

Huygens designed an eye-piece to improve the corrections for the spherical as well as for the chromatic aberrations present in the Ramsden eye-piece. This eye-piece, which enjoys the widest use, consists of two thin plano-convex lenses, made of the same material, the focal length of the field lens being three times that of the eye-piece. The two lenses are coaxially mounted with their curved faces turned towards the incident light and separated by a distance equal to twice the focal length of the eye-piece. Thus if  $f$  is the focal length of the eye-piece, then the focal length of the field lens is  $3f$  and the distance of separation between them is  $2f$  (Fig. 26.10).

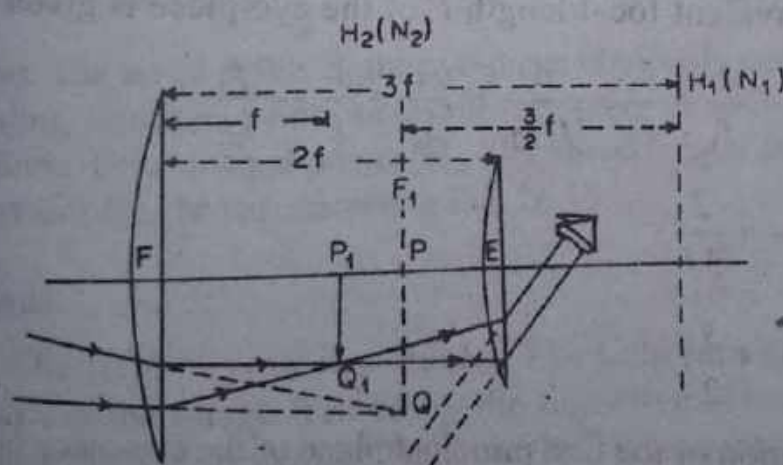


Fig. 26.10



**Action**

The real image PQ which was going to be formed by the objective serves as a virtual object for the eye-piece. For normal vision, this image PQ must be located at the first focal plane  $F_1$  of the eye-piece, so that the final image formed by the objective is at infinity. Actually, however, the eye-piece forms the final image in two steps. In the first step, the converging rays from the objective, before they converge at PQ, are further refracted by the field lens  $F$  and a real image is formed at  $P_1Q_1$  and not at PQ. Thus for the field lens, PQ is the virtual object and  $P_1Q_1$  is the resulting real image. Since the final image must be formed at infinity, the eye-piece should be so designed that  $P_1Q_1$  must be situated at the first focal plane of the eye-lens E.

For distinct vision, the image PQ due to the objective must be formed within the first focal length of the eye-piece, so that the final image may be formed at the least distance of distinct vision. In other words the image  $P_1Q_1$  formed by the field lens should be formed within the first focal plane of the eye-lens.

If a micrometer scale or a cross-wire is to be placed for accurate measurement, it should be placed coinciding with  $P_1Q_1$ , where the real image is produced after successive refraction through the objective and the field lens.

**Cardinal points of Huygens' eye-piece**

The equivalent focal length  $F$  of the eye-piece is given by

$$\begin{aligned}\frac{1}{F} &= \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2} = \frac{1}{3f} + \frac{1}{f} - \frac{2f}{3ff} \\ &= +\frac{2}{3f}\end{aligned}$$

$$\text{or, } F = +\frac{3}{2}f$$

The position of the first principal plane of the eye-piece is given by

$$\alpha = +\frac{F.d}{f_2} = +\frac{\frac{3}{2}f \cdot 2f}{f} = 3f$$



Therefore, for final image to be formed at infinity, the equivalent lens must be placed at a distance  $3f$  towards the right of the field lens i.e., a distance  $(3f - 2f) = f$  on the right (or behind) the eye-lens. Since the equivalent focal length of the eye-piece is  $+\frac{2}{3}f$ , the first principal focus

is at a distance  $(\frac{3}{2}f - f) = \frac{1}{2}f$  in front (or left) of the eye-lens or

$(2f - \frac{1}{2}f) = \frac{3}{2}f$  behind (or right of) the field lens. This is where the objective image must be formed for the final image to be formed at infinity.

As the image formed by the objective lies behind the field lens Huygens' eye-piece is often referred to as *negative eye-piece*.

The second principal point lies at a distance  $\beta$  from the eye-lens where

$$\beta = -\frac{F.d}{f_1} = -\frac{\frac{3}{2}f \cdot 2f}{3f} = -f$$

This means the second principal point lies at a distance  $f$  on the left (or in front) of the eye-lens or a distance  $(2f - f) = f$  on the right of (or behind) the field lens.

The second principal focus  $F_2$  is situated at a distance of  $(3/2)f$  (equivalent focal length) on the right of the second principal point. This means  $F_2$  lies at a distance  $(\frac{3}{2}f - \frac{1}{2}f) = (1/2)f$  on the right of (or behind) the eye lens. The nodal points of the eye-piece obviously coincide with the corresponding principal points since the eye-piece is surrounded by the same medium. The principal points ( $H_1, H_2$ ), the principal foci ( $F_1, F_2$ ) and the nodal points ( $N_1, N_2$ ) are shown in Fig. 26.10.

### Advantages

(i) The field of view is widened. The field lens first converges the beam from the image formed by the objective. Then the eye lens converges it further. Hence the field of view is increased.

(ii) The condition for minimum spherical aberration,  $f_1 - f_2 = 3f - f$   
 $= 2f = d$  and the condition for achromatism  $d = \frac{f_1 + f_2}{2} = \frac{3f + f}{2} = 2f$

are both satisfied in Huygens' eye-piece. The spherical aberration is further reduced by making the light to be incident on the curved surfaces. It has been further recommended by Airy that for least spherical aberration, the field lens should be concavo-convex of radii of curvature in the ratio 4:11, with the convex surface facing the incident rays while the eye-lens should be a double convex crossed lens having the radii of curvature of its two surfaces in the ratio 1:6, the more convex surface facing the incident rays.

If the measurement of the final image is required, a cross-wire or a micrometer scale is to be used and it must be placed between the field lens and the eye lens coinciding with  $P_1Q_1$  where the real image is produced after successive refractions through the objective and the field lens. But the cross-wires are viewed through the eye lens only while the distant object is viewed by rays refracted through both the lens. Due to this reason relative lengths of the cross-wires and the image are disproportionate. Moreover, the final image of the object, formed by both the field lens and the eye lens, will be free from aberrations, but the image of the cross-wires or the scale, being formed by the eye lens alone, will not be free from these defects. *Hence cross-wires cannot be used in a Huygens' eye-piece. Therefore, a faithful measurement or location of the object will not be possible. Thus Huygens' eye-piece cannot be used in telescope and other optical instruments with which distance and angles are to be measured. The eye-piece is used in microscope or other optical instruments using white light only.*

#### 26.10 Comparison of Ramsden eye-piece and Huygens eye-piece

(i) The chief advantage of the Ramsden eye-piece over Huygens eye-piece lies in the fact that the former being a positive eye-piece, cross-wires or micrometer screw can be used to carry out accurate measurements of the image (*quantitative use*). Huygens eye-piece, on the other hand, being a negative eye-piece cannot be used for accurate measurements since cross-wires cannot be used for the purpose. It can only examine image (*qualitative use*).

(ii) The condition for minimum spherical aberration is satisfied in Huygens eye-piece while it is not satisfied in Ramsden eye-piece. However, by using plano-convex lenses and spreading the



total deviation over the four lens surfaces more or less equally, spherical aberration is minimised in Ramsden eye-piece.

(iii) Ramsden eye-piece does not satisfy the condition for achromatism but can be made achromatic by using an achromatic doublet as the eye lens. Huygens eye-piece, on the other hand, satisfies the condition for achromatism.

The chief advantage of the Huygens eye-piece lies where the object to be examined is illuminated with white light and the residual lateral chromatic aberration, if Ramsden eye-piece were used, would be objectionable. This is precisely the case when biological slides are examined. Thus, in microscopes used in biological work, Huygens eye-piece is invariably employed. Since only one wavelength region is viewed at a time in telescope fitted in spectrometers and in other kinds of spectroscopes, residual lateral chromatic aberration becomes unimportant. In such instruments Ramsden eye-piece is employed.

(iv) The other aberrations like distortion which is of the pin-cushion type is not well removed in Huygens eye-piece. There is no coma and distortion is less than 5%.

(v) The field of view is slightly better in the Huygens eye-piece than in Ramsden eye-piece.

(vi) Huygens eye-piece can not be used as a simple microscope because the first focal plane lies to the right of the field lens and the focal plane is virtual. On the other hand, Ramsden eye-piece can be used as a simple microscope since the first principal plane lies to the left of the field lens and the focal plane is real.

### 26.11 Binocular microscope

If a microscope is employed to make an observation for a long time as in the case of study of biological samples in detail, it becomes tiring to the eyes. To overcome this disadvantage, binocular microscope was designed.

It employs one objective and two eye-piece. The object  $A$  is well illuminated and placed in front of the objective  $O$ . The beam of light emerging through the objective is split up into two beams by the reflecting prism  $P_1$ . One of the beams after passing through the prism  $P_2$  in the manner shown in figure 26.11 forms the image of  $A$  at the first



principal focal plane of one of the eye-piece  $E_1$ . Similarly the other beam passes through another prism  $P_2$  and forms the image of A in the first principal focal plane of the other eye-piece  $E_2$ . Both the eye-piece can be adjusted as desired by a rack and pinion arrangement. A binocular microscope is less tiring to the eyes and is more comfortable when an observation is made for a long time.

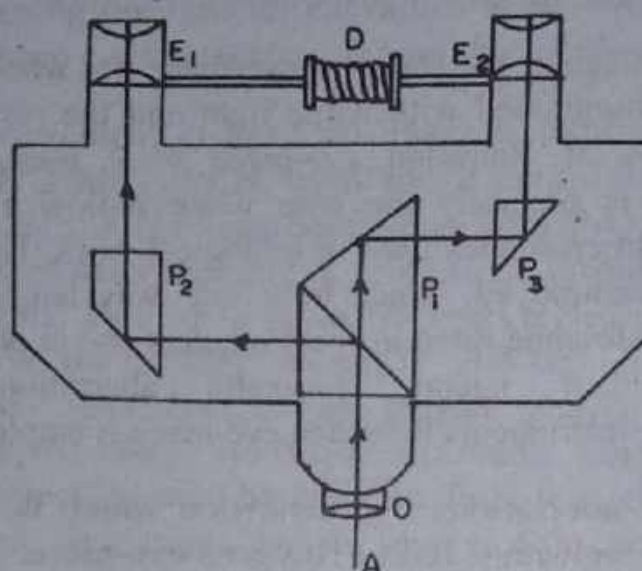


Fig. 26.11

**Example 26.2.** A convex lens of focal length 12.5 cm is used as a magnifying glass. Find the magnifying power when (i) the image is formed at infinity and (ii) the image is formed at the least distance of distinct vision of 25 cm.

**Soln.**

(i) When the image is formed at infinity

$$M = \frac{D}{f} = \frac{25}{12.5} = 2.0$$

(ii) When the image is formed at the near point

$$M = 1 + \frac{D}{f} = 1 + 2 = 3.0$$

**Example 26.3.** The ratio of the magnifying power of a converging lens when it is used to form an image at the least distance of distinct

vision to that when it is used to form the image at infinity is 1.50. What is the least distance of distinct vision if the focal length of the lens is 12 cm and at what distance from the lens must an object be placed so as to form the image at this least distance of distinct vision.

**Soln.**

Magnification when the image is at least distance of distinct

$$\text{vision} = 1 + \frac{D}{f}$$

$$\text{Magnification when the image is at infinity} = \frac{D}{f}$$

$$\therefore \text{the ratio} = \frac{1 + \frac{D}{f}}{\frac{D}{f}} = 1.50$$

$$\text{or, } 1 + \frac{f}{D} = 1.50$$

$$\text{or, } 1 + \frac{12}{D} = 1.50; \quad \text{or, } D = 24 \text{ cm}$$

So, the least distance of distinct vision = 24 cm. This is also the image distance in the second case.

$$\therefore v = -24 \text{ cm, } f = 12 \text{ cm, } u = ?$$

$$\text{From } \frac{1}{v} - \frac{1}{u} = \frac{1}{f}, \text{ we get}$$

$$-\frac{1}{24} - \frac{1}{u} = \frac{1}{12}; \quad \text{or, } u = -8 \text{ cm}$$

The object must be placed at a distance of 8 cm from the lens.

**Example 26.4.** A person has his near point at 20 cm and a range of distinct vision of 80 cm. What will be his range of vision if he wears close fitting spectacles having a power of - 0.5 dioptre?

**Soln.**

$$f = \frac{100}{P} \text{ cm} = -\frac{100}{0.5} \text{ cm} = -200 \text{ cm}$$

Let the near point with the spectacle be at a distance  $u_1$  from the eye. Since the near point without spectacles is 20 cm the image distance,  $v = -20$  cm.

From  $\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$ , we have

$$-\frac{1}{20} - \frac{1}{u_1} = \frac{1}{200}$$

or,  $u_1 = -22.22$  cm.

For the second case let the far point be at a distance  $u_2$  from the eye. The image this time must be formed at a distance  $(20 + 80) = 100$  cm from the eye. Hence  $v = -100$  cm.

From  $\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$ , we have

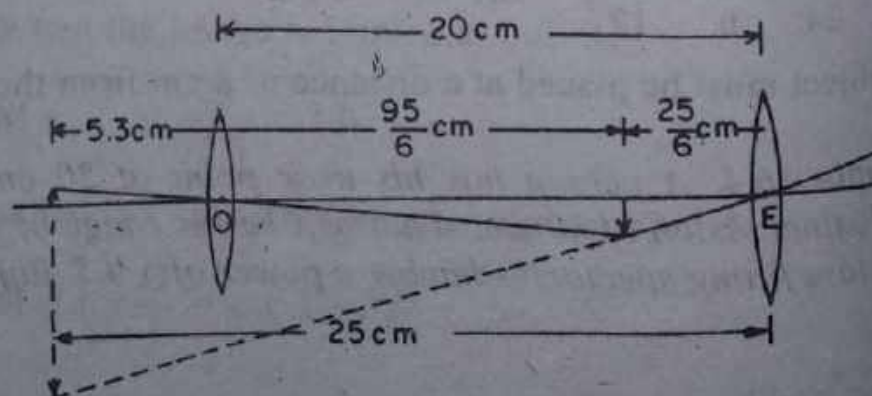
$$-\frac{1}{100} - \frac{1}{u_2} = \frac{1}{200}$$

or,  $u_2 = -200$  cm.

Hence the range of vision

$$= (200 - 22.22) = 177.78 \text{ cm}$$

**Example 26.5.** Two converging lenses of focal lengths 4 cm and 5 cm form the objective and the eye-piece respectively of a compound microscope. The lenses are placed 20 cm apart. Where must an object be placed so that image is formed at a distance of 25 cm from the eye and is virtual?



**Soln.**



In compound microscope, the image of an object is first formed by the objective. This acts as an object for the eye-piece.

Since the final image formed by the eye-piece is on its left,

$$v_2 = -25 \text{ cm}$$

From  $\frac{1}{v_2} - \frac{1}{u_2} = \frac{1}{f_2}$ , we have

$$-\frac{1}{25} - \frac{1}{u_2} = \frac{1}{5}$$

$$\text{or, } u_2 = -\frac{25}{6} \text{ cm.}$$

This means that the image of the object, as formed by the objective, must be formed at a distance of  $\frac{25}{6}$  cm to the left of the eye-piece. Thus the distance for the objective is

$$v_1 = (20 - \frac{25}{6}) = +\frac{95}{6} \text{ cm}$$

$$\text{From } \frac{1}{v_1} - \frac{1}{u_1} = \frac{1}{f_1}$$

we, therefore, have

$$-\frac{1}{95/6} - \frac{1}{u_1} = \frac{1}{4}$$

$$\text{or, } u_1 = -5.3 \text{ cm.}$$

Thus the object must be placed at a distance of 5.3 cm in front of the objective.

**Example 26.6.** Convex lenses of focal lengths 3 and 5 cm are used respectively as the objective and eye-piece of a microscope. If the object is 3.5 cm from the objective and the final image is 25 cm from the eye-piece, what is the distance between the centres of the lenses?

**Soln.**

The optical diagram of this problem is similar to problem 26.4.

*objective*

$$\frac{1}{v_1} - \frac{1}{u_1} = \frac{1}{f_1}; \quad u_1 = -3.5 \text{ cm}, f_1 = 3 \text{ cm.}$$

$$\text{or, } \frac{1}{v_1} + \frac{2}{7} = \frac{1}{3}; \quad v_1 = 21 \text{ cm.}$$

*eye-piece*

$$\frac{1}{v_2} - \frac{1}{u_2} = \frac{1}{f_2}; \quad u_2 = -25 \text{ cm}, f_2 = 5 \text{ cm.}$$

$$\text{or, } -\frac{1}{25} - \frac{1}{u_2} = \frac{2}{5}; \quad \text{or, } u_2 = 4.16 \text{ cm}$$

The image formed by the objective at a distance of 21 cm from the objective must be situated at a distance of 4.16 cm from the eye-piece.

Separation between the lenses

$$= 21 + 4.16 = 25.16 \text{ cm.}$$

**Example 26.7.** A compound microscope has lenses of focal lengths 1 cm and 3 cm. An object is placed 1.2 cm from the object lens when a virtual image is formed at a distance of 25 cm from the eye-piece. Calculate the separation of the lenses and the resultant magnification of the microscope.

**Soln.**

Optical diagram similar to that of problem 26.4.

*Objective*

$$\frac{1}{v_1} - \frac{1}{u_1} = \frac{1}{f_1}$$

$$u_1 = -1.2 \text{ cm}, f_1 = 1 \text{ cm.}$$

$$\therefore \frac{1}{v_1} + \frac{1}{6} = 1; \quad \text{or, } v_1 = 6 \text{ cm.}$$

*eye-piece*

$$\frac{1}{v_2} - \frac{1}{u_2} = \frac{1}{f_2}$$

$$v_2 = -25 \text{ cm}, f_2 = 3 \text{ cm.}$$

$$\text{or, } \frac{1}{25} - \frac{1}{u_1} = \frac{1}{3}; \quad \text{or, } u_2 = -2.7 \text{ cm}$$

Separation of the lenses =  $6 + 2.7 = 8.7 \text{ cm}$ .

Magnification produced by the objective

$$m_1 = \frac{v_1}{u_1} = 6 \times \frac{5}{6} = 5$$

Magnification produced by the eye-piece

$$m_2 = \frac{v_2}{u_2} = 25 \times \frac{25}{67.7} = \frac{28}{3}$$

Resultant magnification of the microscope.

$$M = m_1 \times m_2 = 5 \times \frac{28}{3} = 46.6$$

## 26.12 Telescopes

Jan Lippershey, in 1608, invented the first telescope. One year later, Galileo made a telescope to observe stars. He observed the satellites of Jupiter and the rings of Saturn through his telescope. Kepler's telescope, invented in 1611, was fundamentally similar to an astronomical telescope. James Gregory's invention of achromatic doublet in 1733 made it possible to improve the refracting telescopes. James Gregory also invented the reflecting telescope.

Telescopes have three main uses :

- (i) To make distant objects appear magnified, so that they may be seen distinctly.
- (ii) To collect light over a wider area than that of the pupil of the eye. Distant faint stars thus appear much brighter when viewed through a telescope.
- (iii) Instruments like *theodolites*, *catheometers*, etc. use telescope as a sighting device for defining a line in space relative to the instrument.

Angular magnification  $M$  of a telescope is defined as the ratio of the angle,  $\beta$ , subtended by the image at the eye through the telescope to the angle,  $\alpha$ , subtended by the distant object to the unaided eye. Hence



$$M = \frac{\beta}{\alpha}$$

*It should be remembered that  $\alpha$  here is not the angle subtended by the object at the least distant of distinct vision as in the case of a microscope.*

### 26.13 Astronomical telescope

An astronomical telescope is used to view heavenly bodies. It consists of an objective or object glass and an eye-piece. The eye-piece is a Huygens or a Ramsden eye-piece of small focal length  $f$ . The eye-piece and the objective are mounted coaxially with a draw-tube or rack and pinion arrangement for focussing.

#### Action

Parallel rays from a distant object after refraction through the achromatic objective  $O$  form a real, diminished and inverted image  $PQ$  at the focal plane of the objective which serves as an object to the eye-piece (Figs. 26.12 & 26.14). In the figure both the objective and the eye-piece have been represented by a single lens. For *normal vision*, the eye-piece is so adjusted that the image  $PQ$  formed by the objective is situated at the principal focus of the eye-piece  $E$  (Fig. 26.12). Then the rays after refraction through the eye-piece become

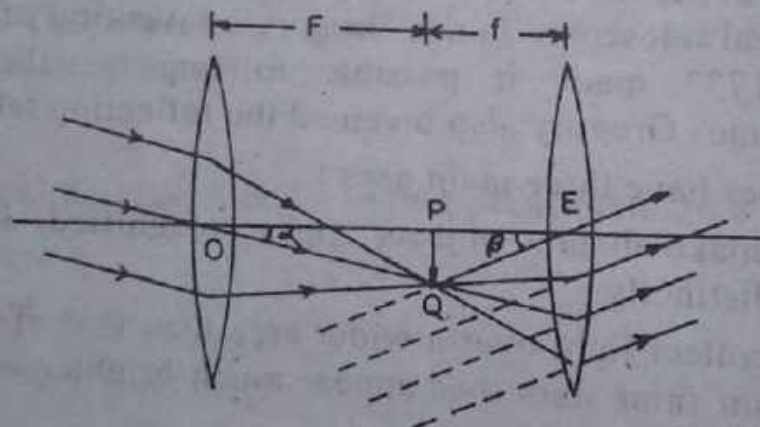


Fig. 26.12

parallel and the final image is formed at infinity. This is the adjustment usually done with an astronomical telescope. When *distinct vision* is desired  $PQ$  must be within the focal length of the eye-piece  $E$ , when the final image is formed at  $P_1Q_1$  where  $EP_1$  is

the least distance of distinct vision (Fig. 26.14). The cross-wires of micrometer scale should be placed coincident with PQ, the focal plane of the objective.

### Magnification for normal vision

When the telescope is in normal adjustment, the distance between the objective and the eye-piece is equal to the sum of their focal lengths (Fig. 26.12). To calculate the angular magnification  $M$  of the telescope, it is assumed that the eye is very close to the eye-piece. Moreover, when compared with the distance of the object from the objective or the eye-piece, the distance between the objective and the eye-piece is very small. The angle  $\alpha$  subtended at the unaided eye by the object can, therefore, be taken as the angle subtended at the objective by the object. The angular magnification, as defined in Art. 26.2, is given by

$$M = \frac{\beta}{\alpha} = \frac{\tan \beta}{\tan \alpha}$$

since the angles  $\alpha$  and  $\beta$  are very small (the moon subtends an angle  $1/2^\circ$  at the naked eye).

In the triangle PQE

$$\tan \beta = \frac{PQ}{PE'} \quad \text{or, } \beta = \frac{PQ}{PE}$$

In the triangle OPQ,

$$\tan \alpha = \frac{PQ}{OP} \quad \text{or, } \alpha = \frac{PQ}{OP}$$

Therefore,

$$M = \frac{\beta}{\alpha} = \frac{PQ/PE}{PQ/OP} = \frac{OP}{PE}$$

But  $OP$  = focal length of the objective =  $F$

and  $PE$  = focal length of the eye-piece =  $f$

$$\therefore M = \frac{F}{f} \quad (26.3)$$



Thus the magnification is equal to the ratio of the focal lengths of the objective and the eye-piece. To increase the magnification of the telescope, the objective focal length should be as large as possible while the eye-piece should be of short focal length. Eqn. (26.3) shows that apparently unlimited values of  $M$  may be obtained by making  $F$  very large and  $f$  very small. Other factors, however, limit the values employed in practice, so that magnifications greater than about 1000 are rarely used in astronomy.

### Magnification in terms of the entrance and exit pupils of an astronomical telescope

The diameter or aperture  $D$  of the objective limits the size of the pencil of rays entering the telescope and is consequently known as the entrance pupil of the telescope. The exit pupil or the eye-ring  $R$  which is the circular image of the objective formed by eye-piece is the best position for the eye for observing the image (Fig. 26.13). Suppose the exit pupil is of the diameter  $d$  and is situated at a distance  $v$  from the eye-piece. When the telescope is adjusted for normal vision, the distance between the objective and the eye-piece is

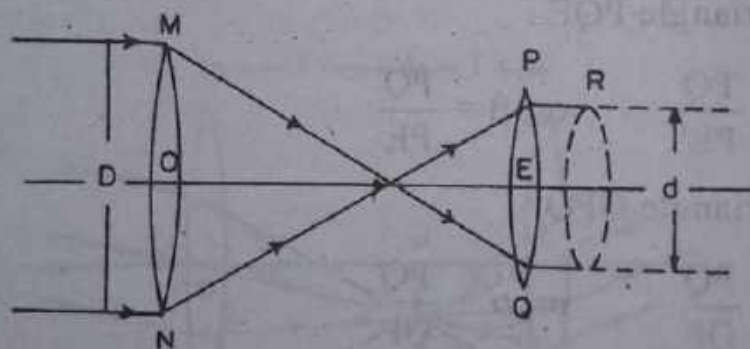


Fig. 26.13

$(F + f)$  and this is the object distance in the present case. Making due allowance for proper sign, we get from the lens formula.

$$\frac{1}{v} + \frac{1}{(F + f)} = \frac{1}{f}$$



$$\text{or, } \frac{1}{v} = \frac{1}{f} - \frac{1}{(F+f)} = \frac{F+f-f}{f(F+f)} = \frac{F}{f(F+f)}$$

$$\text{whence } v = \frac{f(F+f)}{F}$$

Now the linear magnification of the eye-piece is given by

$$m = \frac{\text{size of the image}}{\text{size of the object}} = \frac{\text{image distance}}{\text{object distance}}$$

$$\text{or, } \frac{d}{D} = \frac{v}{u} = \frac{f(F+f)}{F(F+f)} = \frac{f}{F}$$

But from eqn. (26.3) we may write

angular magnification,

$$M = \frac{F}{f} = \frac{D}{d} = \frac{\text{diameter of the entrance pupil}}{\text{diameter of the exit pupil}}$$

For normal adjustment the diameter of the eye pupil is the same as the diameter of the exit pupil. It can further be seen from the above equations that the angular magnification

$$M = \frac{D}{d} = \frac{1}{d/D} = \frac{1}{\text{linear magnification}}$$

Thus the angular magnification produced by an astronomical telescope for normal adjustment is the reciprocal of the linear magnification produced by it.

### Magnification for distinct vision

In normal adjustment, the image is formed at infinity and can be viewed distinctly with the eye unaccommodated. In distinct vision adjustment, the image is formed at the near point and can be viewed clearly. But this time the eye has to be accommodated.

The magnification  $M$  is given by the ratio  $\beta/\alpha$  where  $\alpha$  and  $\beta$  are the angles subtended by the object and the image respectively.

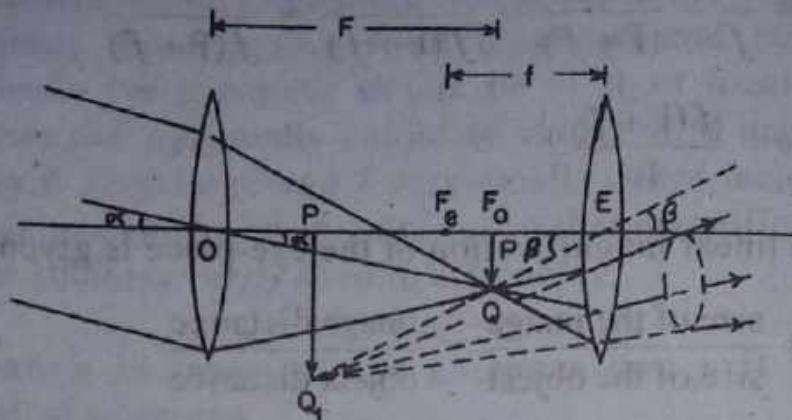


Fig. 26.14

Referring to Fig. 26.14 we get from the triangle  $OPQ$ ,

$$\tan \alpha = \frac{PQ}{OP}; \quad \text{or, } \alpha = \frac{PQ}{F}$$

since  $\alpha$  is small and  $OP$  is the focal length  $F$  of the objective. Similarly from the triangle  $PQE$ ,

$$\tan \beta = \frac{PQ}{PE}; \quad \text{or, } \beta = \frac{PQ}{I}$$

Again  $\beta$  is small and  $PE$  is the distance  $I$  of  $PQ$  from the eye-piece. Hence, magnification

$$M = \frac{\beta}{\alpha} = \frac{PQ}{I} \times \frac{F}{PQ} = \frac{F}{I}$$

Applying the lens equation to the eye-piece, we have

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

As the final image is formed at a distance  $D$  from the eye-piece,  $v = -D$ ,  $u = -I$  (applying sign convention).

$$-\frac{1}{D} + \frac{1}{I} = \frac{1}{f}$$

$$\text{or, } \frac{1}{I} = \frac{1}{D} + \frac{1}{f} = \frac{f+D}{fD}$$

$$\text{or, } I = \frac{fD}{f+D}$$

$$\therefore M = \frac{F}{I} = \frac{F(f+D)}{fD} = \frac{F}{f} \left(1 + \frac{f}{D}\right) \quad (26.4)$$

When the telescope is in normal adjustment, the magnification is given by  $\frac{F}{f}$ . Hence, the angular magnification is greater when the final image is formed at the near point.

For normal vision  $D = \infty$ . Hence eqn. (26.4) reduces to  $\frac{F}{f}$  which is the magnification for normal vision.

#### 26.14 Terrestrial telescope

The image formed by an astronomical telescope is inverted with respect to the object. This, however, presents no difficulty in viewing heavenly bodies. But it appears to be useless for viewing objects on the surface of the earth in which case an erect image is a necessity. The terrestrial telescope provides an erect image for the purpose. In addition to the objective and the eye-piece, the terrestrial telescope consists another lens of focal length  $f_1$  which erects the image. The objective forms, an inverted image PQ at its principal focus (Fig. 26.15). The lens L is so placed that PQ is at a distance  $2f_1$  from it. Therefore, an erect image  $P'Q'$  is formed at a distance  $2f_1$

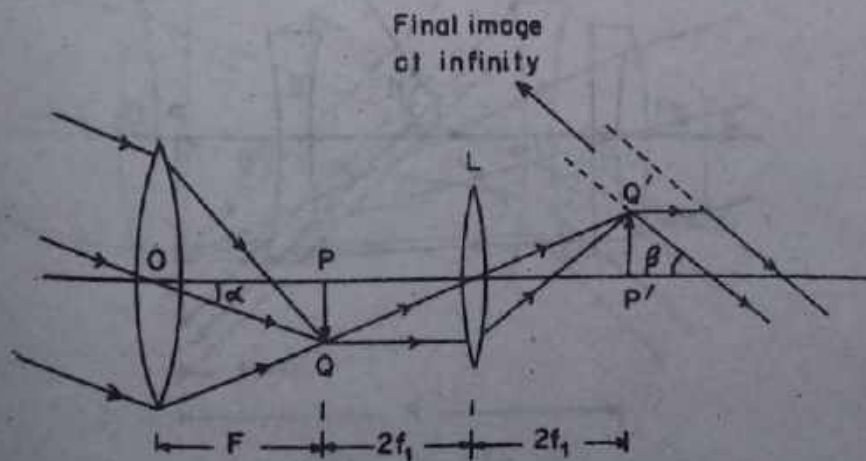


Fig. 26.15



from  $L$  on the other side of it. The eye-piece is so adjusted that the point  $P'$  is its principal focus. As a result the final image, which is erect, is formed at infinity.

The length of the terrestrial telescope is equal to  $(F + f + 4f_1)$  and its angular magnification is  $\frac{F}{f}$ .

### 26.15 Galilean telescope

Galileo devised this telescope in 1609 which provides an erect image of an object with the aid of only two lenses. It consists of an objective  $O$  which is usually an achromatic doublet of large focal length and small aperture. The two lenses are placed coaxially in a tube and may be focussed with a rack and pinion arrangement.

#### Action

Parallel rays from a distant object are converged by the objective. Had there been no eye-piece to intercept them, these rays would have formed a real, diminished and inverted image at the focal plane of the objective. But the concave eye-piece is here pushed inside towards the objective. For normal vision, the eye-piece is so adjusted that the distance  $EP$  is equal to the focal length of the eye-piece and the final image is formed at infinity (Fig. 26.16). The final image which is always erect and virtual can be seen by the eye placed close to the eye-piece. For distinct vision, the eye-piece is

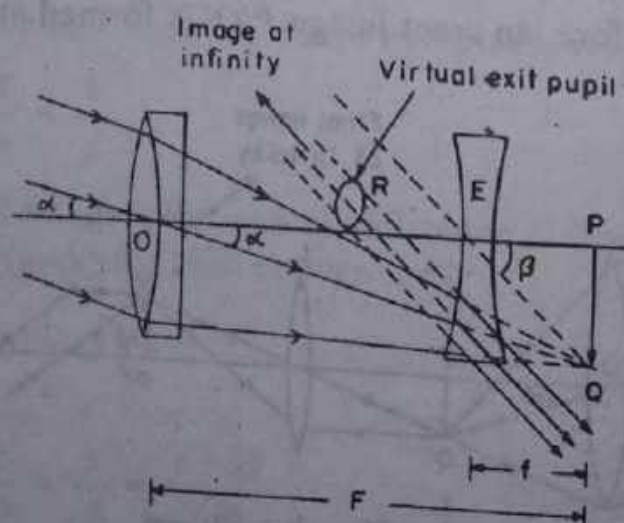


Fig. 26.16

so adjusted that the virtual object is just outside the first principal focus of the eye-piece. The final image is then formed at the least distance of distinct vision (Fig. 26.17).

Since the final image in this telescope is erect with respect to the object, it is suitable for viewing terrestrial objects. For this reason it is sometimes referred to as *terrestrial telescope*.

When the telescope is focussed for normal vision, the length of the telescope tube is  $OE = F - f$  where  $F$  and  $f$  are the focal lengths of the objective and the eye-piece respectively. In case of distinct vision it is less than  $(F - f)$ .

### Magnification

(i) *normal vision*

Angular magnification,

$$M = \frac{\beta}{\alpha} = \frac{\tan \beta}{\tan \alpha} = \frac{PQ/PE}{PQ/OP}$$

$$= \frac{OP}{PE} = \frac{F}{f} \quad (26.5)$$

since  $OP = F$  = focal length of the objective and  $PE = f$  = focal length of the eye-piece. Thus for high magnification, an objective of long focal length and an eye-piece of short focal length are required.

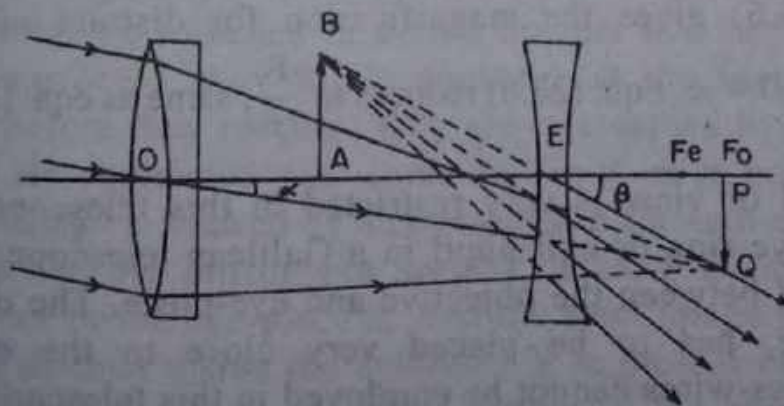


Fig. 26.17

(ii) *distinct vision*

Let the image  $AB$  be formed at the position of least distance of distinct vision so that the distance  $EA = D$ . Applying lens equation to the eye-piece we get,

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

Applying sign convention

$$u = +EP, v = -EA = -D, f \text{ is } -ve$$

$$\therefore -\frac{1}{D} - \frac{1}{EP} = -\frac{1}{f}$$

$$\text{or, } \frac{1}{EP} = \frac{1}{f} - \frac{1}{D} = \frac{D-f}{fD}$$

$$\text{or, } EP = \frac{fD}{D-f}$$

Angular magnification,

$$\begin{aligned} M &= \frac{\tan \beta}{\tan \alpha} = \frac{PQ/EP}{PQ/OP} = \frac{OP}{EP} = \frac{F(D-f)}{fD} \\ &= \frac{F}{f} \left( \frac{D-f}{D} \right) = \frac{F}{f} \left( 1 - \frac{f}{D} \right) \end{aligned} \quad (26.6)$$

Eqn. (26.6) gives the magnification for distinct vision. For normal vision  $D = \infty$ . Eqn. (26.6) reduces to  $\frac{F}{f}$ ; same as eqn. (26.5).

The field of view is very restricted in this telescope. This is because the eye-ring or exit pupil in a Galilean telescope is virtual and is situated between the objective and eye-piece. The observer's eye, therefore, has to be placed very close to the eye-piece. Moreover, cross-wires cannot be employed in this telescope because the eye-piece is negative.



## 26.16 Reflecting telescope

If a bright image and fine details therein are required, the objective of the telescope should have a large aperture so as to collect as much light as possible from the object. The objectives of refracting telescopes are combination of lenses and as such cannot be manufactured with very large apertures. This difficulty is overcome in a reflecting telescope by employing a paraboloidal mirror of large focal length instead of a converging lens. There are other advantages in using mirror. The mirrors are free from chromatic aberration since the image is formed by reflecting and all colours obey the same laws of reflection. Because of this, a comparatively large aperture, admitting more light, can be used in the reflecting telescope. Spherical aberration in the case of a mirror can be avoided by the use of a paraboloidal mirror. Moreover, the mounting of a mirror is easier than that of a lens because the lens can be supported only at the rim. Since much of the weight of a reflecting telescope is located at the base of the telescope, where the heavy mirror is placed, the telescope has greater stability. The objective of the early reflecting telescopes were made of *speculum*, an alloy of copper and tin. Later, objective made of glass silvered on its surface were used. The recent telescopes employ objectives made of pyrex coated with aluminium.

### Action

There are different ways in which a mirror can be arranged in a reflecting telescope. The one described here is due to *Cassegrain*.

The light entering the telescope from a distant light source is considered to be parallel. When these rays strike the large concave mirror  $AB$ , they are reflected in such a manner that had there been no convex mirror  $M$ , they would converge at the focus  $F$  of the mirror. But before they reach  $F$ , they are intercepted by the convex mirror and are redirected and come to focus at a point  $P$  (Fig. 26.18). The image is viewed by an eye-piece  $E$  through a hole in the centre of the concave mirror. For normal vision,  $P$  is situated at the principal focus of the eye-piece so that the final image is formed at infinity. For distinct vision the eye-piece  $E$  is pushed inside a little so that  $P$  may fall within the focal length of the eye-piece. The image is then formed at the least distance of distinct vision. In practice, a camera or some other recording device such as a

spectrograph is placed at this point  $P$ . As very long exposures are required for the purpose, the telescope is mounted so as to turn about

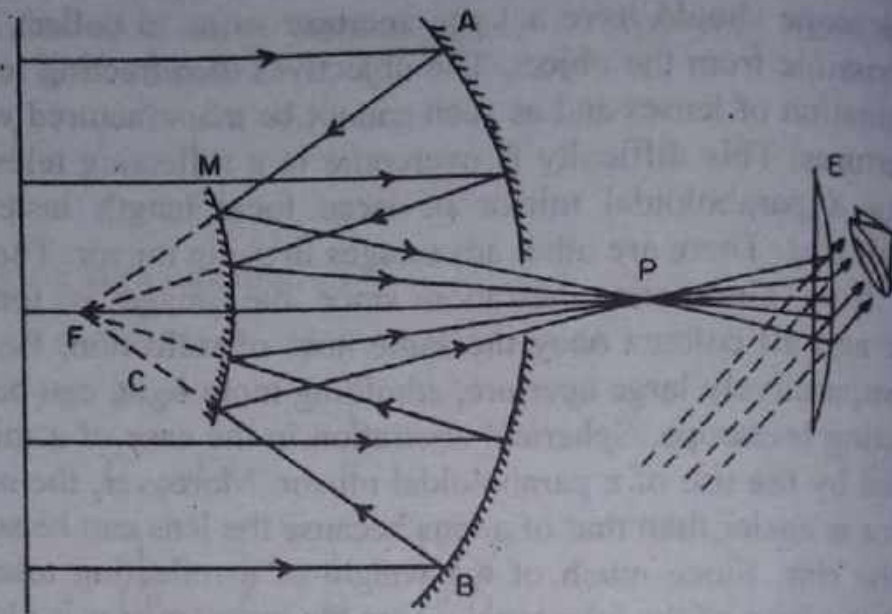


Fig. 26.18

the polar axis by clockwork mechanism so that the star observed is always in the centre of the field. It is possible in practice to make reflecting mirrors of considerably larger diameters than lenses. Thus, much brighter images of stars can be obtained with reflecting telescope than with refracting ones. That is why nearly all of the larger astronomical telescopes today are of the reflecting type. Perhaps the best known reflecting telescope is the 200 - inch telescope located at Mount Palomar. A reflecting telescope has the disadvantage that it is more liable to experience the problems of astigmatism and coma than do refracting telescopes.

### 26.17 The prism binocular

The prism binocular consists of two astronomical telescopes mounted parallel to each other on a frame at such a distance that the eye-pieces are at average distance between the two eyes of a man. The objectives are achromatic doublets and the eye-pieces may be Huygens' eye-piece. In an ordinary astronomical telescope the final image is inverted. But here, in each telescope, two totally reflecting right angled prisms are used. The two prisms are set with half the hypotenuse face of one in contact with half the hypotenuse face of



the other and with the long dimensions of these faces at right angles to each other (Fig. 26.19). One of the prism produces a lateral inversion (exchanging of left and right sides) while the other produces an upside down inversion of the image. The image formed by the objective lenses and prisms is therefore real and has the same

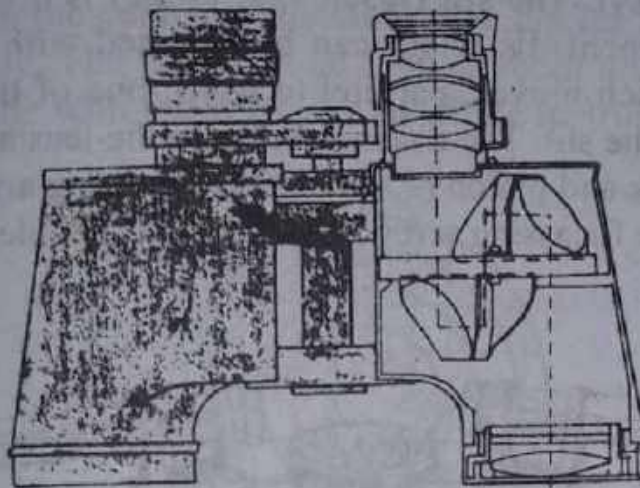


Fig. 26.19

orientation as the object itself, and so does the final virtual image formed by the eye-piece. The eye-piece forms the final image either at the least distance of distinct vision or more usually at a great distance (practically infinity). Due to reflections at the two prisms, the ray turns twice on its own direction. Thus the effective tube length is made almost three times its actual length. The instrument is, therefore, very compact in comparison with the astronomical telescope and hence handy and portable. The field of view is rather wide and the image is also bright. With this construction, advantage is taken of the fact that the distance between the objectives can readily be made greater than the distance between the eyes, thus enhancing the stereoscopic effect as an aid to the perception of distances. This instrument is widely used as a view finder by tourists, by seamen, by military men for observation and by sport fans.

#### 26.18 The Spectrometer

Spectrometer is one of the most important optical instruments. It is usually employed for the study of spectrum produced by light transmitted through devices like a prism or a grating.



The spectrometer (Fig. 26.20) essentially consists of a collimator (C), a prism table (P), a telescope (T) and a circular scale. These parts are described below.

(a) **Collimator** : This is a horizontal tube having at its outer end a narrow rectangular vertical slit (S) of adjustable width. The inner end of the tube, which faces the prism table, has an achromatic convergent lens ( $L_1$ ). The slit (inset, Fig. 26.20) is a very important part of the instrument. Its width can be adjusted with the help of a side screw ( $f$ ) which moves, parallel to itself, one of the two jaws  $j_1$  and  $j_2$  making up the slit. The distance between the lens and the slit can be altered by a rack and pinion or some other focussing arrangement. In the figure  $F_1$  is the focussing screw. Light from a suitable source, such

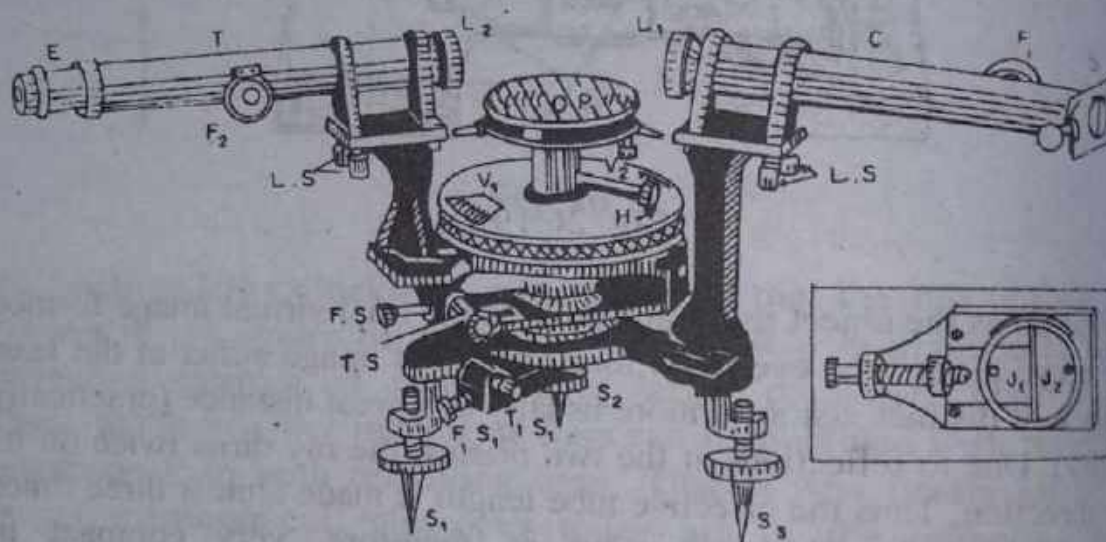


Fig. 26.20

as a sodium flame, illuminates the slit. When properly focussed, the slit can be made to lie in the focal plane of the convergent lens and as a result the rays coming from the slit are rendered parallel by the lens. Thus the collimator gives rise to a parallel beam of light which falls on the prism on the prism table. The collimator is rigidly fixed to the main frame of the instrument with its axis pointing towards the axis of the prism table.

(b) **Prism Table** : It is a small circular table with a flat top so that it can be raised or lowered. The screw  $H$  (Fig. 26.20) fixes the

table to any given height and also keeps it fixed to the verniers. The table is provided with three levelling screws E, F and G (Fig. 26.21). The prism is placed on the table with its refracting edge vertical. On the surface of the table there are sets of equidistant lines which are parallel and perpendicular to the line joining two of the levelling screws say E and F. These lines are required for adjusting the prism. There are also concentric circles, the common centre of which coincides with the centre of the table. The circles help in placing the prism correctly. The table can be rotated about a vertical axis which should coincide with the vertical axis of the instrument.

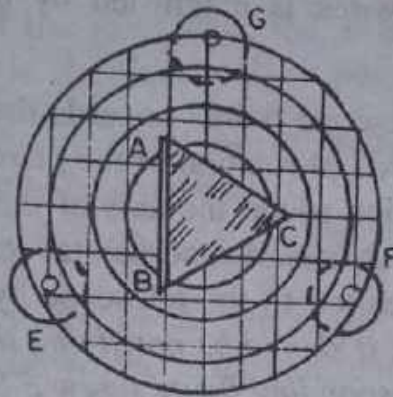


Fig. 26.21

**Note :** In many instruments there are just two lines on the surface of the prism table one parallel to a line joining the two screws and the other perpendicular.

**(c) Telescope :** It is a small astronomical telescope T (shown separately in Fig. 26.22) having at the inner end an achromatic objective  $L_2$  facing the prism table and at the outer end a Ramsden type eye-piece E, fitted with cross-wires. The eye-piece slides in a tube

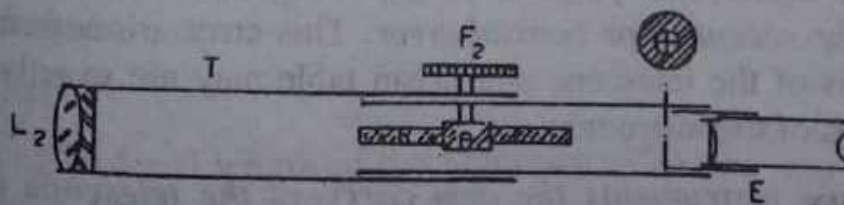


Fig. 26.22

which carries the cross-wires. This tube slides in another tube which carries the objective. The telescope can be focussed by means of a rack and pinion arrangement (RP) which alters the distance between the



objective and the eye-piece. This arrangement is on the outer side of the inner tube and therefore does not obstruct the passage of light. The eye-piece can be focussed on the cross-wires by drawing it in or out. When this is done the cross-wires lie in the focal plane of the eye-piece. Then the eye can see the cross-wires without applying accommodation i.e., without any strain on the eye. The image of the cross-wires is formed at infinity. The focussing screw ( $F_2$ ) is then turned till there is no parallax between the cross-wires and the image of the slit formed by the objective of the telescope. This ensures that the image of the slit has been formed in the plane of the cross-wire. The image of the slit which is formed in the plane of the cross-wires is magnified by the eye-piece. The final image is at infinity.

The axis of the telescope points towards the axis of the prism table and passes through the axis of rotation of the prism table. It is mounted on an arm, capable of rotation round the axis of the prism table and can be fixed at any position by means of a horizontal screw  $F.S.$  (Fig. 26.20). A slow motion can be imparted to the telescope by means of a screw  $T.S.$  called the *tangent screw*. *It should be noted that the tangent screw will not function unless the corresponding fixing screw  $F.S.$  is tightened.* The prism table is also provided with similar fixing and slow motion screws  $F_1S_1$  and  $T_1S_1$  respectively. The telescope (as also the collimator tube) is supported near one end on two screws, each marked ( $L.S.$ ) in Fig. 26.20. By means of these screws, the axis of the telescope may be made exactly horizontal.

**(d) Circular scale :** The spectrometer has a circular scale, which is graduated in degrees. The scale is coaxial with the axis of rotation of the prism table and the telescope. A vernier scale is also provided along with main circular scale. In most of the spectrometers there are two vernier scales  $V_1$  and  $V_2$  (Fig. 26.20)  $180^\circ$  apart, which can move over the circular scale. The purpose of providing two vernier scales is to *eliminate the eccentric or coaxial error*. This error arises from the fact that the axis of the telescope and prism table may not exactly coincide with the axis of the instrument.

*In many instruments the arm carrying the telescope is rigidly connected to the circular scale whereas the prism table can be clamped to the spindle of a circular plate which carries the two verniers. Thus the prism table as well as the telescope can be rotated, independent of each other, and their rotations can be easily read off independently from the circular scale.*



Quantitatively, a spectrometer is always employed for the measurement of an angle, for example, the angle of the prism, the angle of minimum deviation or the angle of diffraction of light due to transmission of light through a plane diffraction grating. But before any measurement is made, certain adjustments must be made in the instrument. These are

(i) *levelling of the instrument (i.e., of the collimator, the prism table and the telescope)* : The axis of the instrument i.e., the axis of rotation of the telescope should be made vertical, making it to coincide with the vertical axis of rotation of the prism table. Besides, the axis of the telescope, the axis of the collimator and the top of the prism table should also be made horizontal.

(ii) *the collimator must be adjusted for rendering the rays from the illuminating slit parallel.*

(iii) *the telescope must be focussed for parallel rays.*

(iv) *the optical axis of the telescope and that of the collimator must be perpendicular to the principal axis of the instrument.*

(v) *when a prism is employed, its refracting edge must be made parallel to the vertical axis of rotation of the prism table and of the telescope.*

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**Example 26.8.** An astronomical telescope is made of an objective of focal length 20 cm and an eye-piece of 5 cm. both converging lenses. Find the magnifying power (a) when the eye is focussed to receive parallel rays and (b) when the image is formed at the least distance of distinct vision of 25 cm.

**Soln.**

Let

$F$  = focal length of the objective = 20 cm.

$f$  = focal length of the eye-piece = 5 cm.

$D$  = least distance of distinct vision = 25 cm.

(a) Image is formed at infinity.

$$M = \frac{F}{f} = \frac{20}{5} = 4$$

(b) Image is formed at least distance of distinct vision.

$$M = F \left( \frac{1}{f} + \frac{1}{D} \right) = 20 \left( \frac{1}{5} + \frac{1}{25} \right) = 4.8$$

**Example 26.9.** A Galilean telescope consists of an objective of focal length 18 cm and a concave lens of focal length 6 cm. What should be the separation of the two lenses so as to form a virtual image of a distant object at a distance of 24 cm from the eye lens? What is the magnifying power under these condition?

**Soln.**

Optical diagram is similar to Fig. 26.17.

Considering refraction through the eye lens, we have from

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}, \quad v = -24 \text{ cm}, f = -6 \text{ cm}$$

$$\text{or, } -\frac{1}{24} - \frac{1}{u} = -\frac{1}{6}; \quad \text{or, } u = 8 \text{ cm}$$

Separation between the two lenses =  $18 - 8 = 10 \text{ cm}$ .

$$\begin{aligned} \text{Magnifying power} &= \frac{\beta}{\alpha} = \frac{\tan \beta}{\tan \alpha} = \frac{F}{f} \left( 1 - \frac{f}{D} \right) \\ &= \frac{18}{6} \left( 1 - \frac{6}{24} \right) = 2.25 \end{aligned}$$

**Example 26.10.** The focal length of the concave mirror objective of a Cassegrain's reflecting telescope is 6 m. This reflects light into a convex mirror placed on its principal axis at a distance of 4 m. The combination forms a real image of a distant object at the pole of the concave mirror. The image is then seen by an eye-piece of focal length 25 cm. Determine the focal length of the convex mirror and the magnifying power of the telescope.



**Note :** For reflection at spherical surfaces, all real distances are considered positive while all virtual distances are considered negative.

**Soln.**

Image formed by the concave mirror (Fig. 26.18) serves as a virtual object for the convex mirror. It lies at a distance of  $6 - 4 = 2$  m from it.

Hence, for the convex mirror, we have from the relation

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f}, \quad u = -2 \text{ m}, v = +4 \text{ m}$$

$$\text{or, } -\frac{1}{2} + \frac{1}{4} = \frac{1}{f}; \quad \text{or, } f = -4 \text{ m}$$

$$\text{Magnification produced by the convex mirror} = \frac{v}{u} = \frac{4}{2} = 2$$

Total magnification

$$= 2 \times \frac{\text{focal length of the objective}}{\text{focal length of the eye-piece}}$$

$$= 2 \times \frac{6}{25/100} = \frac{2 \times 6 \times 100}{25} = 48$$

**Example 26.11.** Light from the sun is falling on the Ramsden's eye-piece. Locate the position of the image thus formed and also find the point from which the distance of the image is to be measured. The focal length of each lens of the eye-piece is 30 mm.

**Soln.**

$$f_2 = f_1 = 3 \text{ cm}$$

$$\text{Distance between the two lenses, } d = \frac{2}{3} \times 3 = 2 \text{ cm}$$

Equivalent focal length

$$F = \frac{f_1 \times f_2}{f_1 + f_2 - d} = \frac{3 \times 3}{3 + 3 - 2} = 2.25 \text{ cm}$$



Since the object *i.e.*, the sun is at infinity, the image is formed in the second focal plane of the eye-piece. The position of the second principal plane is given by

$$\beta = -\frac{F.d}{f_1} = -\frac{2 \times 2.25}{3} = -1.5 \text{ cm}$$

The second principal plane lies at a distance of 1.5 cm on the left of the eye lens. Hence the position of the image is  $(-1.5 + 2.25) = +0.75 \text{ cm}$  behind the eye lens.

### EXERCISES

- [1] Describe a photographic camera and explain its action. Explain the terms stop number and depth of focus. Give the construction of a telephoto lens and explain its working.
- [2] What do you understand by visual angle and angular magnification? What is a simple microscope? Explain its action with a ray diagram and obtain an expression for its magnification.
- [3] Describe the construction of a compound microscope and explain its action with a simple ray diagram. What are the advantages of a compound microscope over a simple microscope? Calculate its magnifying power.
- [4] What are the necessary conditions of a high power microscope? Explain in brief how they are obtained in practice.
- [5] Explain the principle of an oil-immersion objective.
- [6] Explain the function of an eye-piece in an optical instrument. Describe the construction of a Huygens eye-piece and calculate the positions of the cardinal points. Discuss its merits and demerits.
- [7] Describe the construction and working of a Ramsden eye-piece. How are chromatic and spherical aberrations minimised in this eye-piece. Calculate and indicate in a diagram the positions of the cardinal points. Discuss its merits and demerits relative to the Huygens eye-piece.
- [8] Describe a refracting astronomical telescope. Deduce an expression for its magnifying power. Discuss its merits and demerits. What modification is necessary to see terrestrial objects with it?

- [9] Describe a Galilean telescope. Obtain an expression for its magnification. Compare it with an astronomical telescope.
- [10] Briefly describe the construction and working of a Cassegrain's reflecting telescope.
- [11] What are binoculars? Describe the construction and working of a prism binocular.
- [12] Write short notes on
- (i) Telephoto lens
  - (ii) Visual angle
  - (iii) The prism binocular
  - (iv) The spectrometer
- [13] The ratio of the magnifying power of a convex lens when the image is formed at the least distance of distinct vision to that when the image is formed at infinity is 1.20. If the focal length of the lens is 4.80 cm, calculate (i) the least distance of distinct vision for the eye and (ii) the distance from the lens where an object must be placed so as to form its image at the least distance of distinct vision calculated in (i). [24 cm; 4 cm from the lens].
- [14] The focal length of the object glass of a microscope is  $\frac{1}{2}$  inch and that of the eye-piece is 1 inch. Taking the least distance of distinct vision to be 12 inches, find the distance between the object glass and the eye-piece when the object viewed is  $\frac{3}{4}$  inch from the object glass. [2.42 inch].
- [15] Two lenses of focal lengths 1 cm and 5 cm are arranged as in a compound microscope and a small object is placed 1.1 cm in front of the first lens. The final image is observed at a distance of 25 cm from the second lens. Find the resultant magnification and the separation between the lenses. [60; 15.16 cm].
- [16] Draw a diagram showing the path of rays through a compound microscope from an object point not on the axis, when the image is formed coincident with the object at the distance of most distinct vision from the eye. If this distance is 28 cm, the distance of the object from the objective 4 cm, and the magnifying power of the instrument 14, find the focal length of the lenses.  
[objective 3.2 cm : eye-piece 11.2 cm].



- [17] A compound microscope is constructed with an objective of focal length 1 cm and an eye-piece of focal length 5 cm. If the overall magnification of the microscope is 15, what should be the separation between the objective and the eye-piece for the final image to be formed at the least distance of distinct vision of an observer which is 25 cm. [7.7 cm].
- [18] A Cassegrain reflecting telescope has an objective concave mirror of focal length 4 m which reflects light into a convex mirror placed on its principal axis at a distance of 3 m. The combination from a real image of a distant object at the pole of the concave mirror, and the image is then seen by an eye-piece of focal length 20 cm. Find the focal length of the convex mirror and the magnifying power of the telescope. [1.5; 60].
- [19] A camera A, having an  $f/8$  lens 2.5 cm in diameter, photographs an object using the correct exposure of  $1/1000$  sec. What exposure should another camera B use in photographing the same object if it has an  $f/4$  lens 5 cm in diameter? [ $\frac{1}{400}$  sec.].
- [20] The focal length of an  $f/2.8$  camera lens is 8 cm.
- (a) What is the diameter of the lens?
- (b) If the correct exposure of a certain scene is  $\frac{1}{200}$  sec. at  $f/2.8$ , what would be the correct exposure at  $f/5.6$ ? [2.85 cm;  $\frac{1}{50}$  sec.].
- [21] A telescope with an objective of focal length 50 cm is used to bring into view an object at a distance of 150 cm. The magnification, produced by the telescope when the eye-piece is adjusted to form the image at infinity is 5. Calculate (i) the focal length of the eye-piece and (ii) the magnifying power of the telescope when the eye-piece is adjusted to view objects at infinity. [15 cm, 3.3 sec.].
- [22] A reading telescope comprising an objective of focal length 30 cm and eye-piece of focal length 3 cm is focussed on a scale 3 m away. Determine (i) the length of the telescope and (ii) magnification produced.
- [23] A Galilean telescope consists of an objective of focal length 12 cm and a concave lens of focal length 4 cm. What should be the separation of the two lenses when a virtual image of a distant object is formed at a distance of 24 cm from the eye lens? What is the magnification produced? [7.2 cm, 2.5].



## CHAPTER - XXVII

## INTERFERENCE OF LIGHT

*Nature of light – Wave theory of light – Huygens' principle – Principle of superposition – Interference of light – Young's double-slit experiment – Coherent sources – Interference of two light waves: analytical treatment – Energy distribution – Theory of interference fringes: Expression for the width of a fringe – Conditions for interference – Shape of the interference pattern – Interference with white light – Displacement of the fringes – Production of interference fringes – Fresnel biprism – Phase change on reflection: Stoke's treatment – Interference in thin films – Film of varying thickness – Newton's rings – Interferometry – Michelson interferometer – Solved problems – Exercises.*

## 27.1 Nature of light

The branch of optics that deals with attempts to find out the nature of light, i.e., what light is and how light energy is propagated through space is called *physical optics*. Until about the later half of the seventeenth century, the general belief was that light was *corpuscular* in nature. Perhaps the most important experimental fact which led to this early belief in the corpuscular model was the propagation of light through vacuum. It was thought that any wave phenomenon would require a material medium which would be responsible for the propagation of the wave. On the other hand, all the phenomena of light like *rectilinear propagation, reflection and refraction*, known until that time, could be satisfactorily explained on the basis of corpuscular theory. According to this theory, a luminous body continuously emits invisible, tiny, light and elastic particles, called *corpuscles*. These corpuscles are emitted in all directions with tremendous velocity. Since the corpuscles are emitted with high velocity, they, in the absence of other forces, travel in straight lines according to Newton's law of motion, which explains rectilinear propagation of light. Their mechanical impact on the retina produces the *sensation of vision*. The different colours were attributed to the different sizes of the corpuscles. In reflection, the particles were supposed to act like small elastic spheres reflected from the surface like marble balls from a concrete floor. In refraction, however, it was necessary to assume

that the component of the velocity perpendicular to the surface suddenly increased if the second medium was denser than the first. It may be mentioned here that, contrary to popular belief, the *corpuscular* model of light is primarily due to Descartes rather than to Newton. Newton, however, developed further and strongly championed the cause of the model and is now, generally, associated with the model.

Beginning from the middle of the seventeenth century, a large number of experimental observations were reported. These are the *Newton's rings* (*interference of light*) as observed by Boyle and Hooke, the phenomenon of *diffraction* as observed by Grimaldi and the phenomenon of *double refraction* (*polarization of light*) as observed by Bartholinus. Corpuscular theory could not explain these phenomena satisfactorily. To overcome this difficulty, the wave model was first propounded by Huygens around 1679. Using the wave model, Huygens could explain the laws of reflection and refraction. The changes in colours were attributed to the difference in the wavelengths. In explaining refraction it was necessary to assume that the velocity of light was less in an optically denser medium than in an optically rarer medium – an assumption which was just the opposite of that in the corpuscular theory. Foucault repeated his experiment on velocity of light by placing a water tube in the path of light and found that the velocity of light was less in water than in air. The result of this experiment, therefore, was in favour of the wave theory of light. However, so compelling was Newton's authority that people around Newton had more faith on his corpuscular theory than Newton himself and no one really believed in Huygen's wave theory at the time. The acceptance of the wave theory had to wait until the beginning of the nineteenth century. Around 1802, Young performed his famous *double-slit* experiment on interference of light which could only be explained on assuming a wave model of light. Using the wave model, he also gave a satisfactory explanation of the formation of Newton's rings. Around 1816, Fresnel gave a satisfactory explanation of the diffraction pattern. The phenomenon of polarization was successfully explained by Young by assuming that light waves are transverse in character. Thus, around the second quarter of the nineteenth century, the wave theory seemed to be well established. There was, however,



one difficulty regarding the theory. This was the *ether hypothesis*. Since it was thought that a wave requires a medium for its propagation, the elastic ether theory was developed. Ether was supposed to fill all empty space and to *permeate* into all material bodies. On account of the exceedingly high velocity of light it was necessary to assume the ether to have a high elastic coefficient almost like a rigid body. This assumption was extremely inconsistent and regarded as a major drawback for the wave theory.

The nineteenth century also saw the development of electricity and magnetism. In 1820, Oersted discovered the magnetic effect of currents. Shortly afterwards, Ampere found that two parallel wires, carrying currents, attract or repulse each other, depending on the direction of currents. Faraday carried out his famous experiments on electro-magnetic induction around 1831 which showed that a varying magnetic field induces an electromotive force. All these laws of electricity and magnetism were summed up by Maxwell in the form of equations which are now referred to as *Maxwell's equations*. From these equations, he derived a wave equation and showed the possibility of the existence of electro-magnetic waves. In 1873, Maxwell, working on the hypothesis of displacement currents in dielectrics arrived at the conclusion that an electro-magnetic disturbance should travel out in space with a velocity given by  $c = 1/\sqrt{\mu_0 \epsilon_0}$  where  $\mu_0$  is the magnetic permeability and  $\epsilon_0$  specific inductive capacity of the free space. Weber and Kohlrausch by their careful experiments in 1865, on the ratio of the units, found that  $c = 3.1074 \times 10^8$  m/sec. Thus in free space the velocity of electromagnetic waves is equal to  $3.1074 \times 10^8$  m/sec. This value was found to be very close to the speed of light as measured by Fizeau in 1849, which is  $3.14858 \times 10^8$  m/sec. The sole fact that the two values are very close to each other led Maxwell to propound his famous electromagnetic theory of light according to which light waves are electro-magnetic waves.

In 1888 Hertz produced and detected electromagnetic waves of frequencies much smaller than that of light. Using a collimated beam of electro-magnetic waves, Hertz was able to demonstrate that they obey the laws of reflection. The electro-magnetic waves



were also found to exhibit phenomena like interference, diffraction and polarization. All these provide a dramatic confirmation of Maxwell's electro-magnetic theory. Moreover, it was not necessary to assume the mechanical ether theory for the propagation of these electro-magnetic radiations. In addition, there were so many other experimental results which were quantitatively explained by using Maxwell's theory that around the end of the nineteenth century physicists thought that one finally understood what light really is.

It was a great irony that Hertz, whose experiments provided dramatic confirmation of Maxwell's electro-magnetic theory, should be the one to discover in 1887, another phenomenon called *photoelectric effect*, which was destined to provide the greatest setback to the established supremacy of the wave theory and which, ultimately brought into light once again the corpuscular picture of radiant energy but in slightly modified form. This phenomenon consists in the instantaneous emission of electrons when a metal surface – specially an alkaline metal surface, is irradiated by a light beam, having a frequency above a certain critical frequency. The phenomenon could not be explained on the wave model, since according to the model, it would take several days for the atom to absorb the requisite amount of energy from a beam of light of moderate intensity to bring about this expulsion. Actually the emission starts the moment light strikes the metal surface.

In 1905, Einstein interpreted the photo-electric effect by putting forward his famous *photon theory* according to which the energy in a light beam of frequency  $\nu$  was concentrated in corpuscles of energy  $h\nu$ , where  $h$  represents the Planck's constant; these corpuscles are known as *photons*. The vestige of the wave picture was retained in as much as the energy of photon was assumed to be proportional to the frequency,  $\nu$ , of the associated light wave, i.e.,  $E = h\nu$ . In reality, Einstein took his cue from the epoch making hypothesis put forward in December, 1900 by Max Planck in connection with the explanation of the energy distribution in the *black body radiation*. According to this hypothesis, every atom, usually called the *oscillator*, absorbs or gives off energy in intermittent, and discontinuous amounts equal to integral multiple of a certain energy unit,  $h\nu$ , which Planck called *quantum* (of energy) where  $\nu$  is the oscillation frequency of the oscillator. According to the wave theory the

process of absorption or radiation of energy is continuous. Planck's theory, was therefore, a new form of the old corpuscular theory and is now known as the *quantum theory*. The quantum theory can explain the phenomena involving interaction of light with material bodies like *photo-electric emission*, *Bohr's atom model* and *Compton effect*. But it cannot explain *interference*, *diffraction* and *polarization*. These phenomena can only be explained by the wave theory.

Thus in the early part of twentieth century, there were two contradictory theories of light, equally strong in their respective spheres. Attempts have been made to reconcile the two opposite ideas into one unified theory. Wave nature of particles and corpuscular nature of waves have been experimentally demonstrated and theoretically explained. So upto now the wave nature and the corpuscular nature of light are supposed to be the two observable aspects of a single phenomenon.

## 27.2 Wave theory of light

Wave theory of light was first proposed by Huygens in 1679, but it was left to Young, Fresnel and others to develop this theory. In wave theory, light energy was supposed to be radiated from a source in the form of spherical waves with the source as the centre. According to this mode of propagation a vibrating particle placed at a point in a homogeneous medium extending in all directions, will communicate its motion to all its neighbouring particles in space. As a result these particles are disturbed from their mean positions of rest, but due to elasticity of the medium, gravity and surface tension, restoring forces come into play and the particles execute a periodic motion. Due to the periodic motion of the particles of the medium, a wave motion is produced. Thus the disturbance produced by the vibrating particle will travel with equal velocity in every direction and will, therefore, simultaneously reach all particles which are at equal distances from the point. The position of all these particles can be represented by the surface of a sphere drawn with the position of the vibrating particle as the centre. If  $v$  is the velocity of light in the medium considered and  $t$  is the time for the disturbance to reach these particles, then the radius of the sphere will be  $vt$ . With time the wave advances into spheres of gradually increasing radius. Such a sphere is known as a *wavefront*. The wavefront at any instant of time



may, therefore, be defined as the locus of all the neighbouring particles in the medium which are just being disturbed at that instant of time and are consequently in the same state of vibration.

In a homogeneous medium, the wavefronts are always actually spherical. But if we consider a wavefront at a considerable distance from the source, then any small portion of the wavefront can be considered plane. Thus waves from the sun and the stars are taken as plane waves.

A *light ray* is the direction of propagation of the light energy and is normal to a wave front.

### 27.3 Huygens' principle or construction

To explain the propagation of light waves through ether, Huygens proposed the following principles for the construction of a wavefront at a subsequent time from the known position of the wavefront at any given instant :

- (i) every point on a primary wavefront may be considered as a secondary source of disturbance.
- (ii) secondary waves or wavelets spread out from each one of these secondary source into the medium with the same velocity as the original wave.
- (iii) the envelope of all the secondary wavefronts or wavelets (in the forward direction) after any given interval of time gives rise to the secondary wavefront.

Huygens' principle is illustrated in Fig. 27.1 (a) & (b). The original wavefront  $XY$  is travelling in the direction as indicated by the arrows. We wish to find the shape of the wavefront after a time interval  $t$ . Let  $v$  be the velocity of propagation. According to Huygens' principle every point 1, 2, 3, etc., in  $XY$  should be looked upon as the new sources of secondary wavelets. After an interval of time  $t$  the secondary waves travel a distance  $vt$ . With the points 1, 2, 3, ..... etc., as centres, let us draw spheres of radius  $vt$ . The surface  $X_1Y_1$ , which is the trace of envelopes of these wavelets in the forward direction is the new wavefront. If the medium is not homogeneous, then the velocity is not constant at every point of the medium and appropriate velocity must be used for the various wavelets.



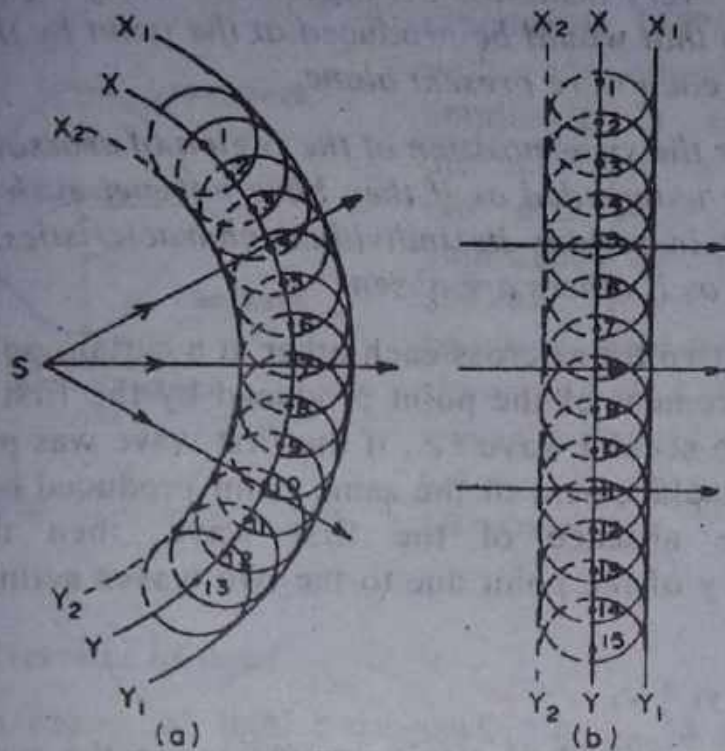


Fig. 27.1

In the simple form given above, Huygens' principle is not fully satisfactory. There is one drawback in the above model; we also obtain a wavefront  $X_2Y_2$  in the backward direction which is not present in practice. It was later shown by Stokes that the rear envelope of the wavelets does not represent the new wavefront. According to Stokes' law the intensity at each point of the secondary wave is proportional to  $(1 + \cos \theta)$  where  $\theta$  is the angle between the wave normal and the line joining the point of the secondary wave to its centre. For the point of the secondary wave which is directly behind the wave,  $\theta = \pi$  and hence  $(1 + \cos \theta)$  is zero. Thus the intensity of the secondary waves in the rear is zero and hence the rear envelope of the circle cannot represent the section of the wavefront.

#### 27.4 Principle of superposition

The principle of superposition of wave motions, first enunciated by Thomas Young in 1801, states that:

- (i) When a medium is disturbed simultaneously by more than one wave, the instantaneous resultant displacement of the medium at

every point at every instant is the algebraic sum of the displacements of the medium that would be produced at the point by the individual wave trains if each were present alone.

(ii) After the superposition at the region of crossover, the wave trains emerge unimpeded as if they have not met each other at all. Each wave train retains its individual characteristics. Each wave train behaves as if others are absent.

Suppose two trains cross each other at a certain point and let  $y_1$  be the displacement of the point produced by the first wave in the absence of the second wave i.e., if the first wave was present alone. If  $y_2$  is the displacement of the same point produced by the second wave in the absence of the first wave, then the resultant displacement  $y$  of the point due to the two waves acting together is expressed by

$$y = y_1 + y_2$$

If the two individual displacements are in the same direction, the resultant displacement will be enhanced; if they are in the opposite direction the resultant displacement will be diminished. In Fig. 27.2, the two waves are of the same frequency but of different amplitudes, say  $a$  and  $b$  where  $a > b$ . When they reach a certain point in phase with each other as in Fig. 27.2 (a), then the resultant displacement or amplitude is equal to the sum of the two amplitudes i.e.,  $(a + b)$ . The two waves reinforce each other and are said to produce *constructive interference*.

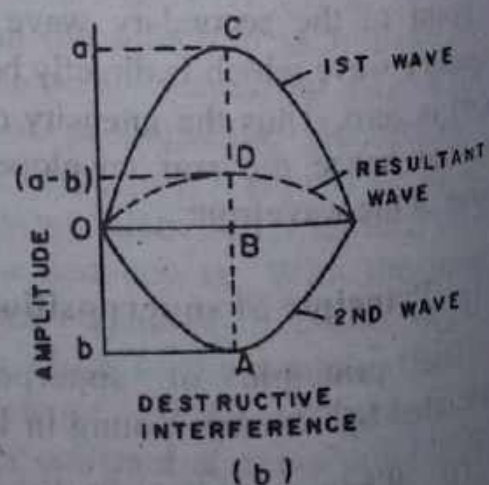
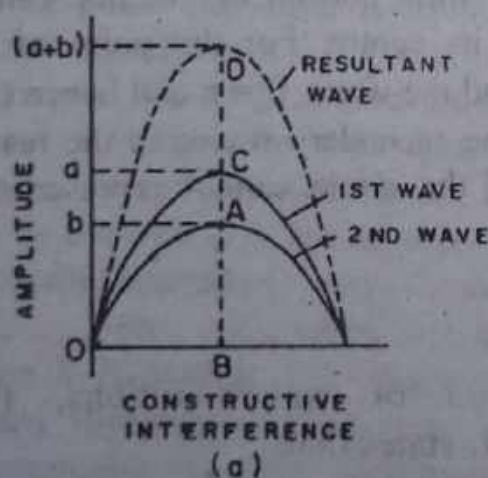


Fig. 27.2 (a) and (b)



However, as shown in Fig. 27.2 (b), if the two waves are  $\pi$

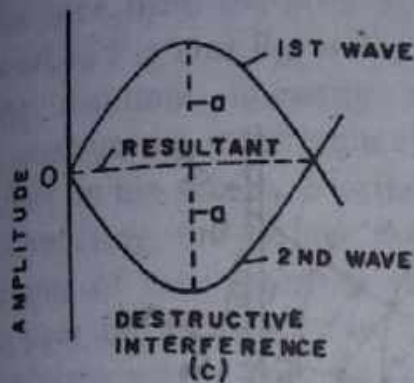


Fig. 27.2 (c)

radians or  $180^\circ$  out of phase with each other, the resultant amplitude is equal to the difference of the two equal to the difference of the two amplitudes *i.e.*,  $(a - b)$ . If in addition,  $a = b$ , then the resultant amplitude is zero as shown in Fig. 27.2 (c). The two waves neutralize each other and are said to produce *destructive interference*.

### 27.5 Interference of light

Let two beams of light cross each other at a certain point. According to principle of superposition the two light waves will continue their motion unimpeded *i.e.*, without being influenced by each other in any way. However, in the region of cross over where both the beams are acting simultaneously, a modification in their intensity is expected. The resultant intensity will be either great or less than that which would be given by one *beam alone*. This modification of intensity due to superposition of two or more beams of light is known as *interference of light*. The interference is said to be constructive or destructive when the resultant intensity is respectively more or less than that given by one beam alone.

### 27.6 Young's double-slit experiment

Historically, the phenomenon of interference of light was first demonstrated by Thomas Young in about 1801 by a simple experiment. Young allowed sunlight to pass through a pin hole  $S$  and then at some distance through two sufficiently close pin holes  $S_1$  and  $S_2$  in an opaque screen. Finally the light was received on a screen on which he observed an uneven distribution of light intensity consisting of many alternate bright and dark spots. The corpuscular theory was found to be totally inadequate to explain this. On the other hand, Young was able to explain this due to



superposition of two light waves. This experiment, described below, was regarded as a crucial one at that time, since it definitely established the wave nature of light.

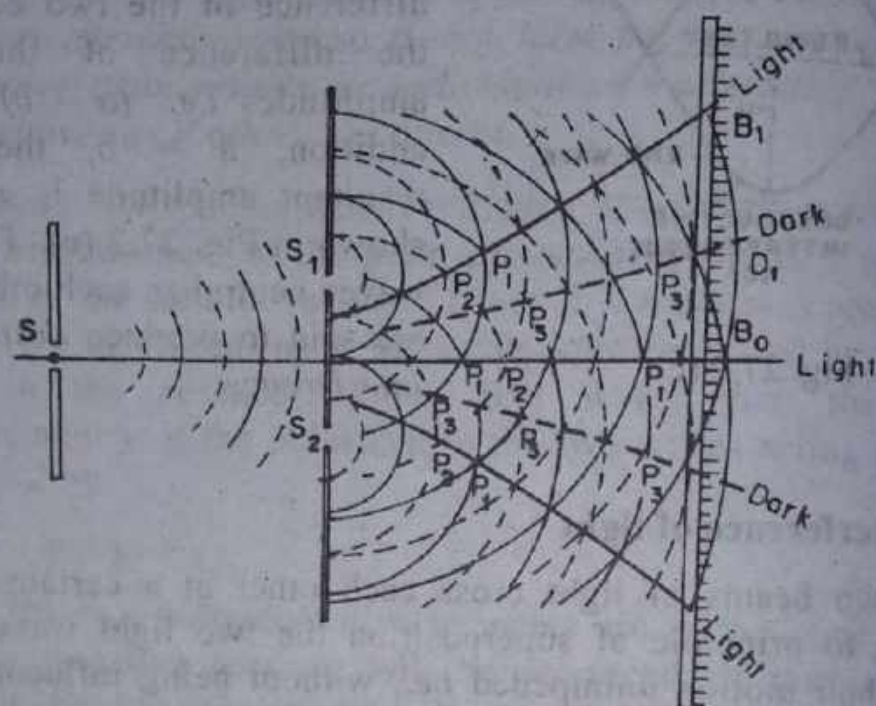


Fig. 27.3

In accordance with the modern laboratory techniques Young's experiment is performed by illuminating a narrow parallel slit  $S$  with monochromatic light of wavelength  $\lambda$ . The light coming out of the slit  $S$  is then allowed to fall on two more narrow parallel equidistant slits  $S_1$  and  $S_2$  on a screen placed at a certain distance to the right of  $S$  (Fig. 27.3). According to Huygens' principle, cylindrical waves spread out from slit  $S$  and reach the slits  $S_1$  and  $S_2$ . As the slits  $S_1$  and  $S_2$  are equidistant from  $S$  the waves reach the slits at the same time i.e.,  $S_1$  and  $S_2$  are on the same wavefront. A train of secondary wavelets, having the same amplitude, velocity, wavelength and *precisely the same phase at the start*, therefore, diverge to the right from both of these slits. Let the crest and the trough in each wave be represented by continuous and dotted circular arcs respectively. Furthermore, let the points where a crest of one wave is superposed on the crest of another wave or a trough of one wave is superposed on the trough of another wave be marked by  $P_1$  and  $P_2$  respectively and

the points where the crest of one wave is superposed on the trough of another wave be marked by  $P_3$ . If a screen be placed at a certain distance from the slits  $S_1$  and  $S_2$ , solid lines connecting the points marked  $P_1$ s and  $P_2$ s will intersect the screen at points  $B_0, B_1$ . Since the resultant intensity along these lines is always maximum (constructive interference), the points  $B_0, B_1$  will appear as bright lines on the screen. Similarly  $D_1$ s, the points of intersection of lines, connecting the points marked  $P_3$ s with the screen will represent points of zero intensity (destructive interference) and consequently appear on the screen as dark lines. Thus the result of interference between waves coming from the slits  $S_1$  and  $S_2$  will appear on the screen as alternate bright and dark lines. As long as the experimental arrangement remains undisturbed, the alternate bright and dark lines on the screen remain *stationary*. This is known as *interference pattern*. That the observed pattern is truly due to interference of two waves of light can be demonstrated by covering one of the slits. Then the well defined dark and bright lines on the screen are replaced by a pattern much coarser due to diffraction of light by the uncovered single slit. Thus a point on the screen, bright when only one slit is uncovered changes to dark when both the slits are uncovered. This cannot be explained on the basis of corpuscular theory of light, but can be readily explained on the basis of interference of two waves of light. The dark or bright lines are usually referred to as *fringes*.

### 27.7 Coherent sources – Fundamental condition of interference

For stationary interference pattern to be observed, the two sources must start either exactly in phase or with a constant phase difference. This can be possible only if the two sources are derived from a single parent source.

If two conventional light sources like two candles are used to illuminate the two slits  $S_1$  and  $S_2$  no interference pattern will be observed on the screen. This is because in a conventional light source, light comes from a large number of independent atoms each of which emits light for about  $10^{-9}$  seconds. Thus, light wave from any source is not an infinite train of wave but essentially a pulse lasting for  $10^{-9}$  seconds. Consequently, even if the two emitting atoms from the two independent sources or even different parts of



the same source are in phase at the beginning of a second, they will change their phases billion times during that second. Hence the interference pattern will keep on changing every billionth of a second. Due to such rapid changes in the phases of the two interfering waves, the points of maximum brightness and darkness would also shift their positions many million times per second, thus producing a *fleeting* pattern of interference fringes. Since the human eye cannot follow such rapid changes in intensity (it can notice changes which last at most for a tenth of a second), the net impression gained would be that of average uniform illumination. However, if we have a camera whose time of shutter opening is less than  $10^{-9}$  sec., then the film will record an interference pattern. This may be summarized by noting that light beams from two independent sources do not have any fixed phase relationship and as such do not produce any stationary interference pattern. But if the two interfering waves can be derived from the same wavefront (same emitting atom), as was done by Young in his double-slit experiment, then they will both change phase simultaneously and will always have a constant phase relationship. When these two split wavefronts, acting as if they have emanated from two sources, interfere, the points of maximum brightness and darkness will be fixed in position and hence we will obtain a stationary interference pattern. It is imperative of any interference experiment with light that the interfering sources must have this *point-to-point* phase relationship. Sources that have this relationship are called *coherent sources* and two sources can only be coherent when both become the sources of light under the influence of light waves originating from the same source. A real source and its virtual-image, two virtual sources formed due to a single source are examples of coherent sources and are employed in practice to produce interference.

### 27.8 Interference of two light waves : analytical treatment

Let us consider two sources of light  $S_1$  and  $S_2$ . Let  $P$  be any particle in the medium at distances  $x_1$  and  $x_2$  from  $S_1$  and  $S_2$  respectively (Fig. 27.4).

Let  $y_1$  be the displacement of the particle  $P$  due to waves emanating from the source  $S_1$  alone (in the absence of waves coming from  $S_2$ ) at any instant of time and let  $y_2$  be the same due to waves coming



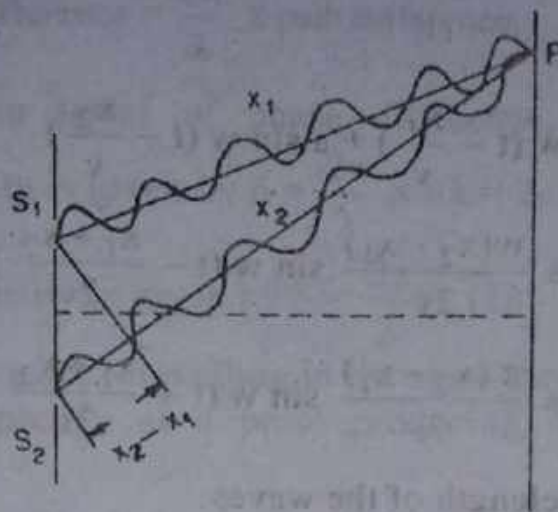


Fig. 27.4

from  $S_2$  alone at the same instant. If  $y$  be the resultant displacement of the particle  $P$  at that instant of time then according to principle of superposition

$$y = y_1 + y_2$$

If the phase of the waves emanating from both  $S_1$  and  $S_2$  at the instant of their moving out be  $\omega t$ , then according to the equation of a progressive wave

$$y_1 = a \sin \omega \left( t - \frac{x_1}{v} \right) \quad (27.1)$$

and

$$y_2 = a \sin \omega \left( t - \frac{x_2}{v} \right) \quad (27.2)$$

where  $a$  and  $v$  are the amplitude and velocity respectively of both sets of waves.

It can be seen from equations (27.1) and (27.2) that although the waves had the same phase at the instant of their moving out, the phases are different when they arrive at the point  $P$ . The difference in phase arises due to the difference in paths ( $x_1 \neq x_2$ ) traversed by them in

reaching the point  $P$ . The new phases are  $\omega \left( t - \frac{x_1}{v} \right)$  and  $\omega \left( t - \frac{x_2}{v} \right)$

Now

$$\begin{aligned}
 y &= y_1 + y_2 \\
 &= a \sin w \left( t - \frac{x_1}{v} \right) + a \sin w \left( t - \frac{x_2}{v} \right) \\
 &= 2a \cos \frac{w(x_2 - x_1)}{2v} \sin w \left( t - \frac{x_1 + x_2}{2v} \right) \\
 &= 2a \cos \frac{\pi (x_2 - x_1)}{\lambda} \sin w \left( t - \frac{x_1 + x_2}{2v} \right) \quad (27.3)
 \end{aligned}$$

where  $\lambda$  is the wavelength of the waves.

As can be seen from eqn. (27.3), the expression for the resultant motion of the particle P is also simple harmonic and the amplitude of motion is given by  $A = 2a \cos \frac{\pi (x_2 - x_1)}{\lambda}$  which is variable. If  $\lambda$  is constant, the variation of the amplitude between  $+2a$  to  $-2a$  through zero is due to the variation of the path difference  $(x_2 - x_1)$ .

For those points for which the path difference

$$x_2 - x_1 = 0, \lambda, 2\lambda, \dots = n\lambda$$

where  $n = 0, 1, 2, \dots, n$ ,  $A$  is maximum and is equal to  $\pm 2a$ . At these points the particle is violently disturbed producing what is known as *constructive interference*.

Again, for points where

$$x_2 - x_1 = \frac{\lambda}{2}, \frac{3\lambda}{2}, \frac{5\lambda}{2}, \dots = (2n - 1) \frac{\lambda}{2}$$

where  $n = 1, 2, 3, \dots, n$  (but not zero),  $A = 0$ . This means that at these points the intensity is *minimum* and the particle remains stationary. The effect produced at these points is called *destructive interference*.

Now for a path difference  $\lambda$  between two similar waves, the phase difference is  $2\pi$ . So for a path difference  $(x_2 - x_1)$ , the phase difference  $\delta$  is,  $\frac{2\pi}{\lambda} (x_2 - x_1)$ , i.e.,

$$\text{phase difference} = \frac{2\pi}{\lambda} \times \text{path difference.}$$

Therefore, in terms of phase difference, the condition of maximum intensity is given by  $\delta = \frac{2\pi}{\lambda} \times n\lambda = 2n\pi$  and the condition

of minimum intensity is given by  $\delta = \frac{2\pi}{\lambda} \times (2n - 1) \frac{\lambda}{2} = (2n - 1) \pi$ .

Hence, two light waves travelling in the same medium will reinforce each other completely at a point producing maximum resultant intensity when.

$$\begin{aligned} \text{path difference} &= n\lambda \\ \text{and} & \quad (\text{maxima}) \end{aligned} \quad (27.4)$$

$$\text{phase difference} = 2n\pi$$

where  $n = 0, 1, 2, \dots$  etc.

and will interfere destructively, when

$$\begin{aligned} \text{path difference} &= (2n - 1) \frac{\lambda}{2} \\ \text{and} & \quad (\text{minima}) \end{aligned} \quad (27.5)$$

$$\text{phase difference} = (2n - 1) \pi$$

where  $n = 1, 2, 3, \dots$  etc.

### 23.9 Energy distribution

Writing the displacements  $y_1$  and  $y_2$  produced by the individual waves in the following manner, we can obtain an expression for the intensity at a particular point on the screen.

$$\text{Let } y_1 + y_2 = a \sin wt + a \sin (wt + \delta)$$

$$\text{or, } y = a \sin wt + a \sin wt \cos \delta + a \cos wt \sin \delta$$

$$= a \sin wt (1 + \cos \delta) + a \cos wt \sin \delta$$

$$\text{Let } a (1 + \cos \delta) = R \cos \theta \quad (27.6)$$

$$\text{and } a \sin \delta = R \sin \theta \quad (27.7)$$



then

$$y = R \sin \omega t \cos \theta + R \cos \omega t \sin \theta$$

$$= R \sin (\omega t + \theta) \quad (27.8)$$

Thus the resultant displacement of the point is simple harmonic of amplitude  $R$ . Squaring (27.6) and (27.7) and adding,

$$R^2 \sin^2 \theta + R^2 \cos^2 \theta = a^2 \sin^2 \delta + a^2 (1 + \cos \delta)^2$$

$$\text{or, } R^2 = a^2 \sin^2 \delta + a^2 [1 + 2 \cos \delta + \cos^2 \delta]$$

$$= a^2 [\sin^2 \delta + \cos^2 \delta] + 2a^2 \cos \delta + a^2$$

$$= 2a^2 + 2a^2 \cos \delta$$

$$= 2a^2 (1 + \cos \delta) \quad (27.9)$$

Now intensity is equal to square of the amplitude. Hence the intensity at a certain point on the screen is given by

$$I = R^2 = 2a^2 (1 + \cos \delta)$$

$$= 2a^2 \cdot 2 \cos^2 \frac{\delta}{2} = 4a^2 \cos^2 \frac{\delta}{2} \quad (27.10)$$

where  $a$  is the amplitude of each of the individual waves and  $\delta$  is the phase difference between the waves.

When the phase difference between the waves is  $\delta = 2n\pi$  (or the path difference is a whole number multiple of wavelength ( $x_2 - x_1 = n\lambda$ ), intensity is maximum and is given by

$$I = 4a^2 \quad (27.11)$$

Again when the phase difference is  $\delta = (2n - 1)\pi$  [or the path difference  $x_2 - x_1 = (2n - 1) \frac{\lambda}{2}$ ] intensity is minimum and is given by

$$I = 0 \quad (27.12)$$

The two conditions discussed above obviously refers to the condition of *bright* and *dark* fringes.

We shall further show that the spacings of the fringes on the screen is constant for given values of  $\lambda$ ,  $d$  and  $D$  (Art. 27.10). Thus

if we mark the values of  $\delta$  viz.  $0, \pi, 2\pi, 3\pi, 4\pi$ , etc. at equidistant points along a horizontal line and take the normal line through  $\delta = 0$  as the axis of  $I$ , then for  $\delta = 0, \pm 2\pi, \pm 4\pi$ , etc.,  $I = 4a^2$  and for  $\delta = \pm \pi, \pm 3\pi, \pm 5\pi$ , etc.,  $I = 0$ . In fact as  $\delta$  gradually increases from  $0$  to  $\pi$ ,  $\cos \delta$  gradually decreases from  $+1$  through  $0$  to  $-1$  and as a consequence,  $I$  gradually diminishes from  $4a^2$  to  $0$ . This explains the shape of the distribution curve as depicted in Fig. 27.5. It may appear from the curve that energy is destroyed at the points of minima: however, it should be emphasized that no destruction of light energy occurs in the phenomenon of interference of light. Since light energy is not transformed into any other form in interference of light, according to law of conservation of energy the total amount of light energy must remain constant. What has happened is merely a *redistribution of energy*, the energy has been transferred from the points of minimum intensity to the points of maximum intensity, where the energy is  $4a^2$  instead of  $2a^2 = (a^2 + a^2)$ , the sum of the intensities of the individual waves in the absence of interference. However, the average value of the energy over any number of fringes is the same as if the interference effects were absent. For example the average value of the intensity on the screen

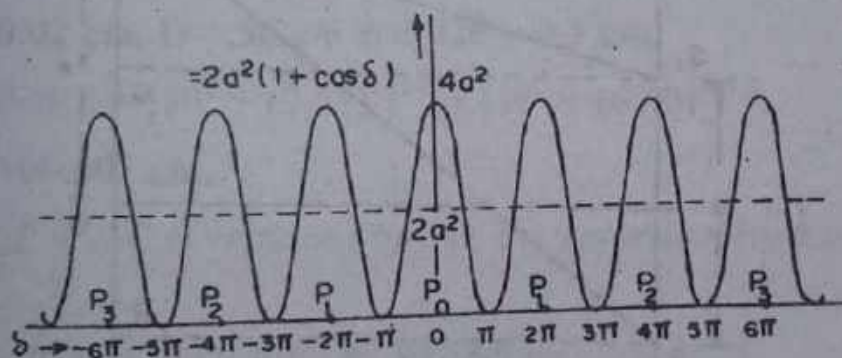


Fig. 27.5

over the range  $\delta = 0$  to  $\delta = 2\pi$  is given by

$$I_{\text{average}} = \frac{\int_0^{2\pi} I d\delta}{\int_0^{2\pi} d\delta} = 2a^2 \frac{\int_0^{2\pi} (1 + \cos \delta) d\delta}{\int_0^{2\pi} d\delta} = 2a^2 \quad (27.13)$$



This justifies the statement made above. The average intensity is equal to uniform intensity  $2a^2$  which will be present in the absence of interference phenomenon due to the two waves. Therefore, the formation of interference fringes is in accordance with the law of conservation of energy.

### 27.10 Theory of interference fringes : expression for the width of a fringe

In Fig. 27.6,  $S$  is a narrow monochromatic source of light.  $S_1$  and  $S_2$  are two narrow parallel slits separated by a distance  $d$  and are equidistant from  $S$ .  $S_1$  and  $S_2$  act as two coherent sources of light (i.e., they emit waves of same wavelength, period and phase). Let  $PR$  be a screen placed at a distance  $D$  from the sources. At point  $O$  on the screen, which is equidistant from the sources, the path difference of the waves is zero and they arrive in phase. Hence  $O$  will be a bright point (maximum intensity).

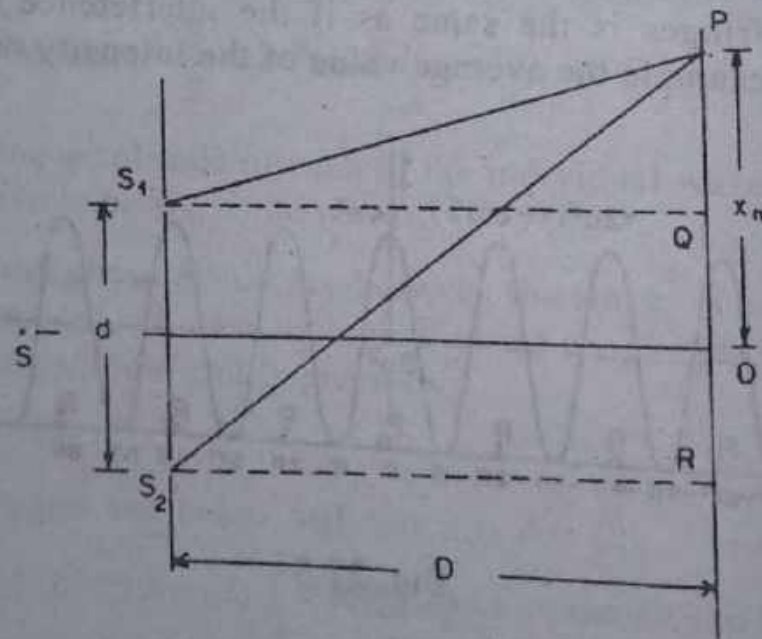


Fig. 27.6

Let the point  $P$ , located on the screen at a distance  $x_n$  from the point  $O$ , represent the  $n^{\text{th}}$  bright fringe.  $S_1Q$  and  $S_2R$  are perpendiculars on the screen from the points  $S_1$  and  $S_2$  respectively.

In the right angled triangle  $S_1PQ$ ,



$$S_1P^2 = S_1Q^2 + PQ^2 = D^2 + \left(x_n - \frac{d}{2}\right)^2 \quad (27.14)$$

Similarly from the right-angled triangle  $S_2PR$ ,

$$S_2P^2 = D^2 + \left(x_n + \frac{d}{2}\right)^2 \quad (27.15)$$

Subtracting eqn. (27.14) from (27.15)

$$S_2P^2 - S_1P^2 = \left[D^2 + \left(x_n + \frac{d}{2}\right)^2\right] - \left[D^2 + \left(x_n - \frac{d}{2}\right)^2\right] = 2x_nd$$

$$\text{or, } (S_2P - S_1P)(S_2P + S_1P) = 2x_nd$$

$$\text{or, } S_2P - S_1P = \frac{2x_nd}{S_2P + S_1P} \quad (27.16)$$

Both  $x_n$  and  $d$  are small compared to  $D$ , which is usually several thousand times longer than  $x_n$  or  $d$ . One can, therefore, write approximately

$$S_2P = S_1P \cong D \quad \text{or, } S_2P + S_1P \cong 2D$$

[In a typical interference experiment,

$$d = 0.02 \text{ cm, } D = 50 \text{ cm and } OP = 0.5 \text{ cm,}$$

$$\begin{aligned} S_2P + S_1P &= [50^2 + (0.51)^2]^{1/2} + [50^2 + (0.49)^2]^{1/2} \\ &= 100.005 \text{ cm.} \end{aligned}$$

Thus if  $S_2P + S_1P$  is replaced by  $2D$ , the error involved is about 0.005%]

Eqn. (27.16) can, therefore, be written as

$$S_2P - S_1P = \frac{2x_nd}{2D} = \frac{x_nd}{D} \quad (27.17)$$

Now  $(S_2P - S_1P)$  is the path difference of the light waves at the point  $P$  and according to the condition of interference, must be equal to  $n\lambda$ , since  $P$  represents the  $n^{\text{th}}$  bright fringe.

$$\text{Thus, } S_2P - S_1P = \frac{x_nd}{D} = n\lambda$$

$$\text{or, } x_n = n\lambda \frac{D}{d} \quad (27.18)$$

when  $n = 0, 1, 2, \dots$

$n = 0$  corresponds to the central bright fringe,  $n = 1$  gives the distance of the first bright fringe,  $n = 2$  of the second bright fringe and so on. Substituting  $n = n + 1$  in eqn. (27.18)

$$x_{n+1} = (n + 1) \lambda \frac{D}{d} \quad (27.19)$$

where  $x_{n+1}$  is the distance of the  $(n + 1)^{\text{th}}$  bright fringe from the central bright fringe. Subtracting eqn. (27.18) from (27.19), we get

$$x_{n+1} - x_n = [n + 1 - n] \lambda \frac{D}{d} = \frac{\lambda D}{d} \quad (27.20)$$

Now  $(x_{n+1} - x_n)$  is the distance of separation between the  $n^{\text{th}}$  and  $(n + 1)^{\text{th}}$  bright fringe. Substituting  $n = 1, 2, 3, 4, \dots$

it can be seen that

$$x_4 - x_3 = x_3 - x_2 = x_2 - x_1 = x_{n+1} - x_n = \frac{\lambda D}{d}$$

= constant and independent of the fringe number.

Thus the distance between any two consecutive bright fringes is same or all bright fringes are equally spaced.

If P be a point where the  $n^{\text{th}}$  dark fringe, instead of the bright fringe appears and if  $x'_n$  be its distance from 0, then the path difference

$$S_2P - S_1P = \frac{x'_n d}{D} = (2n - 1) \frac{\lambda}{2}$$

$$\text{or, } x'_n = (2n - 1) \frac{\lambda D}{2d} \quad (27.21)$$

Substituting  $n = n + 1$ , gives us the distance of the  $(n + 1)^{\text{th}}$  dark fringe

$$x_{n+1} = \{2(n + 1) - 1\} \frac{\lambda D}{2d} \quad (27.22)$$

Subtracting eqn. (27.21) from (27.22), we get for the distance of separation between the  $n^{\text{th}}$  dark fringe

$$\begin{aligned} x'_{n+1} - x'_n &= (2n + 2 - 1 - 2n + 1) \frac{\lambda D}{2d} \\ &= \frac{2\lambda D}{2d} = \frac{\lambda D}{d} = \text{constant} \end{aligned} \quad (27.23)$$

Thus we see from eqn. (27.23) that the distance between two consecutive dark fringes is constant and independent of the fringe number and is the same as the distance between two consecutive bright fringes.

Thus the distance between any two consecutive bright or dark fringes which includes one dark and one bright fringe is same. This distance is called the *fringe-width* ( $X$ ).

Then

$$X = \frac{\lambda D}{d} \quad (27.24)$$

From eqn. (27.24) it can be seen that

$$\lambda = \frac{Xd}{D} \quad (27.25)$$

Thus if the fringe-width, the distance of separation of the sources and the distance of the screen from the sources are known, then the phenomenon of interference can be employed to determine the wavelength of unknown monochromatic light.

### 27.11 Conditions for interference

In a well defined interference pattern the intensity at regions corresponding to destructive interference must remain zero as long as the experimental arrangement remains undisturbed. Furthermore, each fringe should be distinctly visible which means that the spacings between the fringes should be reasonably large. To achieve these objectives, it is very essential that the following conditions must be fulfilled in every experimental arrangement.

- (i) The two beams of light which interfere must be coherent i.e., must originate from the same source of light.



This is important since the total phase difference,  $\delta$ , between two interfering light waves depends on two factors:

(a) The initial phase difference  $\delta_1$ , between the two sources and (b) phase difference due to optical path difference ( $S_2P - S_1P$ ) namely,

$$\delta_2 = \frac{2\pi}{\lambda} \times (S_2P - S_1P).$$

Obviously  $\delta_2$  remains constant throughout the experiment. Hence the constancy of  $\delta$  will depend on the constancy of  $\delta_1$ . As explained in Art. 27.7 the initial phase difference between the two interfering light waves will remain constant only if the sources are coherent.

(ii) *The two interfering waves must have the same amplitude; otherwise the intensity will not be zero at the regions of destructive interference.*

(iii) *The original source must be monochromatic.* Otherwise it may so happen that a particular point will satisfy the condition of destructive interference for a particular wavelength while satisfying the condition of constructive interference for some other wavelengths present in the source. Thus the interference pattern, instead of consisting of alternate bright and dark lines, will be coloured.

(iv) *The fringe-width should reasonably be as large as possible so that each fringe can be recognized distinctly.* From the expression for fringe-width,  $X = \frac{\lambda D}{d}$ , it can be seen that for this to happen, the separation between the two sources should be as small as possible while the distance of the screen from the sources should be as large as possible.

(v) *The two interfering waves must be propagated in almost the same direction or the two interfering wavefronts must intersect at a very small angle. The small separation between the two sources ensures this.*

## 27.12 Shape of the interference pattern

Eqn. (27.14) may be rewritten as

$$S_1P = [D^2 + (x_n - \frac{d}{2})^2]^{1/2}$$

and

$$S_2P = [D^2 + (x_n + \frac{d}{2})^2]^{1/2}$$

Thus the expression for the path difference,  $\Delta = S_2P - S_1P$ , reduces to

$$\Delta = S_2P - S_1P = [D^2 + (x_n + \frac{d}{2})^2]^{1/2} - [D^2 + (x_n - \frac{d}{2})^2]^{1/2}$$

$$\text{or, } \Delta + [D^2 + (x_n - \frac{d}{2})^2]^{1/2} = [D^2 + (x_n + \frac{d}{2})^2]^{1/2}$$

which can be rearranged in the form:

$$\frac{x_n^2}{\Delta^2/4} - \frac{D^2}{(d^2 - \Delta^2/4)} = 1 \quad (27.26)$$

Eqn. (27.26) represents the equation of a hyperbola in the  $X - Y$  plane with  $S_1$  and  $S_2$  as the two foci so long as  $\Delta$ , the path difference is a constant. Thus the loci of the points of constant path difference are *confocal hyperbolae* with  $S_1$  and  $S_2$  as the common foci.

The curves given by eqn. (27.26), when rotated with the line  $S_1S_2$  as the axis, will trace out hyperboloids in space. The shape of the fringe observed will depend upon the position of the screen. If the screen is held parallel to the line  $S_1S_2$  then the fringe will be approximately *straight lines*. If the screen be placed at right angles to the line  $S_1S_2$ , then the fringes will be cross-sections of the hyperboloids on the screen and hence appear *circular* in shape.

**Example 27.1.** In a Young's double slit experiment, the separation between the sources are 0.18 mm. The fringes are observed on a screen 90 cm away. If with certain monochromatic source of light, the third bright fringe is situated at a distance of 8.1 mm from the central bright fringe, find the wavelength of light.

**Soln.**

$$X = \frac{8.1}{3} \text{ mm} = 2.7 \text{ mm} = 2.7 \times 10^{-1} \text{ cm}$$



$$d = 0.18 \text{ mm} = 0.18 \times 10^{-1} \text{ cm},$$

$$D = 90 \text{ cm}, \lambda = ?$$

$$X = \frac{\lambda D}{d} \quad \text{or,} \quad \lambda = \frac{Xd}{D}$$

$$= \frac{2.7 \times 10^{-1} \times 0.18 \times 10^{-1}}{90} \text{ cm}$$

$$= 54 \times 10^{-6} \text{ cm} = 5400 \text{ A.U.}$$

**Example 27.2.** In a double slit experiment using sodium light of wavelength  $5.89 \times 10^{-5} \text{ cm}$ , an interference pattern is obtained in which 20 equally spaced fringes occupied a distance of 2.0 cm on a screen. On replacing the sodium lamp with another monochromatic source but making no other changes, 30 fringes are found to occupy a distance of 2.4 cm, on the screen. What is the wavelength of light from this source?

**Soln.**

$$X = \frac{\lambda D}{d}$$

Since  $D$  and  $d$  are constants  $X$  is obviously proportional to  $\lambda$ .

In the first case  $X_1 \propto \lambda_1$

In the second case  $X_2 \propto \lambda_2$

$$\therefore \frac{\lambda_2}{\lambda_1} = \frac{X_2}{X_1}; \quad \text{or,} \quad \lambda_2 = \lambda_1 \cdot \frac{X_2}{X_1}$$

$$\text{Now } X_1 = \frac{2.0}{20} \text{ cm} = 0.1 \text{ cm and}$$

$$X_2 = \frac{2.4}{30} \text{ cm} = .08 \text{ cm}$$

$$\text{Hence } \lambda_2 = \lambda_1 \cdot \frac{.08}{0.1}$$

$$= 5.89 \times 10^{-5} \times 0.8 \text{ cm}$$

$$= 4.712 \times 10^{-5} \text{ cm} = 4712 \text{ A.U.}$$



**Example 27.3.** In a double-slit experiment the separation between the slits is 2.5 mm and the distance of the screen from the slits is 50 cm. If the arrangement is illuminated with sodium light of wavelength 5890 Å, calculate (i) the angular position of the first maxima and (ii) the linear separation between two adjacent minima. What will be the fringe-width for bright fringes?

**Soln.**

The linear distance,  $x_n$  of the  $n^{\text{th}}$  bright fringe from the central bright fringe is given by the relation  $\frac{x_n d}{D} = n\lambda$ .

Hence for the first bright fringe, we have

$$x_1 \frac{D}{d} = \lambda; \quad D = 50 \text{ cm}, d = 2.5 \text{ mm} = 2.5 \times 10^{-1} \text{ cm}$$

$$\lambda = 5890 \text{ Å} = 5890 \times 10^{-8} \text{ cm} \quad (1 \text{ Å} = 10^{-8} \text{ cm} = 1 \text{ A.U.})$$

$$\therefore x_1 = \frac{50}{2.5 \times 10^{-1}} \times 5890 \times 10^{-8} \text{ cm}$$

$$= 11.78 \times 10^{-3} \text{ cm}$$

$$\text{now } \tan \theta = \frac{x_1}{D} = \frac{11.78 \times 10^{-3}}{50}$$

$$= 0.000236$$

$$\text{or, } \theta = \tan^{-1} (0.000236) \cong 14'$$

The linear separation between two adjacent minima or the fringe-width

$$X = \frac{\lambda D}{d} = \frac{5890 \times 10^{-8} \times 50}{2.5 \times 10^{-1}}$$

$$= 0.011780 \text{ cm}$$

$$= 0.1178 \text{ mm}$$

The separation between two adjacent maxima is the same

**Example 27.4.** Light from a sodium vapour lamp ( $\lambda = 589 \text{ nm}$ ) forms an interference pattern on a screen  $0.8 \text{ m}$  from a pair of slits. The bright fringes in the pattern are  $0.35 \text{ cm}$  apart. What is the slit separation.

**Soln.**

$$X = \frac{\lambda D}{d}$$

Here

$$\lambda = 589 \text{ nm} = 589 \times 10^{-9} \text{ m}$$

$$D = 0.8 \text{ m}$$

$$X = 0.35 \text{ cm} = 0.35 \times 10^{-2} \text{ m}$$

$$\text{or, } d = \frac{\lambda D}{X}$$

$$\therefore d = \frac{(589 \times 10^{-9})(0.8)}{(0.35 \times 10^{-2})} \text{ m}$$

$$= 1.35 \times 10^{-4} \text{ m} = 0.135 \text{ mm.}$$

**Example 27.5.** In a Young's double-slit experiment, the slits are  $2 \text{ mm}$  apart and are illuminated with a mixture of two wavelengths,  $\lambda = 750 \text{ nm}$  and  $\lambda' = 900 \text{ nm}$ . At what distance from the common central bright fringe on a screen  $2 \text{ m}$  from the slits will a bright fringe from one interference pattern coincide with a bright fringe from the other?

**Soln.**

Let the  $m$ 'th bright fringe from the  $\lambda$  pattern coincide with the  $m$ 'th bright fringe from the  $\lambda'$  pattern.

The distance of the  $m$ 'th bright fringe from the central bright fringe

$$Y_m = mX = \frac{m\lambda D}{d}$$

Similarly the distance

$$Y_{m'} = m'X' = \frac{m'\lambda'D}{d}$$

The two distances are equal.

$$\text{So } \frac{m\lambda D}{d} = \frac{m'\lambda'D}{d}$$

$$\text{or, } \frac{m}{m'} = \frac{\lambda'}{\lambda} = \frac{900}{750} = \frac{6}{5}$$

Hence the first position at which overlapping occurs is

$$y_6 = y'_5 = \frac{(6)(2)(750 \times 10^{-9})}{2 \times 10^{-3}} = 4.5 \text{ mm.}$$

### 27.13 Interference with white light

We shall now discuss the interference pattern when the slit  $S$  is illuminated by white light. The central fringe formed at  $O$  will clearly be white since the geometrical path difference of this point from  $S_1$  and  $S_2$  is zero and hence all the wavelengths will constructively interfere here. Let us now consider a point  $P$  slightly above or below the point  $O$ . If  $(S_2P - S_1P)$  is approximately  $2 \times 10^{-5}$  cm, we will see that

$$S_2P - S_1P = 2 \times 10^{-5} \text{ cm} = \frac{\lambda_{\text{violet}}}{2}$$

where the wavelength corresponding to violet end of the visible spectrum is  $4 \times 10^{-5}$  cm. Thus complete destructive interference for the violet colour will occur at this point. Partial destructive interference will occur for other wavelengths. Therefore, at the point  $P$ , we shall have a line devoid of the violet colour and will appear reddish. Let us consider another point  $Q$  such that

$$S_2Q - S_1Q = 3.75 \times 10^{-5} \text{ cm} = \frac{\lambda_{\text{red}}}{2}$$

where  $7.5 \times 10^{-5}$  cm is the wavelength of the red end of the visible spectrum. The point  $Q$  will therefore be completely devoid of the red colour but will correspond to almost constructive interference for the violet colour. Thus following the white central fringe we will have coloured fringes; when the path difference is about  $2 \times 10^{-5}$  cm the fringe will be red in colour, then the colour will gradually change to violet. The coloured fringes will soon disappear because at point far away from  $O$ , so many wavelengths in the visible region of the spectrum will constructively interfere that we will observe uniform white illumination. For example let us consider a point  $R$



such that  $S_2R - S_1R = 30 \times 10^{-5}$  cm. Wavelengths corresponding to  $\frac{30 \times 10^{-5}}{n}$  where  $n = 1, 2, \dots$  will constructively interfere at this point. In the visible region these wavelengths are  $7.5 \times 10^{-5}$  cm (red),  $6 \times 10^{-5}$  cm (yellow),  $5 \times 10^{-5}$  cm (greenish yellow) and  $4.4 \times 10^{-5}$  cm (violet). Again wavelengths corresponding to

$$\frac{30 \times 10^{-5} \text{ cm}}{(n + \frac{1}{2})} \text{ where } n = 1, 2, \dots \text{ will destructively}$$

interfere and are absent at this point.

These wavelengths in the visible region are  $6.67 \times 10^{-5}$  cm (orange) and  $5.5 \times 10^{-5}$  cm (indigo). Hence the colour of the line formed at the point R, as seen by unaided eye, will be white. Thus, gets a white central fringe at the point of zero along with a few coloured fringes on both the sides, the colour soon fading off to white.

#### 27.14 Displacement of the fringes (measurement of the thickness of a very thin plate)

We shall now investigate the effect on the interference pattern of introducing a thin transparent plate in the path of one of the two interfering beams of monochromatic light. Let  $e$  be the thickness of the plate and  $\mu$  its refractive index for the mono-chromatic light used. From Fig. 27.7 it is clear that a light wave in travelling from  $S_1$  to P has to traverse a distance  $(S_1P - e)$  in air and a distance  $e$  in the plate. The total time,  $t$ , required for this journey is, therefore, given by

$$\begin{aligned} t &= \frac{S_1P - e}{c} + \frac{e}{v} \\ &= \frac{1}{c} [S_1P - e + \frac{c}{v} e] \\ &= \frac{1}{c} [S_1P + (\mu - 1) e] \end{aligned} \quad (27.27)$$

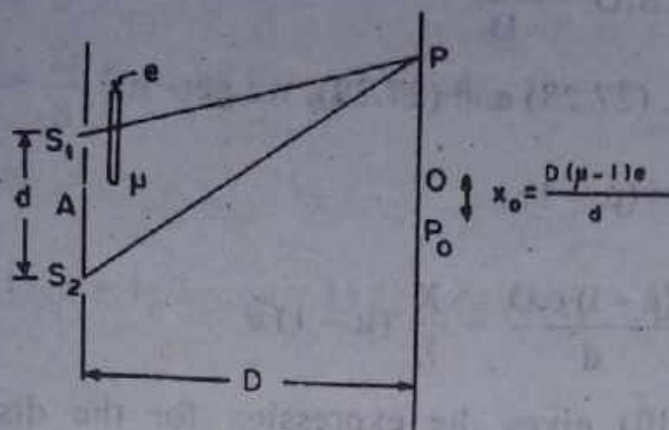


Fig. 27.7

where  $c$  and  $v$  are the velocities of light in air and in the plate respectively.

The physical interpretation of eqn. (27.27) is that due to the introduction of the thin plate into the path of the light wave considered, the effective path or the optical path in air from  $S_1$  to  $P$  becomes

$$c \times t = S_1P + (\mu - 1)e$$

Similarly the effective path in air from  $S_1$  to  $P_0$ , the point equidistant from  $S_1$  and  $S_2$  becomes  $S_1P_0 + (\mu - 1)e$ .

Since  $[S_1P_0 + (\mu - 1)e] > S_2P_0$ , the central bright fringe of zero order is not formed at  $P_0$  which of course, is the normal position of the central fringe in the absence of the plate. To locate the new position of the central fringe, one should move along the screen in such a direction that the left side of the above inequality may decrease while the right side may increase so that the two sides may eventually become equal at some point. Suppose in the presence of the plate, the central fringe of zero optical path difference is formed at  $O$ ; we, therefore, write,

$$S_2O = S_1O + (\mu - 1)e$$

$$\text{or, } S_2O - S_1O = (\mu - 1)e \quad (27.28)$$

If  $P_0O = x_0$  then the geometrical path difference, as given by eqn. (27.17), is



$$S_2O - S_1O = \frac{x_0 d}{D} \quad (27.29)$$

From eqns. (27.28) and (27.29), we get

$$\begin{aligned} \frac{x_0 d}{D} &= (\mu - 1) e \\ \text{or, } x_0 &= \frac{(\mu - 1) e D}{d} = \frac{X}{\lambda} (\mu - 1) e \end{aligned} \quad (27.30)$$

Eqn. (27.30) gives the expression for the distance through which the fringe system has been displaced as a result of the introduction of the thin plate.

Furthermore, let the point O where the central fringe shifts after the introduction of the plate be occupied by the  $n^{\text{th}}$  bright fringe in the original system without the plate, then

$$\begin{aligned} S_2O - S_1O &= n\lambda \\ \text{Hence } (\mu - 1) e &= n\lambda \end{aligned} \quad (27.31)$$

It can be easily shown that the spacings of the interference fringes, however, remains unaffected due to the introduction of the plate. If, in the presence of the plate, bright fringe of the  $n^{\text{th}}$  order is formed at the point P, then the optical path difference

$$\begin{aligned} S_2P - [S_1P + (\mu - 1) e] &= n\lambda \\ \text{or, } S_2P - S_1P &= n\lambda + (\mu - 1) e \end{aligned}$$

If  $PP_0 = x_n$ , then the geometrical path difference

$$\begin{aligned} S_2P - S_1P &= \frac{x_n d}{D} \\ \text{or, } x_n &= \frac{D}{d} (S_2P - S_1P) \\ &= \frac{D}{d} [n\lambda + (\mu - 1) e] \end{aligned} \quad (27.32)$$

Similarly for the  $(n + 1)^{\text{th}}$  bright fringe, we have



$$x_{n+1} = \frac{D}{d} [(n+1)\lambda + (\mu-1)e] \quad (27.33)$$

$$\begin{aligned} \text{Hence } X &= x_{n+1} - x_n \\ &= \frac{D}{d} [\{(n+1)\lambda + (\mu-1)e\} - \{n\lambda + (\mu-1)e\}] \\ &= \frac{D}{d} \lambda \end{aligned} \quad (27.34)$$

Eqn. (27.34) is a constant, independent of the thickness of the plate and is same as that derived in the absence of the plate (Eqn. 27.20). Same can be proved in the case of dark fringes as well.

Thus the effect of introducing a thin transparent plate into the path of one of the interfering beams is to shift the entire interference pattern laterally through a distance  $[\frac{D}{d} (\mu-1)e]$  without altering the spacings of the fringes. On putting  $n = 0$  in eqn. (27.32), one can again get  $x_0$ , the distance through which the fringes have been shifted. Therefore, knowing  $x_0$ ,  $D$ ,  $d$  and  $\lambda$ , the thickness of the transparent plate can be calculated.

**Note :** If a monochromatic source of light is used, the fringes will be similar and it is difficult to locate the central fringe and hence the position where the central fringe shifts after the introduction of the transparent plate. Therefore, white light is used. The fringe will be coloured but the central fringe will be white (Art. 27.13). When the cross-wire is at the central white fringe without the transparent plate in the path, the reading is noted. When the plate is introduced, the position to which the central white fringe shifts is observed. The difference between the two positions on the micrometer scale of the eye-piece gives the value of the shift,  $x_0$ . Now with the monochromatic source of light, the micrometer eye-piece is moved through the same distance  $x_0$  and the number of fringes that cross the field of view is observed. The value of  $e$  can thus be calculated from either eqns. (27.30) or (27.31). This principle of interference is very useful for measuring very small thicknesses, which are otherwise very difficult to measure by conventional means.

**Example 27.6.** A thin sheet of mica ( $\mu = 1.58$ ) is used to cover one slit of the double-slit arrangement. The central point on the screen is now occupied by what used to be the seventh bright fringe. If the wavelength of the light used is  $5500 \text{ \AA}$ , what is the thickness of the mica sheet?

**Soln.**

If  $e$  is the thickness of the mica sheet, then we have

$$e(\mu - 1) = n\lambda$$

$$\mu = 1.58, n = 7$$

$$\lambda = 5500 \times 10^{-8} \text{ cm}$$

$$\text{or, } e(1.58 - 1) = 7 \times 5500 \times 10^{-8}$$

$$\text{or, } e = \frac{7 \times 5500 \times 10^{-8}}{0.58} \text{ cm.} = 6.64 \times 10^{-4} \text{ cm}$$

**Example 27.7.** On introducing a transparent plate of thickness  $5.89 \times 10^{-4} \text{ cm}$  in the path of one of the interfering beams of a double-slit experiment, the central bright fringe is found to shift to position previously occupied by the sixth bright fringe. If sodium light of wavelength  $5890 \text{ \AA}$  is used to illuminate the slits, calculate the refractive index of the material of the plate.

**Soln.**

We have

$$e(\mu - 1) = n\lambda$$

$$e = 5.89 \times 10^{-4} \text{ cm}$$

$$\lambda = 5890 \text{ \AA} = 5890 \times 10^{-8} \text{ cm}$$

$$n = 6$$

$$\therefore 5.89 \times 10^{-4} (\mu - 1) = 6 \times 5890 \times 10^{-8}$$

$$\text{or, } (\mu - 1) = \frac{7 \times 5500 \times 10^{-8}}{0.58} = 0.6$$

$$\text{or, } \mu = 1 + 0.6 = 1.6$$

**Example 27.8.** In a double-slit experiment the separation between the sources is  $2.5 \text{ mm}$  and the distance of the screen from the sources is  $100 \text{ cm}$ . When the arrangement is illuminated with



monochromatic light, a well-defined interference pattern is obtained. On now introducing a transparent plate of .05 mm thickness, the central fringe is found to move across the screen through a distance of 1 cm. Calculate the refractive index of the material of the plate.

**Soln.**

We have

$$x_0 = \frac{D(\mu - 1)e}{d} \quad \text{or,} \quad (\mu - 1) = \frac{x_0 d}{D.e}$$

where  $x_0 = 1$  cm,  $D = 100$  cm,

$$d = 2.5 \times 10^{-1} \text{ cm}$$

and  $e = .05$  mm =  $5 \times 10^{-3}$  cm

$$\therefore (\mu - 1) = \frac{1 \times 2.5 \times 10^{-1}}{100 \times 5 \times 10^{-3}} = \frac{1}{2} = 0.5$$

$$\text{or, } \mu = 1.5$$

### 27.15 Production of interference fringes

Interference fringes can be obtained by allowing light rays from two coherent sources to interfere on a screen or in space. There are different methods of producing two coherent sources; but all these methods may be conveniently classified under the following two main heads:

(a) *division of wavefront* : In this method a wavefront is divided laterally into two parts by utilizing the phenomenon of reflection, refraction or diffraction in such a way that the directions of the two divided parts are changed. Since they are derived from the same wavefront, they will always have a *point-to-point phase relationship* i.e., they are *coherent*. Hence when these two wavefronts are allowed to interfere after travelling different paths, they will produce an interference pattern. This method demands a very narrow source of light. The formation of interference fringes by (i) *bi-prism* (ii) *Lloyd's single mirror*, (iii) *Fresnel's double mirror*, (iv) *Billet's divided lens* and (v) *Rayleigh's interferometer* belong to this category.

(b) *division of amplitude* : In this method, the wavefront is also divided into two parts by a combination of both reflection and refraction. Since the resulting wavefronts are derived from the same source, they



satisfy the condition of coherence. The two wavefronts maintain the original width but their amplitudes will be reduced. On being reunited, they produce interference fringes. Examples of this class are the interference effects observed in (i) *thin films* (ii) *Newton's rings* (iii) *Michelson's interferometer*, (iv) *Jamin's interferometer* and (v) *Fabry-Perot interferometer*. In these cases, we require an extended source of light and as the interference effects corresponding to different points of the source are superposed, we get brighter fringes.

### 27.16 Fresnel bi-prism

A bi-prism is essentially two prisms, each of very small refracting angle placed base to base. In reality the bi-prism is constructed from a single plate of glass by suitably grinding and polishing it. The obtuse angle of the prism is only slightly less than  $180^\circ$  and the other angles are equal. In a very thin bi-prism the obtuse angle may be  $179^\circ 20'$ , the other two angles being  $20'$  each. In the experimental arrangement the bi-prism is placed with its refracting edge accurately parallel to the slit  $S$  (line source) which is illuminated by a source of monochromatic light of wavelength  $\lambda$ . When light is allowed to fall symmetrically on the bi-prism, the refracting edge  $B$  divides the incident wavefront into two parts (Fig. 27.8). Firstly the one which in passing through the upper half  $ABD$  of the bi-prism is deviated through a small angle towards the lower

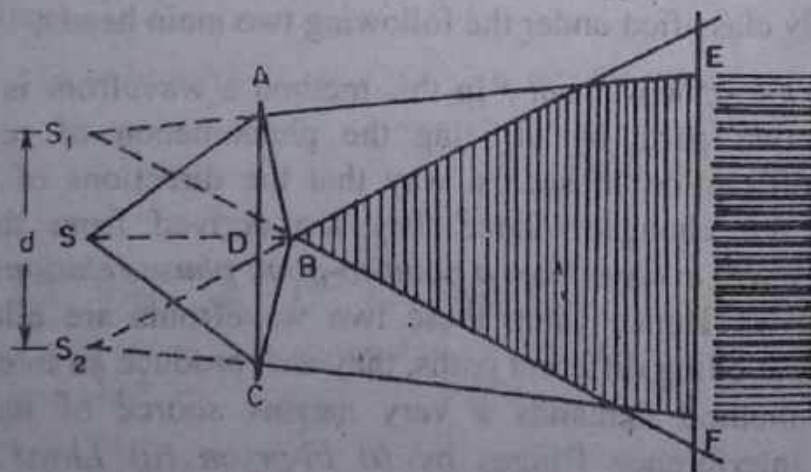


Fig. 27.8

half of the diagram and appears to diverge from the virtual source  $S_1$ . Secondly the one which passes through the lower half is deviated through a small angle towards the upper half and appears to diverge

from the virtual source  $S_2$ .  $S_1$  and  $S_2$  being the image of the slit  $S$ , obviously function as coherent sources in this experiment. Moreover, the two emergent wavefronts intersect at small angles and hence the fundamental condition of interference is satisfied. If a screen is placed on the right of the bi-prism, interference fringes of equal widths are produced in the overlapping region between  $E$  and  $F$  of the two emergent beams. Beyond  $E$  and  $F$  fringes of large width are produced which are due to diffraction. In reality diffraction bands are present over the whole region but equally spaced interference fringes are observed superposed on them in portion between  $E$  and  $F$ . To observe the fringes, the screen can be replaced by an eye-piece or a low power microscope and fringes are seen in the principal plane. The fringes extend into space and are thus non-localized.

Since the angles  $BAC$  and  $BCA$  are small and equal, the virtual sources are coplanar with, extremely close to and equidistant from the source slit  $S$ , i.e., the distance of separation of the virtual sources,  $d = S_1S_2$  is extremely small.

#### determination of the wavelength of light

The bi-prism arrangement can be used to determine the wavelength of monochromatic light or almost monochromatic light like the one coming from a sodium lamp. Light from the sodium lamp illuminates the slit  $S$  and resulting interference fringes can be viewed through the eye-piece. By employing the formula

$$\lambda = \frac{dX}{D} \quad \left[ X = \frac{D}{d} \lambda \right]$$

$\lambda$  can be determined once the values of  $X$ , the fringe-width,  $d$ , the distance of separation of the virtual sources  $S_1$  and  $S_2$  and  $D$ , the normal distance of the plane of observation of the fringes from the slit, are measured with the help of the bi-prism experiment.

The experiment is performed on a heavy metallic optical bench, about 2 metres in length, having a graduated scale on one side. The bench is supported on four levelling screws at the base and carries four uprights for supporting the adjustable slit, the bi-prism, a high power micrometer eye-piece of the Ramsden type and a convergent lens. Each upright, which is capable of movement along and also



perpendicular to the length of the bench, carries a vernier at the base for accurately ascertaining its position on the bench and may be adjusted to any desired height. The two uprights meant for the slit and the bi-prism are in addition provided with *tangent screws* so that the slit and the bi-prism may be rotated in their own plane.

### **adjustments**

It is essential to obtain a fringe system with uniform spacing over the entire field of view before any measurement of the fringe-width can be carried out. To achieve this the following adjustments in the experimental arrangement are carried out.

(i) The bed of the optical bench is first levelled with the help of the levelling screws and a spirit level.

(ii) The eye-piece is focussed on the cross-wires by drawing in and out the tube containing the lenses in the cross-wire tube until they are distinctly visible. One of the cross-wires is then made exactly vertical by observing a plumb line through the eye-piece and rotating the eye-piece about its own axis until the particular cross-wire coincides with the image of the plumb line, which of course is vertical.

(iii) The slit and the eye-piece are mounted on the optical bench and adjusted to the same height. The slit is made quite narrow and illuminated with light whose wavelength is to be determined. A real image of the illuminated slit is formed at the plane of the cross-wires with the help of a convex lens. Using the tangent screw, the slit is then rotated in its own plane until its image coincides with the vertical cross-wire. The slit is then exactly vertical.

(iv) The bi-prism is then mounted between the eye-piece and the slit with its refracting edge nearly vertical. To make this edge exactly parallel to the slit, two real images of the virtual sources  $S_1$  and  $S_2$  are formed at the focal plane of the eye-piece with the help of the convex lens, inserted between the prism and the eye-piece. By lateral movement of the prism, the images are made equally bright *i.e., equally well focussed*. By rotating the prism in the vertical plane with the help of the tangent screw, the images are made of *equal height*. On removing the lens, interference fringes are seen in the focal plane of the eye-piece. The refracting edge can now be made exactly parallel to the



slit by giving finer rotation to the prism until the interference fringes become perfectly distinct and well-defined (Fig. 27.9)

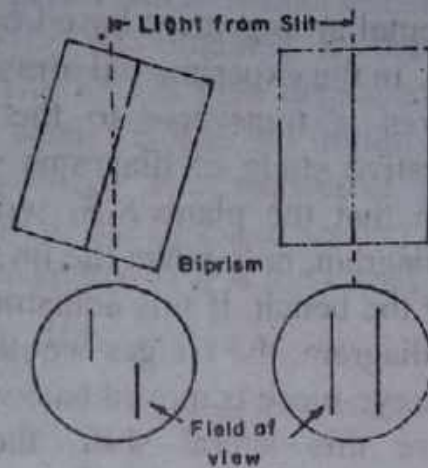


Fig. 27.9

(v) The line joining the obtuse angle of the bi-prism and the slit, i.e., the axis of the experiment must be adjusted exactly parallel to the length of the bench. This is absolutely essential because the expression for the fringe-width  $X = \frac{D\lambda}{d}$  was derived and shown to

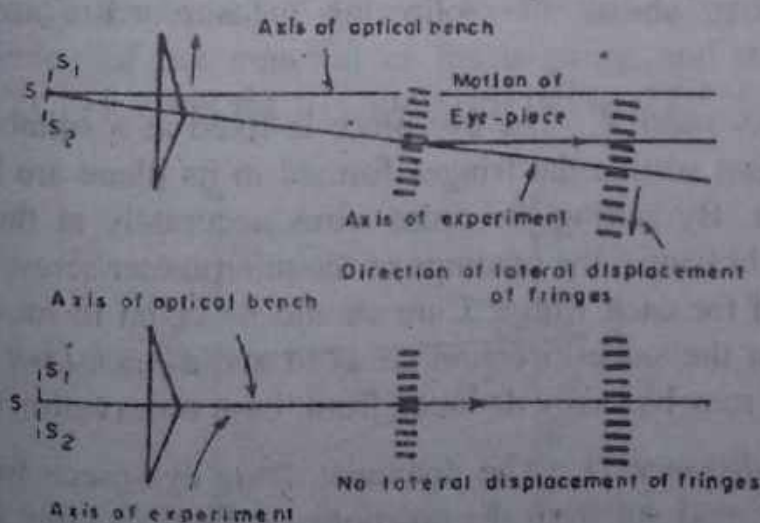


Fig. 27.10

be uniform over the entire field for given  $D$ ,  $\lambda$  and  $d$  on the supposition that the plane of observation of the fringes is exactly parallel to the plane containing the virtual coherent sources  $S_1$  and  $S_2$  – a condition that must be satisfied in experimental arrangement based on *division of wavefront* to obtain correct fringes. In the experimental arrangement described here, the plane of cross-wires is transverse to the length of the bench. Furthermore, a comparative study of diagrams sketched in Fig. 27.10 leads to the conclusion that the plane  $S_1S_2$  will be transverse to the bench, as in the lower diagram, only when the line slit is adjusted exactly parallel to the length of the bench. If this adjustment is not perfect, then as shown in the upper diagram, the fringes would shift laterally relative to the cross-wires as the eye-piece is moved backward (or forward) along the bench. To remove this lateral shift, the bi-prism is moved transversely to the bench in a direction opposite to the direction of the shift. The fringes are again tested for lateral shift by moving the eye-piece to and fro along the bench. Any lateral shift is again corrected in the manner described above. In this manner the position of the bi-prism relative to the slit is so adjusted (as shown in the lower diagram), that the lateral shift of the fringes would vanish for any change in the position of the eye-piece.

### measurements

After making the necessary adjustments in the apparatus in the manner described above, the following measurements are taken in sequence.

(i) *fringe-width  $X$*  : The eye-piece is fixed at a suitable distance from the bi-prism so that the fringes formed in its plane are fairly short and wide apart. By setting the cross-wires accurately at the centre of successive bright fringe, the readings of the micrometer screw of the eye-piece are noted for each fringe. Care should be taken to move the eye-piece always in the same direction so as to avoid *back-lash error*. The fringe-width  $X$  may be easily deduced from these observations.

(ii) *The distance  $D$*  : The distance of the eye-piece from the slit can be directly read out from the positions of the respective uprights on the bench. Appropriate *index correction* should be applied in case the slit and the cross-wires are not exactly at the zero mark of the vernier of their respective uprights.



(iii) measurements of  $d$  :

(a) *magnification method* : The eye-piece is first fixed at a distance from the slit which is greater than four times the focal length of the convex lens to be used. The convex lens is then introduced to its first position nearer to the bi-prism when distinct magnified real images of the virtual sources  $S_1$  and  $S_2$  are obtained in the field of view of the micrometer eye-piece. The lens is then displaced to its second position nearer to the eye-piece when distinct but diminished real images of the virtual sources are obtained. The respective separations  $d_1$  and  $d_2$  between the two sources in the first and second positions are measured with the help of the micrometer. If  $d$  is the actual separation between the sources, then  $d = \sqrt{d_1 d_2}$ . (Fig. 27.11).

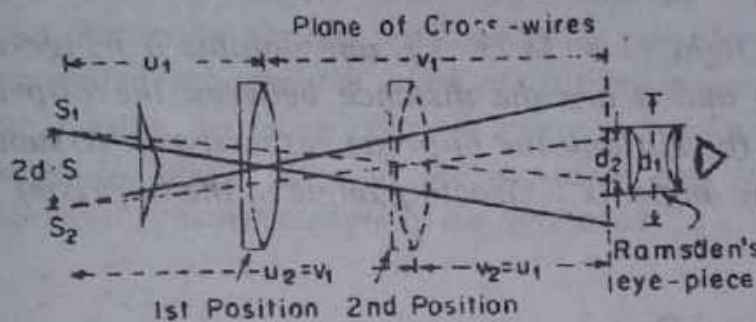


Fig. 27.11

(b) *method of deviation* : Since the refracting angle of the bi-prism is very small, the deviation,  $\delta$ , suffered by the light beam in passing through the bi-prism is given by  $\delta = (\mu - 1) \alpha$  where  $\mu$  is the refractive index of the material of the bi-prism and  $\alpha$  its refracting angle (Art. 21.11). From the triangle  $SBS_1$  in Fig. 27.12

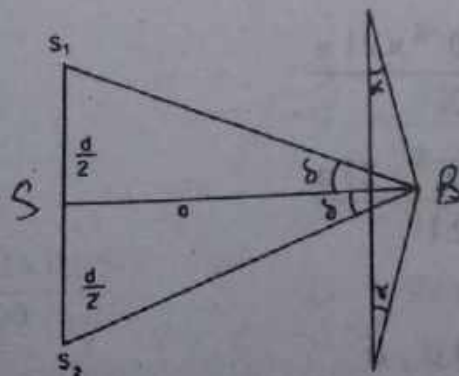


Fig. 27.12



$$\tan \delta = \frac{d/2}{a}$$

where  $d$  is the distance of separation between the sources  $S_1$  and  $S_2$  and  $a$  is the distance between the slit and the refracting edge of the bi-prism. Since  $\delta$  is very small

$$\frac{d}{2} = a \cdot \delta; \text{ or, } d = 2 \cdot a \cdot \delta = 2 (\mu - 1) \alpha \cdot a.$$

Knowing  $\mu$ ,  $a$  and  $\alpha$ ,  $d$  can be determined.

**Example 27.9.** In a typical bi-prism arrangement  $b/a = 20$ , and for sodium light ( $\lambda = 5893 \text{ \AA}$ ), one obtains a fringe-width of 0.1 cm; here  $b$  and  $a$  are the distance between the bi-prism and the screen and the slit and the bi-prism respectively. Assuming  $\mu = 1.5$  calculate the angle ' $\alpha$ ', refracting angle of the bi-prism).

**Soln.**

$$X = \frac{\lambda D}{d}$$

$$\frac{b}{a} = 20; b \neq 20a$$

$$\therefore D = a + b = (20 + 1) a$$

$$\therefore X = \frac{\lambda \cdot (20 + 1)a}{d}$$

$$\begin{aligned} \text{or, } d &= \frac{5893 \times 10^{-8} \times 21 a}{X} \\ &= \frac{5893 \times 10^{-8} \times 21 a}{0.1} \end{aligned}$$

Also

$$\begin{aligned} d &= 2 (\mu - 1) \alpha \cdot a \\ &= 2 (1.5 - 1) \alpha \cdot a \end{aligned}$$

$$= 2 \times 0.5 \alpha \cdot a$$

$$= \alpha \cdot a$$

$$\therefore \alpha \cdot a = \frac{5893 \times 10^{-8} \times 21 a}{0.1} = 0.0123753 a$$

$$\text{or, } \alpha = 0.0123753 \text{ radian}$$

$$= \frac{0.0123753 \times 180}{22} \text{ degrees} \approx 0.7^\circ$$

**Example 27.10.** In an experiment with a bi-prism, the readings on the optical bench of the position of the eye-piece and the two positions of the lens were respectively 100.00, 67.00 and 34.00 cm. The distances between the two images for the two positions of the lens were 0.300 mm and 1.200 mm. respectively and the width of 10 fringes was 9.720 mm. Assuming that there is no index error in any case, calculate (a) the distance between the focal plane of the eye-piece and the plane of interfering sources (b) the wavelength of the light used.

**Soln.**

(a) With the convex lens in the 1st position, the image distance  $v_1 = 100 - 67 = 33$  cm.

With the convex lens in the 2nd position, the image distance  $v_1 = 100.00 - 34 = 66$  cm.

But the image distance in the 2nd position is the object distance in the 1st position. Hence  $u_1 = v_2 = 66$  cm.

Thus the distance between the focal plane of the eye-piece and the plane of the interfering sources (slit)  $= u_1 + v_1 = 66 \text{ cm} + 33 \text{ cm} = 99 \text{ cm}$ .

(b)

$$\lambda = \frac{X d}{D}$$

$$X = \frac{0.972}{10} \text{ cm}$$

$$= 0.0972 \text{ cm}$$

$$= \frac{0.0972 \times 0.06}{99}$$

$$D = 99 \text{ cm}$$

$$= 5891 \times 10^{-8} \text{ cm.}$$

$$d = \sqrt{d_1 d_2}$$

$$= 5891 \text{ A.U.}$$

$$= \sqrt{0.3 \times 1.2}$$

$$= 0.6 \text{ mm}$$

$$= 0.06 \text{ cm.}$$

**Example 27.11.** Using sodium light with a Fresnel bi-prism the fringes were found to have a width of 0.0196 cm when observed at a distance of 100 cm from the slit. When a convex lens was placed between the bi-prism and the observer to give an image of the source at 100.00 cm from the slit, the distance apart of the images was found to be 0.70 cm. Calculate the wavelength. Given, distance from slit to lens 30.0 cm.

**Soln.**

With the convex lens in position, the object distance  $u = 30 \text{ cm}$  and the image distance  $v = 100 - 30 = 70 \text{ cm}$ .

$$\text{Hence magnification, } m = \frac{v}{u} = \frac{70}{30}$$

$$\text{Again magnification, } m = \frac{d_i}{d} = \frac{0.70}{d}$$

Where  $d_i$  is the observed distance of separation between the source and  $d$  is the real distance of separation. Thus

$$m = \frac{0.70}{d} = \frac{70}{30}$$

$$\text{or, } d = \frac{30 \times 0.70}{70} = 0.3 \text{ cm}$$

$$X = 0.0196 \text{ cm, } D = 100 \text{ cm}$$

$$\therefore \lambda = \frac{Xd}{D} = \frac{0.0196 \times 0.3}{100} \text{ cm}$$

$$= 588 \times 10^{-7} \text{ cm} = 5880 \text{ A.U.}$$

**Example 27.12.** When a thin monochromatic source of light was placed at a distance of 50 cm from a Fresnel bi-prism of  $\mu = 1.5$ , the distance between two consecutive bands formed on a screen placed at a distance of 100 cm from the bi-prism was found to be 0.012 cm. If the



wavelength of light was  $5893 \times 10^{-8}$  cm, find the magnitude of the obtuse angle of the bi-prism.

Soln.

$$X = \frac{D\lambda}{d}$$

$$a = 50 \text{ cm}, b = 100 \text{ cm}$$

$$\text{or, } d = \frac{D\lambda}{X}$$

$$D = (a + b) \text{ cm}$$

$$\text{But } d = 2a(\mu - 1)\alpha$$

$$= (50 + 100) \text{ cm} = 150 \text{ cm}$$

$\alpha$  = refracting angle

of the bi-prism.

$$\lambda = 5893 \times 10^{-8} \text{ cm}$$

$$X = 0.012 \text{ cm}$$

$$\text{Hence } 2a(\mu - 1)\alpha = \frac{D\lambda}{X}$$

$$\text{or, } \alpha = \frac{D\lambda}{2a(\mu - 1).X}$$

$$= \frac{(50 + 100) \times 5893 \times 10^{-8}}{2 \times 50 \times (1.5 - 1) \times 0.012} \text{ radians}$$

$$\pi \text{ radian} = 180^\circ; \text{ or, } 1 \text{ radian} = \left( \frac{180^\circ}{\pi} \right)$$

Hence,

$$\alpha = \frac{150 \times 5893 \times 10^{-8} \times 180}{100 \times 0.5 \times 0.012 \times \pi} \text{ degree}$$

$$= \frac{150 \times 5893 \times 10^{-8} \times 180 \times 7}{100 \times 0.5 \times 0.012 \times 22} = 0.844^\circ$$

Hence obtuse angle of the bi-prism

$$= 180^\circ - 2\alpha = 180^\circ - 1.688^\circ = 178.312^\circ.$$

**Example 27.13.** A Fresnel bi-prism having angle of  $1^\circ$  and refractive index 1.5 forms interference fringes on a screen placed 80 cm from the prism. If the distance between the source and the bi-prism is 20 cm, find the fringe separation when the wavelength of light used is (a)  $6900 \text{ \AA}$  and (b)  $4600 \text{ \AA}$ .

**Soln.**

$$D = a + b = 20 + 80 = 100 \text{ cm}$$

$$a = 20 \text{ cm}, \alpha = 1^\circ = \frac{\pi}{180} \text{ radian}$$

$$d = 2(\mu - 1) \cdot \alpha \cdot a;$$

$$= 2(1.5 - 1) \cdot \frac{\pi}{180} \cdot 20$$

$$= 2 \times 0.5 \cdot \frac{\pi}{9} = \frac{22}{63} \text{ cm.}$$

$$(a) \lambda = 6900 \text{ \AA} = 6900 \times 10^{-8} \text{ cm}$$

$$X = \frac{\lambda \cdot D}{d} = \frac{6900 \times 10^{-8} \times 100 \times 63}{22}$$

$$= 1.976 \times 10^{-2} \text{ cm}$$

$$(b) \lambda = 4600 \text{ \AA} = 4600 \times 10^{-8} \text{ cm}$$

$$X = \frac{\lambda \cdot D}{d} = \frac{4600 \times 10^{-8} \times 100 \times 63}{22}$$

$$= 1.317 \times 10^{-2} \text{ cm.}$$

### 27.17 Phase change on reflection ; Stokes' treatment

Let us now investigate the reflection of light at an interface of two media using the principle of *optical reversibility*. Let a ray of monochromatic light AB be incident on the interface GH, separating two media of refractive indices  $\mu_1$  and  $\mu_2$  ( $\mu_2 > \mu_1$ ). Let  $r_1$  and  $t_1$  be the fraction of the incident amplitudes reflected and transmitted respectively. Thus, if the amplitude of the incident ray is  $a$ , then the amplitudes of the reflected ray BC in medium 1 and transmitted ray BD in medium 2 are  $ar_1$  and  $at_1$  respectively (Fig. 27.13).

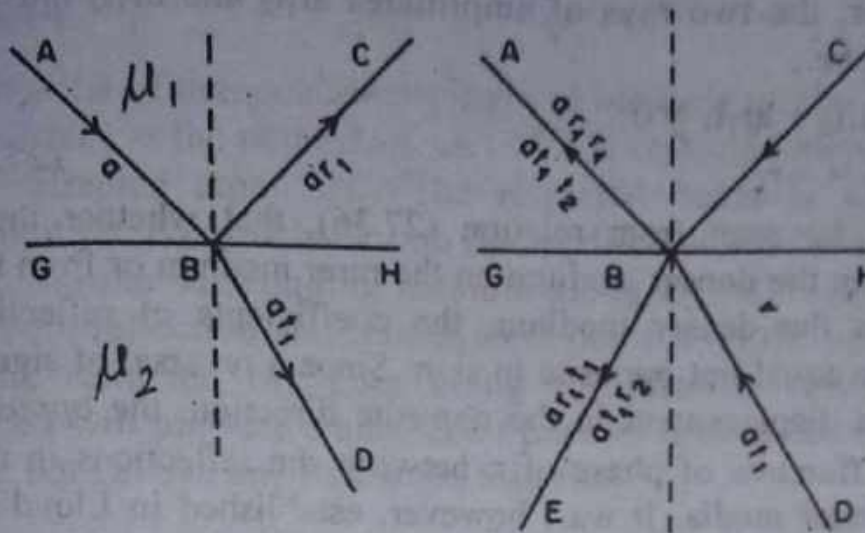


Fig. 27.13

Now, according to the principle of reversibility, in the absence of any absorption, a light ray that is reflected or refracted will retrace its original path if its direction is reversed. Thus, if somehow or other, we reverse the reflected and transmitted waves, then according to this principle, they should together produce, after striking the interface, a wave in medium 1 equal in amplitude to that of the incident wave but travelling in opposite direction *viz.* Along BA.

When the rays are reversed, we shall have to consider a ray of amplitude  $ar_1$  (CB) incident on medium 2 and ray of amplitude  $at_1$  (DB) incident on medium 1 as shown in Fig. 27.13. The ray of amplitude  $ar_1$  will give rise to a ray of amplitude  $ar_1^2$  ( $ar_1.r_1$ ) reflected in the medium 1 along BA and a ray of amplitude  $ar_1.t_1$  refracted or transmitted in medium 2 along BE. Similarly the ray of amplitude  $at_1$  will give rise to a reflected ray of amplitude  $at_1.r_2$  along BE in medium 2 and a transmitted ray of amplitude  $at_1.t_2$  along BA in medium 1 where  $r_2$  and  $t_2$  are fraction of the incident amplitudes reflected and transmitted respectively when the ray is incident from medium 2 on medium 1.

Since the resultant effect of the reversal of BC and BD must be only a single wave of amplitude  $a$  in the direction BA, the two rays of amplitudes  $a.r_1^2$  and  $at_1.t_2$  must combine to give the incident ray of amplitude  $a$ ; thus

$$ar_1^2 + at_1t_2 = a$$

$$\text{or, } t_1t_2 = 1 - r_1^2$$

(27.35)



Further, the two rays of amplitudes  $at_1t_2$  and  $ar_1t_2$  must cancel each other; i.e.,

$$at_1t_2 + ar_1t_1 = 0$$

$$\text{or, } r_2 = -r_1$$

(27.36)

It can be seen from relation (27.36), that whether the ray is incident from the denser medium on the rarer medium or from the rarer medium on the denser medium, the coefficients of reflections are numerically equal but opposite in sign. Since a reversal of sign simply illustrates a displacement in the opposite direction, the *opposite sign* means a difference of phase of  $\pi$  between the reflections in the rarer and the denser media. It was, however, established in Lloyd's single mirror experiment and Wiener's experiment on stationary waves that *an abrupt phase change of  $\pi$  occurs in the light waves at reflections in the rarer medium i.e., a surface backed by a denser medium*. Thus there will not be any phase change at reflections in the denser medium i.e., when the wave is reflected from a surface backed by a rarer medium. Furthermore, since  $(r_1)^2 = (-r_2)^2$ , the fraction of intensity reflected is the same in the two cases under consideration. *It should, however, be emphasized that for the relation  $r_2 = -r_1$  to hold good, the angle of incidence of the wave incident in medium 2 (denser) must be equal to the angle of refraction of the wave incident in medium 1 (rarer).*

### 27.18 Interference in thin films

Everyone is familiar with the brilliant colours produced by a thin film of oil on the surface of water, a thin film of a soap bubble and coatings of oxides on heated materials. Hooke observed such colours in thin films of mica and similar thin transparent plates. Newton was able to show the interference rings when a convex lens was placed on a plane glass plate. The explanation of the origin of this colour phenomenon was given by Young, in 1802, in terms of the interference of light waves reflected from the upper and the lower surface of the thin film. It has been observed that interference in the case of thin film takes place due to both reflected as well as transmitted light.

#### Interference due to reflected light from a plane parallel (thin) film

Fig. 27.14 shows a thin transparent film of uniform thickness  $t$  and refractive index  $\mu$ , bounded by two parallel surfaces  $MN$  and

*PQ*. A ray *AB* of monochromatic light is incident on the upper (Fig. 27.14) surface at the point *B*. A part of it is reflected along *BC* and a part is refracted along *BD*. The refracted beam is again partly reflected at the point *D* back into the medium along *DE* and the rest refracts into the surrounding medium along *DL*. The ray along *DE* suffers both reflection and refraction at point *E* on the upper surface *MN*. The refracted ray goes along *EF*. Similar reflections and refractions will produce a number of parallel rays in the surrounding medium both above and below the thin film.

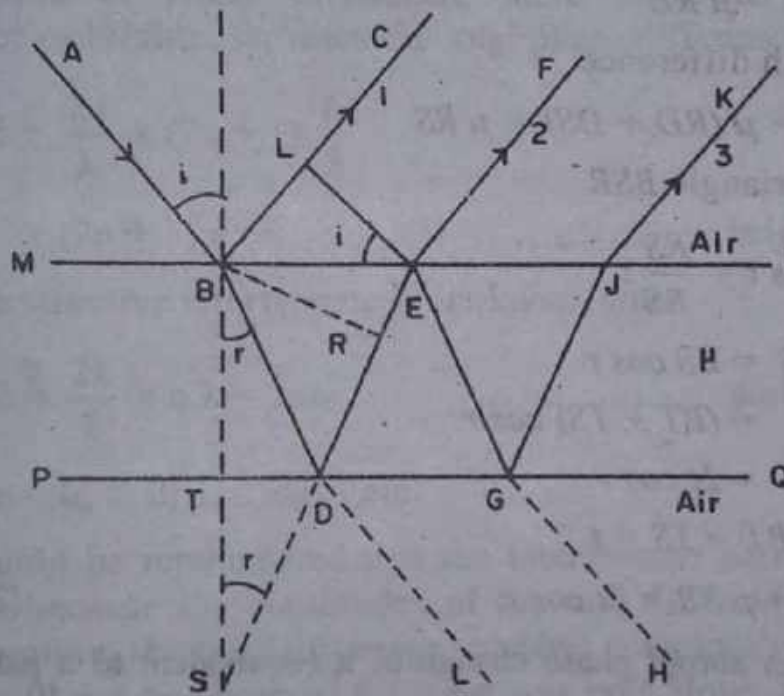


Fig. 27.14

Consider the rays *BC* and *BDEF*. They are both derived from the single beam *AB* and hence satisfy the condition for producing interference (coherent source). Whether they (when collected by a lens or an eye) will interfere constructively or destructively *i.e.*, produce brightness or darkness, will depend on their path difference. To determine this path difference, draw *EL* normal to *BC* and *BR* normal to *DE*. Also produce *ED* to meet *BT* produced at *S*. Let the angle of incidence and refraction be *i* and *r* respectively.



It is obvious that from the points  $L$  and  $E$  onwards, the two waves and rays travel equal distances. Since the path  $(BD + DE)$  within the film is equivalent to a path  $\mu (BD + DE)$  in air, the difference in optical path of the two rays is

$$\begin{aligned} x &= \mu (BD + DE) - BL \\ &= \mu (BD + DR + RE) - BL \end{aligned}$$

Also  $BD = DE$ .

$$\text{and } \mu = \frac{\sin i}{\sin r} = \frac{BL/BE}{RE/BE} = \frac{BL}{RE}$$

$$\text{or, } BL = \mu RE$$

The path difference

$$x = \mu (RD + DS) = \mu RS$$

In the triangle  $BSR$

$$\cos r = \frac{RS}{BS};$$

$$\begin{aligned} \text{or, } RS &= BS \cos r \\ &= (BT + TS) \cos r \\ &= 2t \cos r \end{aligned}$$

where  $BT = TS = t$

$$\therefore x = \mu RS = 2t \cos r \quad (27.37)$$

Since an abrupt phase change of  $\pi$  (equivalent to a path difference of  $\frac{\lambda}{2}$ ) is introduced whenever a ray is reflected from a surface backed by a denser medium (Art. 27.17), eqn. 27.37 does not represent the total path difference between  $BC$  and  $BDEF$ . Taking into account this additional path difference of  $\frac{\lambda}{2}$  for reflection suffered at the point  $D$  the total path difference will be

$$x = 2 \mu t \cos r \pm \frac{\lambda}{2}$$

Therefore, for constructive interference or brightness



$$2 \mu t \cos r \pm \frac{\lambda}{2} = n \lambda$$

$$\text{or, } 2 \mu \cos r = (2n \pm 1) \frac{\lambda}{2} \dots \dots \dots \text{bright}$$

and for destructive interference or darkness

$$2 \mu t \cos r \pm \frac{\lambda}{2} = (2n \pm 1) \frac{\lambda}{2}$$

$$\text{or, } 2 \mu t \cos r = n \lambda \dots \dots \dots \text{dark}$$

In terms of phase difference, there will be constructive interference or brightness, when the total phase difference

$$\delta = \frac{2\pi}{\lambda} \times (2n \pm 1) \frac{\lambda}{2}$$

$$= (2n \pm 1) \pi \dots \dots \dots \text{bright}$$

and destructive interference or darkness, when

$$\delta = \frac{2\pi}{\lambda} \times n \lambda = 2\pi n \dots \dots \dots \text{dark}$$

$$n = 0, 1, 2, 3, \dots \dots \dots \text{etc.}$$

It should be remembered that the interference pattern will not be perfect because the amplitudes of the rays BC and EF are not same. Therefore, the intensity never vanishes completely and perfect darkness will not be observed for these two rays alone. However, in case of multiple reflection, it can be shown that the resultant amplitude of the rays 2, 3, 4 ..... etc., is equal in magnitude to that of the ray 1 but out of phase with it. Therefore, in case of multiple reflection, the intensity of the minima is zero.

### Colour of the films

When white light is incident on a thin film, the light which comes from it will not include the colour whose wavelength satisfies the equation  $2 \mu t \cos r = n \lambda$  in the reflected system. Therefore, the film will appear coloured and the colour will depend upon the thickness and the angle of inclination. If both the incident light as well as the film are parallel i.e., if t

and  $r$  are constant, colours will be seen in the order violet, blue, etc. as the wavelength increases. When the film is parallel but the incident white light is not, the path difference will change due to change in the angle  $r$ . Hence, the film will show different colours when viewed from different directions. When the incident white light is parallel but the film is of varying thickness, even then the film will pass through various colours. In the case of oil on water, different colours are seen because  $r$  and  $t$  vary.

### Necessity of broad source

Coherent sources used to obtain interference fringes in Fresnel bi-prism, inclined mirrors and Lloyd's single mirror are narrow. These fringes can be obtained on a screen or viewed with an eye-piece. But in the case of interference in thin films, the narrow sources limits the visibility of the film.

Consider a narrow source of light and a thin film. Two rays, 1 and 2, from this source are incident on the film at different angles (Fig. 27.15). Ray 1 incident at the region A produces interference since the parallel rays 3 and 4 originating from it as a result of reflections at the upper and lower surfaces of the film reach the eye. But the rays 5 and 6 originating from rays incident at some other angles fail to reach the eye and hence do not produce interference. Thus only the portion A of the film is visible and not the rest.

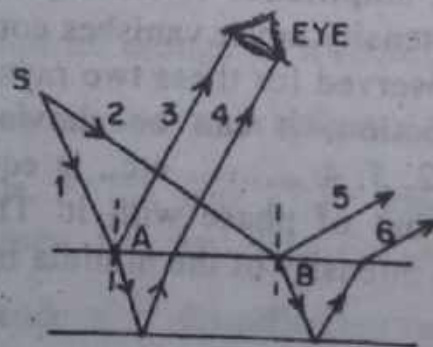


Fig. 27.15

If, however, an extended source of light is used (Fig. 27.16), the whole of the film can be made visible. This is because, the rays 3 and 4, originating from ray 1 coming from a certain portion of the



source, reach the eye making the portion A visible. The rays 5 and 6, originating from ray 2 coming from some other portion of the source, also reach the eye making the portion B visible. In this way each portion of the film can be made visible by rays incident at different angles and coming from different portions of the source.

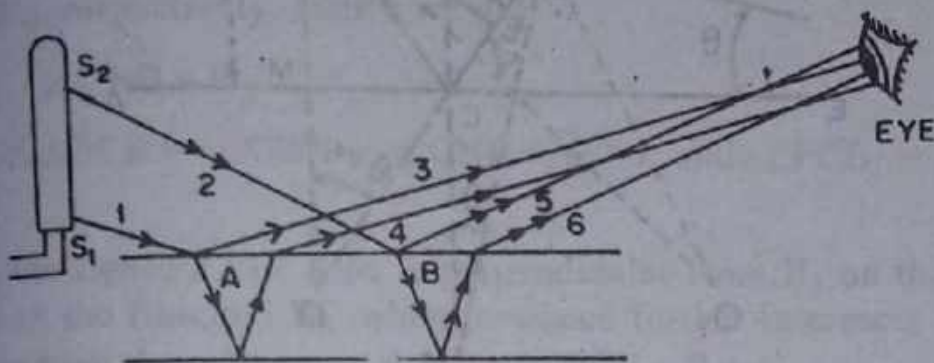


Fig. 27.16

### 27.19 Film of varying thickness (wedge shaped film)

Suppose the film is not parallel sided but is in the shape of thin wedge *i.e.*, its surfaces make an angle  $\theta$  with each other. Such a film can be obtained by placing two accurately plane pieces of glass in contact at one end, the far end being kept apart by a very thin object like a human hair or a cigarette paper. The thickness of the film will continually increase from E to A (Fig. 27.17). consider the film to be illuminated with monochromatic light. The incident light wave propagating along AB will give rise to two light waves – the directly reflected ray along BR and the internally reflected ray along  $B_1R_1$ . The rays BR and  $B_1R_1$  are, therefore, coherent and capable of producing interference. However, the two interfering rays do not reach the eye along parallel paths but appear to diverge from a point Q in the rear of the film. Constructive or destructive interference, therefore, occurs at the point Q, which is, however, *virtual*. If the two rays BR and  $B_1R_1$  are allowed to fall on a lens, they will cross each other at a point which is the focus conjugate to Q, where real constructive or destructive interference would occur. Thus Q will be the point where the interference fringe appears to be formed.





or,  $BD = \mu BE$

Hence eqn. (27.38) becomes

$$\begin{aligned}\Delta &= \mu (BE + EC + CB_1) - \mu BE \\ &= \mu (EC + CB_1).\end{aligned}$$

Let  $BN$  and  $CN$  be the normals to the upper and lower surfaces of the film respectively. Hence we get

$$\angle CNB = \theta$$

and  $\angle PCB = \angle CBN + \angle CNB = \theta + r$ , and  $\angle PCB_1 = \angle PCB = \theta + r$

In the figure 27.17  $B_1M$  is perpendicular from  $B_1$  on the lower surface of the film and  $BC$  when produced further intersects it at  $O$ . Thus we have the relation  $\angle B_1OC = \angle PCB = \theta + r$

Also  $\angle CB_1O = \angle PCB_1 = \theta + r = \angle B_1OC$ .

Thus  $B_1OC$  is an isosceles triangle.

Hence  $CB_1 = CO$

Eqn. (27.38), therefore, reduces to

$$\begin{aligned}\Delta &= \mu (EC + CB_1) \\ &= \mu (EC + CO) = \mu \cdot EO = \mu B_1O \cos (\theta + r)\end{aligned}$$

If  $t$  is the thickness of the film at the point  $B_1$  and since  $B_1M = MO = t$ ,

$$\Delta = \mu 2t \cos (\theta + r) = 2\mu t \cos (\theta + r).$$

The path difference  $\Delta$ , therefore, varies both on account of changing thickness as well as changing angle of incidence, provided the broad light source is at a finite distance from the film.

Eqn. (27.38) does not, however, represent the total path difference between the rays. We still have to consider a path difference of  $\frac{\lambda}{2}$  introduced as a result of reflection at the point  $B$

which represents reflection at a surface backed by a denser medium. Thus the total path difference between the rays is



$$2\mu t \cos(\theta + r) \pm \frac{\lambda}{2}$$

From the condition of constructive interference or brightness

$$2\mu t \cos(\theta + r) \pm \frac{\lambda}{2} = n\lambda$$

$$\text{or, } 2\mu t \cos(\theta + r) = (2n \pm 1) \frac{\lambda}{2} \quad \dots\dots\dots \text{bright}$$

and from the condition of destructive interference or darkness

$$2\mu t \cos(\theta + r) \pm \frac{\lambda}{2} = (2n \pm 1) \frac{\lambda}{2}$$

$$\text{or, } 2\mu t \cos(\theta + r) = n\lambda \quad \dots\dots\dots \text{dark}$$

Consider the wedge shaped film to be illuminated by a parallel beam of monochromatic light of wavelength  $\lambda$ . Then the angle of incidence  $i$ , will be constant at every point of the film and so will be  $r$ , the angle of refraction. The total optical path difference will, therefore, be only due to variation of the thickness,  $t$ , from point to point of the film. At the edge of the wedge, since  $t = 0$ , the film appears perfectly dark and the two interfering waves are  $\pi$  out of phase. At distances from the edge where the total path difference

$$\Delta = \lambda/2, 3\lambda/2, 5\lambda/2, \dots\dots \text{etc. the film is bright,}$$

while at distances where

$$\Delta = \lambda, 2\lambda, 3\lambda, 5\lambda, \dots\dots \text{etc. the film appears to be dark.}$$

Thus as we go along the wedge in the direction of increasing thickness there will be alternate dark and bright bands (Fig. 27.18) parallel to the edge of the film.

### Testing of plane surface

If the two surfaces are perfectly plane, each band or fringe is the locus of the points at which the thickness of the film has a constant value. This is an important application of the phenomenon of



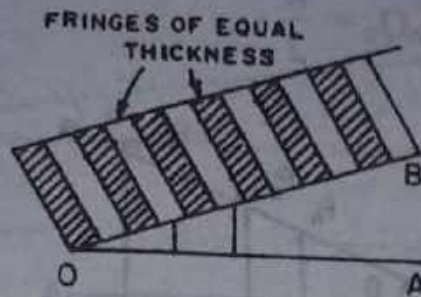


Fig. 27.18

interference. Any deviation from perfect planeness of the surface will be immediately obvious through the curvature of these fringes. That is why these fringes are called fringes of *constant thickness*. The standard method to test the planeness of a surface is to take an optically plane surface OA and the surface to be tested OB. The fringes are observed in the field of view and if they are of equal thickness, the surface OB is plane. If not, the surface OB is not plane. The surface OB is polished and the process repeated until the fringes observed are of equal width which means that the surface has become plane.

Suppose the rays are incident normally on the film which is air, i.e.,  $\mu = 1$ . Then the angle of refraction is very small and since angle  $\theta$  is also very small  $\cos(\theta + r) = 1$ .

Suppose under this condition the  $n^{\text{th}}$  bright fringe occurs at the point  $P_n$  where the thickness of the film,  $t = P_n Q_n$ . (Fig. 27.19)

Applying the relation for a bright fringe,

$$2\mu t \cos(\theta + r) = (2n \pm 1) \frac{\lambda}{2}$$

$$\text{or, } 2\mu P_n Q_n = (2n \pm 1) \frac{\lambda}{2}$$

$$\text{or, } 2 P_n Q_n = (2n \pm 1) \frac{\lambda}{2} \quad (\mu = 1)$$

The next bright fringe  $(n + 1)$  will occur at  $P_{n+1}$  such that

$$2 P_{n+1} Q_{n+1} = [2(n + 1) \pm 1] \frac{\lambda}{2}$$

Subtracting

$$P_{n+1}Q_{n+1} - P_nQ_n = \frac{\lambda}{2}$$

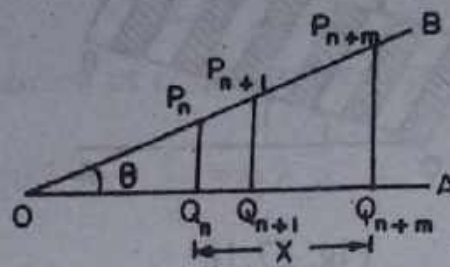


Fig. 27.19

Thus the next bright fringe will occur at the point where the air-film thickness has increased by  $\frac{\lambda}{2}$ . Suppose  $P_{n+m}$  represents the position of the  $(n + m)^{\text{th}}$  bright fringe. Hence, there will be  $m$  bright fringes between  $P_n$  and  $P_{n+m}$

$$\text{or, } P_{n+m} - P_n = m \cdot \frac{\lambda}{2}$$

Let the distance  $Q_nQ_{n+m} = x$

Then the angle of inclination  $\theta$ , between OA and OB is

$$\theta = \frac{P_{n+m}Q_{n+m} - P_nQ_n}{Q_nQ_{n+m}} = \frac{m \lambda}{2x}$$

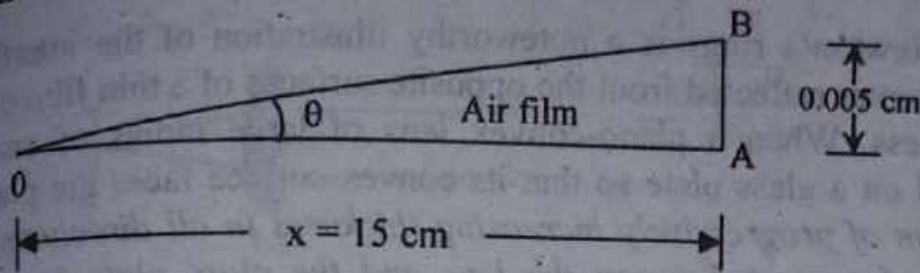
$$\text{or, } x = \frac{m \lambda}{2\theta}$$

Since  $x$  is the distance corresponding to  $m$  fringes, the fringe-width

$$X = \frac{x}{m} = \frac{\lambda}{2\theta}$$

**Example 27.14.** Two glass plates enclose a wedge-shaped air-film, touching at one edge and are separated by a wire of 0.05 mm diameter at a distance of 15 cm from the edge. Calculate the fringe-width. Monochromatic light of  $\lambda = 6,000 \text{ \AA}$  from a broad source falls normally on the film.

Soln.



$$\text{Fringe-width, } X = \frac{\lambda}{2\theta}$$

$$OA = 15 \text{ cm}$$

$$\theta = \frac{AB}{OA} = \frac{0.005}{15}$$

$$AB = 0.005 \text{ cm}$$

$$\lambda = 6000 \times 10^{-8} \text{ cm}$$

$$\therefore X = \frac{(6000 \times 10^{-8})(15)}{(2)(0.005)} \\ = 0.09 \text{ cm.}$$

**Example 27.15.** Interfringes are produced with monochromatic light falling normally on a wedge-shaped film of cellophane whose refractive index is 1.40. The angle of wedge is 10 seconds of an arc and the distance between the successive fringes is 0.5 cm. Calculate the wavelength of light used.

Soln.

$$X = \frac{\lambda}{2\theta\mu}$$

Here

$$\theta = 10 \text{ seconds of an arc}$$

where  $\mu$  is the refractive index of the wedge.

$$= \frac{10 \times \pi}{60 \times 60 \times 180} \text{ radian}$$

$$X = 0.50 \text{ cm} \quad \mu = 1.40$$

$$\therefore \lambda = X 2 \theta \mu$$

$$= \frac{2 \times 10 \times 22 \times 1.40 \times 0.50}{60 \times 60 \times 180 \times 7}$$

$$= 6790 \times 10^{-8} \text{ cm} = 6790 \text{ \AA}$$



### 27.20 Newton's rings

Newton's rings is a noteworthy illustration of the interference of light waves reflected from the opposite surfaces of a thin film of variable thickness. When a plano-convex lens of large radius of curvature is placed on a glass plate so that its convex surface faces the plate, a thin air film of progressively increasing thickness in all directions from the point of contact between the lens and the glass plate is very easily formed. The air film thus possesses a radial symmetry about the point of contact. When it is illuminated normally, preferably with monochromatic light, an interference pattern consisting of a series of alternate dark and bright circular rings, concentric with the point of contact is observed. The fringes are the loci of points of equal optical film thickness and gradually become narrower as their radii increase until the eye or the magnifying instrument can no longer separate them. The circular rings can also be formed by bringing in contact two spherical surfaces of different radii of curvature. The rings are localized in the air film. Since the phenomenon was first examined in detail by Newton, the rings are termed *Newton's rings* (Fig. 27.20). They are frequently used in the laboratory for experimental determination of the wavelength of light.

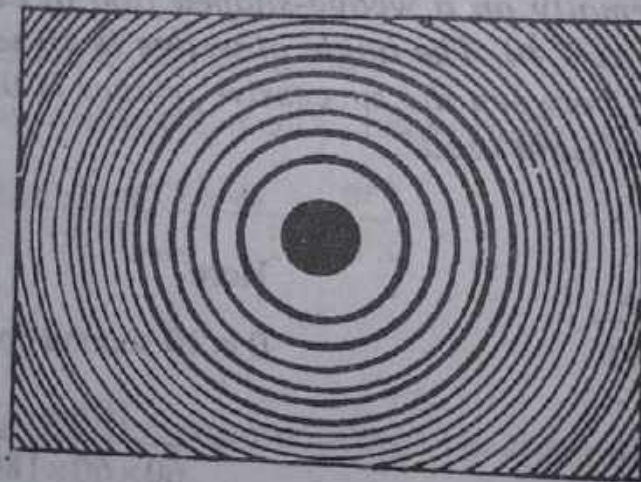


Fig. 27.20

#### experimental arrangement

S is an extended source of monochromatic light at the focus of a lens  $L_1$  (Fig. 27.21). The light is, therefore, rendered parallel by the lens and then falls on a glass plate G fixed at  $45^\circ$  to the vertical. The glass plate partially reflects the light normally on plano-convex lens  $L_2$  placed on an optically flat glass plate P. The lights reflected from the upper and lower surfaces of the air film formed between the convex lens  $L_2$  and the

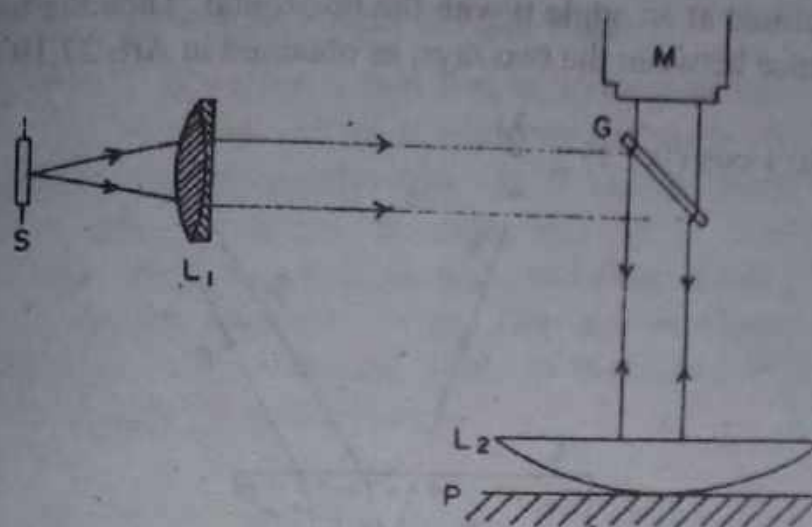


Fig. 27.21

glass plate pursue nearly the same path as that of the incident light and finally received by the eye or in a travelling microscope  $M$ . Interference takes place and alternate dark and bright circular fringes are produced. The diameters of dark or bright rings can now be measured by gradually moving the microscope  $M$  from one side of the ring system to the diametrically opposite side, fixing one of the cross-wires exactly at the centre of width of the successive rings, care being taken to avoid the *back-lash* in the microscope screw.

### Theory of Newton's rings

To discuss theoretically the formation of Newton's rings, let us consider a ray of monochromatic light  $AB$  incident at the point  $B$  on the upper surface of the film (Fig. 27.22). One portion of the ray is reflected from point  $B$  on the glass-air boundary and goes upwards along  $BC$  without any abrupt phase reversal. The other part refracts into the air film along  $BD$ . At point  $D$ , part of the light is again reflected along  $DEF$  but with an abrupt phase reversal of  $\pi$  (or a path difference of  $\frac{\lambda}{2}$ ). The two reflected rays (or wavetrains)  $BC$  and  $BDEF$  are derived from the same source and are *coherent*. They will produce constructive or destructive interference depending on their path difference. Let  $t$  be the thickness of the film at the point  $E$  and let the tangent to the convex surface at the



point be inclined at an angle  $\theta$  with the horizontal. Then the total optical path difference between the two rays, as obtained in Art. 27.19 is

$$2\mu t \cos(\theta + r) \pm \frac{\lambda}{2}$$

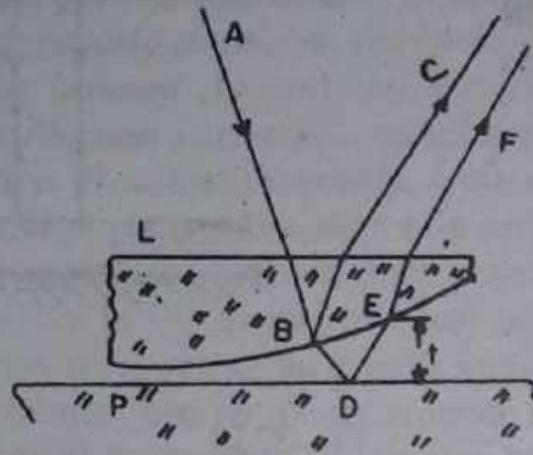


Fig. 27.21

in air where  $r$  is the angle of refraction at  $B$  and  $\mu$  is the refractive index of the film with respect to air. Thus the two rays will interfere constructively if

$$2\mu t \cos(\theta + r) \pm \frac{\lambda}{2} = n\lambda$$

$$\text{or, } 2\mu t \cos(\theta + r)$$

$$= (2n - 1) \frac{\lambda}{2} \dots\dots\dots \text{bright} \quad (27.39)$$

The minus sign has been selected on purpose since  $n$  cannot have a value of zero for bright fringes seen in reflected light. If the positive sign is selected instead,  $n$  in the expression  $(2n + 1) \frac{\lambda}{2}$  can have a value of zero without producing any mathematically absurd results, but this is not allowed. The two rays will interfere destructively, if

$$2\mu t \cos(\theta + r) \pm \frac{\lambda}{2} = (2n \pm 1) \frac{\lambda}{2}$$

$$\text{or, } 2\mu t \cos(\theta + r) = n\lambda \dots\dots\dots \text{dark.} \quad (27.40)$$



where  $\lambda$  is the wavelength of light in air.

In practice, however, a thin lens of extremely small curvature is used in order to keep the film enclosed between the lens and the plane glass plate extremely thin. As a consequence, the angle  $\theta$  becomes negligibly small as compared to  $r$ . Furthermore, the experimental arrangement is so designed (Fig. 27.21) that the light is incident almost normally on the film and is viewed from nearly normal directions by reflected light, so that  $\cos r = 1$ . Accordingly eqns. (23.39) and (23.40) reduce to

$$2\mu t = (2n - 1) \frac{\lambda}{2} \quad \dots\dots\dots \text{bright} \quad (27.41)$$

and

$$2\mu t = n\lambda \quad \dots\dots\dots \text{dark} \quad (27.42)$$

Let us now compute the radius of any ring. Let  $R$  be the radius of curvature of the convex surface which rests on the plane glass surface (Fig. 27.23). From the right angled triangle  $OFB_1$ , we get the relation

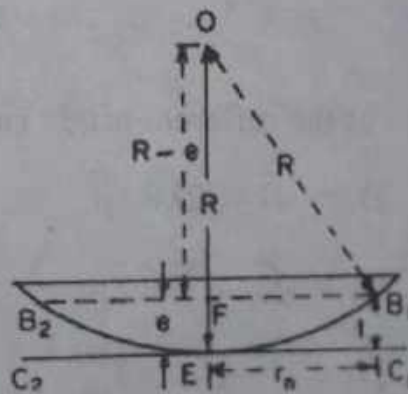


Fig. 27.23

$$R^2 = r_n^2 + (R - t)^2$$

$$\text{or, } r_n^2 = 2Rt - t^2$$

where  $r_n$  is the radius of the circular ring corresponding to the constant film thickness  $t$ . As outlined above the condition of the experiment makes  $t$  extremely small; so to a sufficient degree of accuracy,  $t^2$  may be neglected compared to  $2Rt$ . Then

$$t = \frac{r_n^2}{2R}$$

Substituting the value of  $t$  in eqns. (27.41) and (27.42), we have

$$r_n^2 = (2n - 1) \frac{\lambda R}{2\mu} \dots\dots\dots \text{bright} \quad (27.43)$$

and

$$r_n^2 = \frac{n\lambda R}{\mu} \dots\dots\dots \text{dark} \quad (27.44)$$

The square of the diameters of the bright and dark rings are, therefore, given by the expressions

$$D_m^2 = 2(2n - 1) \frac{\lambda R}{\mu} \dots\dots\dots \text{bright} \quad (27.45)$$

and

$$D_n^2 = \frac{4n\lambda}{\mu} \dots\dots\dots \text{dark} \quad (27.46)$$

Thus the diameters of the different bright rings may be written as

$$\text{1st ring,} \quad D_1 = \sqrt{1} \sqrt{2\lambda R / \mu}$$

$$\text{2nd ring,} \quad D_2 = \sqrt{3} \sqrt{2\lambda R / \mu}$$

$$\text{3rd ring,} \quad D_3 = \sqrt{5} \sqrt{2\lambda R / \mu}$$

$$\text{4th ring,} \quad D_4 = \sqrt{7} \sqrt{2\lambda R / \mu} \text{ and so on.}$$

Hence, it can be seen that the diameters (also radii) of bright rings are proportional to the square root of the odd natural numbers.

Similarly the diameters of the different dark rings can be written as

$$\text{central ring } D_0 = 0$$

$$\text{1st ring,} \quad D_1 = \sqrt{1} \sqrt{2\lambda R / \mu}$$

$$\text{2nd ring, } D_2 = \sqrt{2} \sqrt{2\lambda R / \mu}$$

$$\text{3rd ring, } D_3 = \sqrt{3} \sqrt{2\lambda R / \mu} \text{ and so on.}$$

It is obvious that the diameters (also radii) of the dark rings are proportional to the square root of the natural numbers.

At the point of contact of the lens and the glass plate,  $t = 0$ ; therefore, the total phase difference ( $\frac{2\pi}{\lambda} \cdot 2\mu t + \pi$ ) between the directly and internally reflected rays reduces to  $\pi$ . As a consequence, *when Newton's rings are viewed in reflected light, the central spot appears to be dark.* This central spot is surrounded alternately by a large number of bright and dark rings. Furthermore, if we consider the difference in diameters of the 5<sup>th</sup> and 4<sup>th</sup> dark rings, then

$$\begin{aligned} D_5 - D_4 &= 2 [\sqrt{5} - \sqrt{4}] \sqrt{\lambda R / \mu} \\ &= 0.46 \sqrt{\lambda R / \mu} \end{aligned}$$

and that between the 17<sup>th</sup> and 16<sup>th</sup> dark rings.

$$\begin{aligned} D_{17} - D_{16} &= 2 [\sqrt{17} - \sqrt{16}] \sqrt{\lambda R / \mu} \\ &= 0.26 \sqrt{\lambda R / \mu} \end{aligned}$$

Thus it is clear that the alternate bright and dark rings surrounding the central dark spot in Newton's rings gradually become narrower as their radii increase.

### determination of wavelength

In the laboratory, the diameters of the Newton's rings are measured with a travelling microscope. Usually a little away from the centre, a bright (or dark) ring is chosen which is clearly visible and its diameter measured. Let it be the  $n^{\text{th}}$  order ring. For an air film, since  $\mu = 1$ , we have

$$D_n^2 = 2(2n - 1) \lambda R \dots\dots\dots \text{bright ring} \quad (27.47)$$

and



$$D_n^2 = 4n \lambda R \quad \dots\dots\dots \text{dark ring} \quad (27.48)$$

The wavelength of the monochromatic light employed to illuminate the film can be computed from either of the above equations.

However, in actual practice, another ring,  $p$  rings from this ring onwards, is selected. The diameter of this  $(n + p)^{\text{th}}$  ring is also measured. Thus we have

$$\begin{aligned} D_{n+p}^2 &= 2 [2(n + p) - 1] \lambda R \\ &= 2 (2n + 2p - 1) \lambda R \quad \dots\dots\dots \text{bright} \end{aligned} \quad (27.49)$$

and

$$D_{n+p}^2 = 4 (n + p) \lambda R \quad \dots\dots\dots \text{dark} \quad (27.50)$$

Subtracting either eqn. (27.47) from eqn. (27.49) or eqn. (27.48) from eqn. (27.50), we get for both the dark and the bright rings the relation

$$\begin{aligned} D_{n+p}^2 - D_n^2 &= 4p \lambda R \\ \text{or, } \lambda &= \frac{D_{n+p}^2 - D_n^2}{4p R} \end{aligned} \quad (27.51)$$

In Newton's rings experiment, eqn. (27.51) is invariably employed to compute  $\lambda$ . The advantage of this equation over equations (27.47) and (27.48) can be seen from the following discussion : eqns. (27.47) and (27.48) have been derived on the supposition that the surfaces of the lens and the plate are perfect and that they are in perfect contact, i.e., the thickness of the air film at the point of contact is zero ( $t = 0$ ). This gives rise, in a reflected system, a fringe system of alternate bright and dark rings concentric with a central dark spot. In actual practice, either due to some imperfections in the surfaces in contact or due to encroachment of some dust particles between the lens and the plate, they may not be in perfect contact i.e., the thickness of the film may not be zero at the central point. The order,  $x$ , of the central ring, therefore, is indeterminate, i.e., it is not possible to say with certainty if the

central dark ring corresponds to zero or 1st or 2nd ..... etc. order. The central spot may even be white. As a consequence, the order of every other bright or dark ring advances by this indeterminate number  $x$ . For any one of them, the square of the diameter is not given by eqn. (27.47) or (27.48). But this indeterminacy does not occur in eqn. (27.51) when the difference of the squares of the diameters of the  $n^{\text{th}}$  and  $(n + p)^{\text{th}}$  dark or bright rings are considered, counting the rings  $p$ , between them visually.

### determination of the radius of curvature of the lens

Eqn. (27.51) can be rearranged as

$$R = \frac{D_{n+p}^2 - D_n^2}{4p\lambda}$$

Thus, if the wavelength  $\lambda$  of the light used is known then eqn. (27.51) can be used to determine the radius of curvature of the surface of the lens in contact with the plane glass plate.

### determination of refractive index of a liquid with Newton's rings

It is possible to determine the refractive index of a liquid by Newton's rings method. The diameters of two particular rings, say the  $n^{\text{th}}$  and  $(n + p)^{\text{th}}$ , obtained in Newton's rings with an air film, are measured. Then a drop of liquid, whose refractive index is to be measured, is carefully introduced into the air film. The liquid is drawn in at the centre by the capillary action forming a liquid film between the lens and the plate. When the film is illuminated with the same monochromatic light, another set of Newton's rings is obtained. The diameters of the same two rings [ $n^{\text{th}}$  and  $(n + p)^{\text{th}}$ ] are then measured. The difference in diameters of the two rings for the two films are

$$\text{for air film : } (D_{n+p}^2 - D_n^2)_{\text{air}} = 4p\lambda R$$

$$\begin{aligned} \text{for liquid film : } (D_{n+p}^2 - D_n^2)_{\text{liquid}} \\ = \frac{4p\lambda R}{\mu} \end{aligned}$$



where  $\mu$  is the refractive index of the liquid.

Then,

$$\mu = \frac{(D_{n+p}^2 - D_n^2)_{\text{air}}}{(D_{n+p}^2 - D_n^2)_{\text{liquid}}} \quad (27.52)$$

since  $\mu > 1$ , the rings contract in the ratio  $\sqrt{1/\mu}$  when the air-film is replaced by the liquid film.

**Example 27.16.** The diameter of the tenth dark ring in a Newton's ring system viewed normally by reflected light of wavelength 5900 Å is 5 mm. Calculate the radius of curvature of the lens and the thickness of the air film.

**Soln.**

For dark rings

$$D_n^2 = \frac{4n\lambda R}{\mu}$$

$$\text{or, } (0.5)^2 = \frac{4 \times 10 \times 5900 \times 10^{-8} \times R}{1}$$

$$\text{or, } R = 106 \text{ cm (approximately)}$$

For dark rings, the thickness of the air-film is given by

$$2\mu e = n\lambda$$

$$\text{or, } 2 \times 1 \times e = 10 \times 5900 \times 10^{-8}$$

$$\text{or, } e = 29.5 \times 10^{-5} \text{ cm.}$$

**Example 27.17.** In a Newton's ring experiment, the plano-convex lens and the glass plate are in optical contact and the thickness of the air film at that point is zero. Find out the thickness of the air-wedge at the 4<sup>th</sup> bright ring for light of  $\lambda = 5000 \text{ Å}$ .

**Soln.**

For bright rings the formula in terms of the thickness,  $e$ , of the air film is



$$2\mu e = (2n - 1) \frac{\lambda}{2} \text{ where } n = 4$$

$$\text{or, } 2\mu e = (2n + 1) \frac{\lambda}{2} \text{ where } n = 3$$

$$\therefore e = \frac{7 \times 5 \times 10^{-5}}{4} \quad (\mu = 1)$$

$$= 8.75 \times 10^{-5} \text{ cm.}$$

**Example 27.18.** Newton's rings formed with sodium light between a flat glass plate and a convex lens are viewed normally. What will be the order of the dark ring which will have double the diameter of the 10<sup>th</sup> ring?

**Soln.**

The diameter of the dark rings are proportional to square root of the natural numbers i.e.,  $D_n^2 \propto n$

$\therefore D_n^2 \propto 10$  where  $D_{10}$  is the diameter of the 10<sup>th</sup> dark ring. Let  $n$  be the order of the dark ring whose diameter is twice the diameter of the 10<sup>th</sup> ring i.e.,  $2D_{10}$ .

Hence  $(2D_{10})^2 \propto n$

$$\text{or, } \frac{n}{10} = \frac{(2D_{10})^2}{(D_{10})^2} = 4 \quad \text{or, } n = 40$$

The 40<sup>th</sup> dark ring will have a diameter twice that of the 10<sup>th</sup> dark ring.

**Example 27.19.** Newton's rings are formed with reflected light of  $\lambda = 5890 \times 10^{-6} \text{ cm}$  between a double convex lens and a glass plate with a liquid between them. The diameter of the third bright ring is 0.2 cm. If the radius of curvature of the lens surface in contact with the plane glass surface is 90 cm. find the refractive index of the liquid.

**Soln.**

For bright ring,  $D_n^2 = \frac{2\lambda R}{\mu} (2n - 1)$

$$\begin{aligned} \text{or, } \mu &= \frac{2\lambda R (2n - 1)}{D_n^2} \\ &= \frac{2 \times 5890 \times 10^{-8} \times 90 \times (3 \times 2 - 1)}{(0.2)^2} \end{aligned}$$

or,  $\mu = 1.325$

**Example 27.20.** In a Newton's ring experiment with air film, the diameter of the 15<sup>th</sup> dark ring is 0.59 cm. On introducing a liquid film in between the glass plate and the curved surface plate, its diameter decreases by 0.09 cm. What is the refractive index of the liquid?

**Soln.**

For the air film  $D_{15}^2 = 4n\lambda R$

or,  $(0.59)^2 = 4 \times 15 \times \lambda R$

For liquid film  $D_{15}^2 = \frac{4n\lambda R}{\mu}$

or,  $(0.59 - 0.09)^2 = \frac{4 \times 15 \times \lambda R}{\mu}$

or,  $\mu = \frac{(0.59)^2}{(0.50)^2} = 1.3924$

**Example 27.21.** The diameter of the 10<sup>th</sup> bright ring in a Newton's ring apparatus changes from 1.40 to 1.27 cm as a liquid is introduced between the lens and the plate. Find the index of refraction of the liquid.

**Soln.**

For bright rings

$$(D_n^2)_{\text{air}} = 2\lambda R (2n - 1)$$

$$\text{and } (D_n^2)_{\text{liquid}} = \frac{2\lambda R(2n - 1)}{\mu}$$

$$\text{Hence } \mu = \frac{(D_{10}^2)_{\text{air}}}{(D_{10}^2)_{\text{liquid}}} = \frac{(1.40)^2}{(1.27)^2} = 1.215$$

**Example 27.22.** Newton's rings are formed with sodium light ( $\lambda = 5.9 \times 10^{-5}$  cm) between a plane glass plate and a convex lens surface. The diameters of two successive dark rings are 2.0 mm and 2.236 mm. What is the radius of curvature of the lens surface?

**Soln.**

$$\text{For dark ring } D_n^2 = \frac{4n\lambda R}{\mu} = 4n\lambda R \quad (\mu = 1)$$

$$\therefore D_{n+1}^2 - D_n^2 = 4(n+1)\lambda R - 4n\lambda R = 4\lambda R$$

$$\begin{aligned} \text{or, } R &= \frac{D_{n+1}^2 - D_n^2}{4\lambda} \\ &= \frac{(0.2236)^2 - (0.2)^2}{4 \times 5.9 \times 10^{-5}} = 42.36 \text{ cm} \end{aligned}$$

**Example 27.23.** In a Newton's ring experiment, the diameter of the 4<sup>th</sup> and the 8<sup>th</sup> rings are 3.21 mm and 4.55 mm respectively. If the wavelength of light used is 6145 Å, what is the radius of curvature of the face of the convex lens in contact with plane glass plate?

**Soln.**

$$D_{n+p}^2 - D_n^2 = 4p\lambda R$$

$$\begin{aligned} \text{or, } \frac{D_8^2 - D_4^2}{4p\lambda} &= \frac{(0.455)^2 - (0.32)^2}{4 \times 4 \times 6.145 \times 10^{-5}} \\ &= 105.76 \text{ cm} \end{aligned}$$



**Example 27.24.** In a Newton's ring arrangement with a source emitting two wavelengths  $\lambda_1 = 6 \times 10^{-5}$  cm and  $\lambda_2 = 4.5 \times 10^{-5}$  cm it is found that the  $n^{\text{th}}$  dark ring due to  $\lambda_1$  coincides with  $(n+1)^{\text{th}}$  dark ring for  $\lambda_2$ . If the radius of curvature of the curved surface is 90 cm., find the diameter of the  $n^{\text{th}}$  dark ring for  $\lambda_1$ .

**Soln.**

Let  $D$  be the diameter of the  $n^{\text{th}}$  dark ring when light of wavelength  $\lambda_1$  is employed. According to the condition of the problem,

$$D^2 = 4n\lambda_1 R = 4(n+1)\lambda_2 R$$

On substituting the values of  $\lambda_1$  and  $\lambda_2$  in the above equation, we have

$$4n \times 6 \times 10^{-5} = 4(n+1) \times 4.5 \times 10^{-5};$$

$$\text{or, } n = 3$$

$$\begin{aligned} \text{Hence } D &= \sqrt{4n\lambda_1 R} = \sqrt{4 \times 3 \times 6 \times 10^{-5} \times 90} \\ &= 0.545 \text{ cm} \end{aligned}$$

**Example 27.25.** Light containing two wavelength  $\lambda_1$  and  $\lambda_2$  falls normally on a plano-convex lens of radius of curvature  $R$  resting on a glass plate. If the  $n^{\text{th}}$  dark ring due to  $\lambda_1$  coincides with  $(n+1)^{\text{th}}$  dark ring due to  $\lambda_2$ , prove that the radius of the  $n^{\text{th}}$  dark ring of  $\lambda_1$  is  $\sqrt{\lambda_1 \lambda_2 R / (\lambda_1 - \lambda_2)}$

**Soln.**

\* The radius of the  $n^{\text{th}}$  dark ring due to  $\lambda_1$

$$= \sqrt{n\lambda_1 R} \dots\dots\dots (i)$$

The radius of the  $(n+1)^{\text{th}}$  dark ring due to  $\lambda_2$

$$= \sqrt{(n+1)\lambda_2 R} \dots\dots\dots (ii)$$

As (i) and (ii) are equal

$$r = \sqrt{n\lambda_1 R} = \sqrt{(n+1)\lambda_2 R}$$

$$\text{or, } n\lambda_1 R = (n+1)\lambda_2 R$$

$$\text{or, } n = \frac{\lambda_2}{\lambda_1 - \lambda_2} \dots\dots\dots (iii)$$

Substituting the value of  $n$  in equation (i)

$$r = \sqrt{n\lambda_1 R} = \sqrt{\frac{\lambda_1 \lambda_2 R}{\lambda_1 - \lambda_2}}$$

**Example 27.26.** In Newton's ring experiment, the diameters of the  $n^{\text{th}}$  and  $(n + 8)^{\text{th}}$  bright rings are 4.2 mm and 7 mm respectively. Radius of curvature of the lower surface of the lens is 2 meters. Determine the wavelength of light and the ring number  $n$ .

**Soln.**

$$D_n = 0.42 \text{ cm; } D_{n+8} = 0.7 \text{ cm;}$$

$$R = 200 \text{ cm; } p = 8$$

$$\lambda = \frac{D_{n+8}^2 - D_n^2}{4pR} = \frac{(0.7)^2 - (0.42)^2}{4 \times 8 \times 200}$$

$$\text{or, } \lambda = 4900 \times 10^{-8} \text{ cm}$$

$$\text{Now } D_n^2 = (2n - 1) \frac{2\lambda R}{\mu} \text{ for bright rings}$$

$$\begin{aligned} \text{or, } (0.42)^2 &= (2n - 1) \times 2 \times 4.9 \times 10^{-5} \times 200 \quad (\mu = 1) \\ &= (2n - 1) \times 19.6 \times 10^{-3} \end{aligned}$$

$$\text{or, } (2n - 1) = 9$$

$$\text{or, } n = \frac{9+1}{2} = 5$$

**Example 27.27.** Newton's rings are formed by light of  $\lambda = 5890 \text{ \AA}$  between a double convex lens and a glass plate with a liquid between them. The diameters of the 5<sup>th</sup> and 15<sup>th</sup> rings in the reflected system are 2.78 mm and 4.51 mm respectively. If the radius of curvature of the lens is 80 cm, calculate the refractive index of the liquid.

**Soln.**

$$D_{15}^2 - D_5^2 = \frac{4n\lambda R}{\mu}$$

$$\text{or, } \mu = \frac{4n\lambda R}{D_{15}^2 - D_5^2}$$

$$= \frac{4 \times 10 \times 5890 \times 10^{-9} \times 80}{(0.451)^2 - (0.278)^2} = 1.5$$

### 27.21 Interferometry

The phenomenon of interference has been used to measure the wavelength of light and the thickness of very thin plate, to test the planeness of surfaces, to determine refractive indices of substances and also to determine whether a given source of light is monochromatic or otherwise. Instruments based on the principle of interference of light are known as *interferometers*.

Some of these interferometers can be used in conjunction with a large extended source. Of these, the *Michelson interferometer* has been important in the past and still is. The instrument, designed by Prof. Albert Michelson, is used to determine the wavelength of light, to measure the thickness of thin strips, for the resolution of spectral line and also for the standardization of the metre.

### 27.22 Michelson interferometer

Fig. 27.24 is a schematic representation of the principal features of Michelson interferometer.  $M_1$  and  $M_2$  are two excellent optically plane mirrors, highly silvered on their front surfaces to avoid multiple internal reflections.  $G_1$  and  $G_2$  are two plane parallel glass plates of equal thickness. To ensure equality of thickness and the nature of the material,  $G_1$  and  $G_2$  are cut from the same optically plane-parallel plate. Both the plates are mounted vertically, exactly



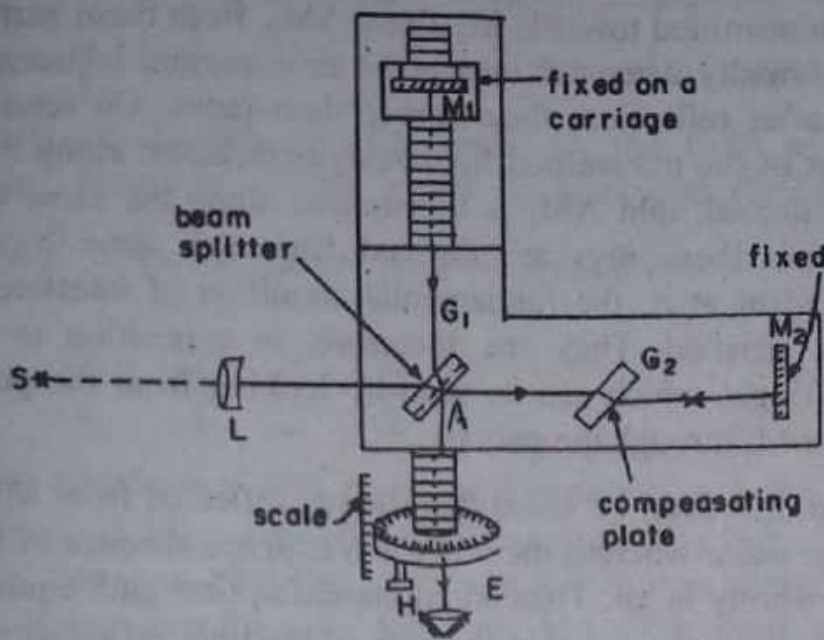


Fig. 27.24

parallel to each other, on a heavy frame and are usually inclined at  $45^\circ$  to the interferometer arm. The right side or rear of the plate  $G_1$  has a thin coating of silver of such a thickness that one half of the amplitude of the incident wave is reflected while the rest is transmitted through it. The mirror  $M_2$  is fixed while the other mirror  $M_1$  is fixed on a carriage, which can be moved slowly and accurately along a precision machined straight and parallel tracks. This motion is communicated by means of a handle  $H$ , which is provided with a wheel and a graduated circle so that the motion of  $M_1$  may be accurately known to the order of  $10^{-5}$  cm. During the motion,  $M_1$  remains exactly parallel to its preceding positions. Mirrors  $M_1$  and  $M_2$  are provided with screws which can give them slight rotation about the vertical as well as the horizontal axis. Thus the planes of the mirrors  $M_1$  and  $M_2$  can be made perfectly perpendicular.

### *working principle*

Light from an extended monochromatic source  $S$  after being rendered nearly parallel by the lens  $L$ , falls on the glass plate  $G_1$ . At the rear of the glass plate (thinly silvered surface), the beam is split into two

parts. One part is reflected along  $AM_1$  towards the mirror  $M_1$  while the other part is transmitted towards  $M_2$  along  $AM_2$ . Both these parts fall on the mirrors normally when the instrument is in normal adjustment, and accordingly, after reflection, they retrace their paths. On return at the point A a part of the transmitted light  $AM_2$  is reflected along  $AE$  and a part of the reflected light  $AM_1$  is transmitted along the same direction  $AE$ . Since both these rays are derived from the same light wave, originally incident at A, the fundamental condition of interference i.e., coherence, is satisfied. They are, therefore, in a position to produce interference fringes which can be seen by looking from the position E into the mirror  $M_1$  through the plate  $G_1$ .

It should, however, be noted that the ray reflected from  $M_1$  crosses the glass plate twice whereas the other wave, in the absence of the glass plate  $G_2$  lies wholly in air. Thus an additional optical path equivalent to  $2(\mu - 1)t$  in air is introduced in the path of the light reflected from  $M_1$ , where  $\mu$  is the refractive index of glass plate  $G_1$  for the wavelength of light employed and  $t$  its thickness. This in itself does not present any serious difficulty when monochromatic light is used. But if white light is used,  $\mu$  will vary with wavelength and the fringes observed will be coloured. Hence, if achromatic fringes are to be observed in this instrument with white light, this additional path must be compensated not only for one wavelength but for all wavelengths. That is why a second plate  $G_2$  of the same material and thickness and placed exactly parallel to  $G_1$  is introduced into the path of the ray reflected from  $M_2$  to equalise the two paths.  $G_2$  is, therefore, known as the *compensating plate* while  $G_1$  is called the *beam splitter*.

When  $G_1$  is exactly parallel to  $G_2$  and the two air paths  $AM_1$  and  $AM_2$  are equal, then the optical path difference between the two rays is  $\frac{\lambda}{2}$  because the ray  $M_2A$  suffers reflection from the back of  $G_1$ , i.e., a surface backed by a denser medium. Hence, the two rays interfere destructively i.e., they annul each other to produce darkness. If now the movable mirror  $M_1$  is advanced slowly parallel to itself by a distance  $\frac{\lambda}{4}$  to its new position  $M'_1$ , then the change in path difference becomes  $\frac{\lambda}{4} + \frac{\lambda}{4} = \frac{\lambda}{2}$  (up and down journey).



Hence, the total optical path difference between the two rays becomes  $\frac{\lambda}{2} + \frac{\lambda}{2} = \lambda$ . The two rays will, therefore, interfere constructively to produce brightness in the field of view of the telescope. However, if the mirror is advanced further by another distance  $\frac{\lambda}{4}$ , then the optical path difference will become equal to

$$\frac{2\lambda}{4} + \frac{2\lambda}{4} + \frac{\lambda}{2} = \frac{3\lambda}{2}$$

The interference will again be destructive and the field of view dark. Thus, as  $M_1$  is moved parallel to itself, the field of view of the telescope becomes alternately bright and dark depending upon the path difference between the rays  $AM_1$  and  $AM_2$ . Therefore, the passage of consecutive bright or dark fringes across the cross-wire of the telescope corresponds to a displacement of the mirror  $M_1$  by a distance  $\frac{\lambda}{2}$ .

#### *nature of fringes*

Looking in the direction  $M_1$  from E, one observes a virtual image  $M'_2$  of  $M_2$  along with  $M_1$ . Therefore, one of the interfering beam arises out of reflection from  $M_1$  and the other, which actually arises out of reflection from  $M_2$  functions as if it had been reflected from  $M'_2$ . The two reflected beams originate from the same incident beam. Thus the mirror  $M_1$  and the virtual image of  $M_2$  play the same roles as the two surface of a thin film (without, of course, the phenomenon of multiple reflections within the film) and the same sort of interference fringes result from the light that is reflected from these surfaces. Depending on the separation between  $M_1$  and  $M_2$  and the angle of inclination between them, the fringes may be *straight, circular, parabolic, hyperbolic or elliptical*. Of these various possibilities, two cases are of practical interest namely, *circular and localized fringes*. The mathematical theory of the shape of these fringes is beyond the scope of this book; only the conditions under which these fringes are obtained may be given.

(i) *circular fringes* : When the interferometer is in *normal adjustment i.e.*, the image  $M'_2$  of  $M_2$  formed in  $G_1$  is exactly parallel to  $M_1$ , concentric circular fringes are produced with monochromatic light.



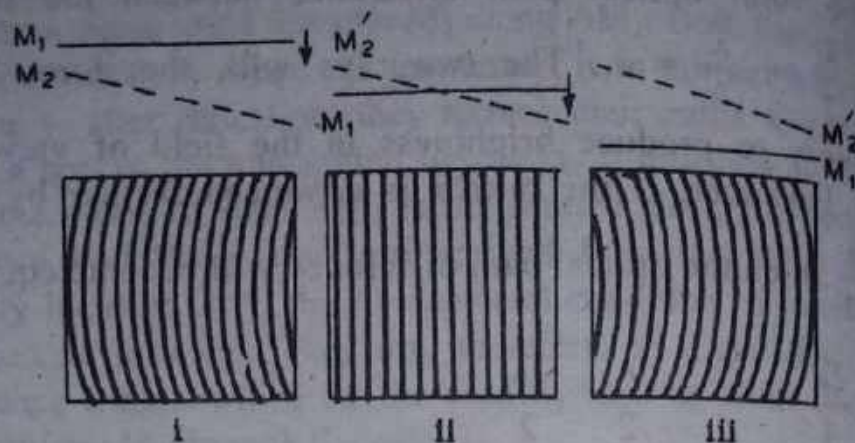


Fig. 27.25

(ii) *Localized fringes* : When the mirror  $M_1$  and the virtual mirror  $M_2'$  are not exactly parallel, the air film enclosed is wedge-shaped. Three distinct cases then arise. If  $M_1$  actually intersects  $M_2'$  in the middle [Fig. 27.25 (ii)], the fringes obtained are perfectly straight. In the other two positions [27.25 (i) and (iii)], the fringes are curved and are always convex towards the thin edge of the wedge.

#### uses of Michelson interferometer

(i) *determination of wavelength* : As discussed above, for every displacement of  $\frac{\lambda}{2}$  by the mirror  $M_1$ , one bright fringe goes past the cross-wire of the telescope. If  $n$  fringes move across, then displacement of  $M_1$  is  $n \frac{\lambda}{2}$ . Denoting this displacement by  $d$ , we get

$$n \cdot \frac{\lambda}{2} = d; \quad \text{or, } \lambda = \frac{2d}{n}$$

Thus, by measuring  $d$  and counting  $n$ ,  $\lambda$  can be found. Conversely, if  $\lambda$  is known, then  $d$  can be found out.

(ii) *determination of refractive index* : Suppose the refractive index  $\mu$  of a thin film of thickness  $t$  is to be determined. For this purpose, the thin film is introduced in the path of one of the interfering beams, thereby increasing the path of that beam by  $(\mu - 1)t$ . The path difference

between the two beams, therefore, becomes  $2(\mu - 1)t$  (up and down journey). Let  $n$  number of fringes move across the cross-wire due to this path difference. Then

$$2(\mu - 1)t = n\lambda$$

$\mu$  can be calculated from the known or measured values of  $t$ ,  $n$  and  $\lambda$ .

(iii) *resolution of spectral lines* : Michelson interferometer can also be used to find (a) whether a given source of light is monochromatic and (b) the difference in the wavelengths of a spectral doublet i.e., two lines having very small difference in wavelength e.g., sodium-D lines.

Suppose the wavelengths of sodium  $D_1$  and  $D_2$  lines are  $\lambda_1$  and  $\lambda_2$  respectively. Each one of them will give rise to its own set of fringes in the Michelson interferometer. But adjusting the position of the mirror  $M_1$ , the bright fringes due to  $D_1$  may be made to coincide with the bright fringe due to  $D_2$  when the fringes are very bright (*consonance*). As the mirror  $M_1$  is moved, the two sets get in step or out of step alternately because  $\lambda_1 \neq \lambda_2$ . Starting from a position of maximum brightness (the two sets of fringes are in step), let the mirror  $M_1$  be moved through a certain distance, when the bright fringe due to one set coincides with the dark fringe of the other set (out of step) and no fringes will be seen in this case (minimum intensity or *dissonance*). By continuing to move the mirror  $M_1$  in the same direction, a position is reached when the bright fringe of one set coincides with the bright fringe of the other set (in step) and the fringes are again very bright and distinct (maximum intensity). This is possible when the  $n^{\text{th}}$  order of the longer wavelength coincides with the  $(n + 1)^{\text{th}}$  order of the shorter wavelength.

Let  $d$  be the distance through which  $M_1$  is moved between two consecutive positions of maximum distinctness of the fringes and let  $n_1$  and  $n_2$  be the corresponding changes in the order at the centre of the field.

Then

$$2d \cos \theta = n_1 \lambda_1 = n_2 \lambda_2$$

$$\text{or, } 2d = n_1 \lambda_1 = n_2 \lambda_2$$

(27.53)



( $\cos \theta = 1$  at the centre of the field)

If  $\lambda_1$  is greater than  $\lambda_2$ , then  $n_2 = n_1 + 1$

$$\therefore 2d = n_1 \lambda_1 = (n_1 + 1) \lambda_2$$

$$\text{or, } n_1 \lambda_1 = (n_1 + 1) \lambda_2$$

$$\text{or, } n_1 = \frac{\lambda_2}{\lambda_1 - \lambda_2} \quad (27.54)$$

Substituting this value of  $n_1$  in (27.53)

$$2d = \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2}$$

$$\text{or, } \lambda_1 - \lambda_2 = \frac{\lambda_1 \lambda_2}{2d}$$

Taking  $\lambda$  as the mean of  $\lambda_1$  and  $\lambda_2$ , we get

$$\Delta \lambda = \frac{\lambda^2}{2d} \quad (27.55)$$

Thus the difference in wavelength between two extremely close spectral lines may be calculated with the help of eqn. (27.55). In actual practice, readings for ten successive positions of maximum distinctness are taken and the mean value of  $d$  is calculated.

#### *calibration of standard metre*

A modified form of the interferometer was used by Michelson and Benoit in 1895 to calibrate the standard metre, kept at Bureau of Weights and Measures at Sevres near Paris. The calibration was made with respect to the red cadmium line of wavelength 6438 Å.U. i.e., how many wavelengths of this line is contained in the metre.

The basic principle consists in counting the number of fringes go past the cross-wire of the telescope when the mirror  $M_1$  is brought in flush with one fiduciary mark of the metre at one end and moved slowly till it is in flush with the fiduciary mark at the other end. Since the shift of one bright fringe corresponds to a displacement of  $\lambda/2$  of the mirror  $M_1$ , the total number of



wavelengths of the cadmium line may be determined from the number of fringes. However, in practice, it is not possible to count the number of fringes which cross the field of view when the mirror  $M_1$  is moved through the whole length of one metre. Moreover, for a path difference of more than 20 cm, it is not possible to obtain the fringes. Realizing the difficulties involved, Michelson divided the metre into a number of intermediate sub-standards called *etalons*, which consisted of two exactly parallel mirrors separated by distances of 10 cm, 5 cm, 1.25 cm, ..... 0.039 cm – the last one being the smallest. The smallest etalon was calibrated in terms of the wavelength by actually counting the number of fringes which crossed field of view as the mirror  $M_1$  was moved from the plane of first mirror of the etalon to the next one. The number of times this shortest etalon was contained in the next longer etalon was then determined. The second etalon was then compared with the third and so on till the 10 cm etalon was evaluated. Finally with the 10 cm etalon, the standard metre was calibrated in terms of cadmium light of red, green and blue colours.

As a result of their experiments Michelson and Benoit found that at 15°C and 760 mm pressure

1 metre

= 1553163.5 wavelengths of cadmium red light.

$$(\lambda_r = 6438.4722 \text{ \AA})$$

= 1966249.7 wavelengths of cadmium green light.

$$(\lambda_g = 5085.8240 \text{ \AA})$$

= 2083372.1 wavelengths of cadmium blue light

$$(\lambda_b = 4799.9107 \text{ \AA})$$

These values are correct up to about 1 part in 1 million. This experiment enables the standard metre to be measured in terms of a constant and easily and accurately reproducible quantity – the wavelength of red cadmium light. The experiment, therefore, is of great significance, since the standard metre can always be reproduced exactly, should the standard metre lying in Paris be lost due to some unforeseen reasons.

**Example 27.28.** In a Michelson interferometer 200 fringes cross the field of view when the movable mirror is displaced through 0.0589 mm. Calculate the wavelength of monochromatic light used.

**Soln.**

We have

$$n \frac{\lambda}{2} = d$$

where

$d$  = displacement of the mirror

$n$  = number of fringes crossing the field of view.

$$\text{or, } 200 \frac{\lambda}{2} = 589 \times 10^{-5} \text{ cm}$$

$$\text{or, } \lambda = 589 \times 10^{-7} \text{ cm.}$$

$$= 5890 \times 10^{-8} \text{ cm.}$$

**Example 27.29.** A Michelson interferometer, illuminated with monochromatic light of wavelength 5890 Å, is used to measure the distance between two points. If 324 fringes sweep across the cross-wire of the telescope while the interferometer mirror moves from one point to other, calculate the distance between the points.

**Soln.**

From  $n \frac{\lambda}{2} = d$ , we have

$$\frac{324 \times 5890 \times 10^{-8}}{2} = d$$

$$\text{or, } d = 0.0095418 \text{ cm.}$$

**Example 27.30.** In an experiment with Michelson interferometer, the scale readings for two consecutive positions of maximum distinctness were found to be 0.6939 mm and 0.9884 mm. If the



mean wavelength for the two components of the D lines be 5893 A.U., deduce the difference between the two wavelengths.

**Soln.**

$$\lambda_1 - \lambda_2 = \frac{\lambda_{av}^2}{2(d_1 - d_2)}$$

$$\begin{aligned} \text{or, } \lambda_1 - \lambda_2 &= \frac{(5893 \times 10^{-8})^2}{2(0.09884 - 0.06939)} \\ &= 6 \times 10^{-8} \text{ cm} = 6 \text{ A.U.} \end{aligned}$$

**Example 27.31.** Two wavelengths  $\lambda_1 = 5890 \text{ A.U.}$  and  $\lambda_2 = 5896 \text{ A.U.}$  are emitted by a source of light. Interference fringes are observed with a certain arrangement when the paths of the interfering beams are exactly equal. How much will the path difference have to be increased so that a bright fringe for  $\lambda_2$  coincides with a dark fringe of  $\lambda_1$ .

**Soln.**

The bright or dark fringes due to  $\lambda_1$  fall over the bright or dark fringes corresponding to  $\lambda_2$  i.e., the fringe systems due to  $\lambda_1$  and  $\lambda_2$  are coincident when the paths of the interfering beams are exactly equal. If  $\Delta$  be the path difference introduced when a bright fringe for  $\lambda_2$  coincides with a dark fringe for  $\lambda_1$ , then we have the relation.

$$\Delta = n\lambda_2 = \left(n + \frac{1}{2}\right) \lambda_1$$

$$\text{or, } \Delta = \frac{\lambda_1 \lambda_2}{2(\lambda_2 - \lambda_1)}$$

$$= \frac{5890 \times 10^{-8} \times 5896 \times 10^{-8}}{2 \times 6 \times 10^{-8}} = 0.0289 \text{ cm.}$$


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## EXERCISES

- [1] What is interference of light? Discuss the important conditions for the interference of light.
- [2] What are coherent sources? Discuss why two independent sources of light of the same wavelength cannot produce interference fringes. Give diagrams showing clearly how coherent sources are produced in (i) Newton's rings arrangement, (ii) bi-prism arrangement.
- [3] Discuss interference of light analytically and obtain the conditions of maximum and minimum intensities.
- [4] Discuss the conditions for interference. Describe Young's double-slit experiment and show how the result can be explained with the help of wave theory of light.
- [5] Describe Young's double-slit experiment and derive an expression for (i) intensity at a point on the screen and (ii) fringe-width.
- [6] State and explain Huygens' principle of secondary waves and Young's principle of superposition.
- [7] Prove that the distance  $X$ , between two successive bright or dark fringes is given by  $X = \frac{D\lambda}{d}$ .
- [8] Derive an expression for the width of a fringe in terms of the wavelength of light used, distance between the two coherent sources and the distance of the screen from the sources.
- [9] Describe, giving experimental details, Fresnel's bi-prism method for determining the wavelength of light. Derive the formula used.
- [10] Explain the formation of coherent sources in the case of bi-prism. How is the separation between such coherent sources measured in the experiment with a bi-prism? Explain the theoretical basis of this measurement.
- [11] Describe an experimental arrangement for observation of interference fringes using a Fresnel bi-prism. Show with necessary theory, how would you determine the wavelength of monochromatic light thereby.
- [12] Calculate the displacement of fringes when a thin transparent lamina is introduced in the path of one of the interfering beams in a bi-prism. Show how this method is used for finding the thickness or refractive index of a mica sheet.

- [13] Describe in detail the experiment to determine the wavelength of sodium light with a Fresnel's bi-prism.
- [14] Describe and explain the phenomenon of interference in thin films. Give the necessary theory. Why do colours appear in thin film in white light?
- [15] Explain why different colours are exhibited by thin films in white light. With suitable diagrams, explain why broad source of light is needed to observe this phenomenon.
- [16] Explain the formation of Newton's rings. Show how you would use them to find (a) refractive index of a liquid and (b) radius of curvature of lens.
- [17] Describe with necessary theory the Newton's rings method of measuring wavelength of monochromatic light.
- [18] Explain with necessary theory how can you determine the refractive index of a liquid by means of Newton's rings. Derive the formula used. Why are the rings circular?
- [19] (a) Describe and explain the formation of Newton's rings in reflected light. Prove that in reflected light (i) diameters of dark rings are proportional to the square roots of natural numbers and (ii) diameters of bright rings are proportional to the square roots of odd numbers.  
(b) Account for the perfect blackness of the central spot in Newton's rings.  
(c) What will happen if a little water is introduced between the lens and the plate?
- [20] Describe a Michelson's interferometer and explain its working.
- [21] Outline the theory of Michelson's interferometer and discuss the nature of the interference pattern produced. Under what conditions will it give circular fringes? Show how it may be used to measure the wavelength of light.
- [22] (a) How can Michelson's interferometer be used for measuring the wavelength of light?  
(b) How will you use Michelson's interferometer to determine the difference between two wavelength very close to each other?  
(c) How will you use Michelson's interferometer to determine the thickness of a thin transparent film or plate?



- (d) How can you use Michelson's interferometer to standardize the metre?
- (e) Mention the various applications of Michelson's interferometer.
- [23] Write short notes on :
- (i) Huygens' Principle
  - (ii) Coherent sources
  - (iii) Testing the planeness of surfaces
  - (iv) Fringes with white light
  - (v) Conditions of interference
  - (vi) Fresnel bi-prism
  - (vii) Stokes' theorem
  - (viii) Colours of thin plates
  - (ix) Newton's rings
  - (x) Michelson interferometer and its uses.
- [24] Two straight and narrow parallel slits 3 mm apart are illuminated with a monochromatic source ( $\lambda = 5.9 \times 10^{-5}$  cm). Fringes are obtained at a distance of 30 cm from the slit. Find the width of fringes. (0.0059 cm).
- [25] Green light of wavelength  $5100 \text{ \AA}$  from a narrow slit is incident on a double slit. If the fringe spacing on a screen 2 metre away is 2 mm, find the separation of the slits. (0.051 cm).
- [26] Two coherent sources are 0.18 mm apart and the fringes are observed on a screen 80 cm away. It is found that with a certain monochromatic sources of light the fourth bright fringe is situated at a distance of 10.8 mm from the central fringe. Calculate the wavelength of light. ( $6075 \text{ \AA}$ ).
- [27] In a bi-prism experiment the eye-piece was placed at a distance of 1.20 m from the source. The distance between the virtual sources was found to be equal to  $7.5 \times 10^{-4}$  m. Find the wavelength of the source of light, if the eye-piece has to be moved transversely through a distance of 1.888 cm for 20 fringes. ( $5900 \text{ \AA}$ ).
- [28] The distance between the slit and the bi-prism and that between the bi-prism and the screen are each 50 cm. The obtuse angle of the bi-prism is  $179^\circ$  and its refractive index is 1.5. If the distance between



the successive bright fringes is 0.135 mm, calculate the wavelength of light.

- [29] The inclined faces of a bi-prism of refractive index 1.5 make angles of  $2^\circ$  with its base. A slit illuminated by a monochromatic light is placed at a distance of 10 cm from the bi-prism. If the distance between two dark fringes, observed at a distance of 1 metre from the bi-prism is 0.18 mm, find the wavelength of light used.

- [30] The double slit arrangement is illuminated with sodium light ( $\lambda = 5890 \text{ \AA}$ ). The slits are having a separation of 2.0 mm and the screen on which interference pattern appears is 20 cm away. Calculate (i) the angular position of the first minima and (ii) the linear separation between two adjacent minima.

What will be the value of separation between two adjacent maxima?

( $0.0084^\circ$ ,  $5.89 \times 10^{-3} \text{ cm}$ , same as that between two adjacent minima)

- [31] In one typical bi-prism experiment, the spacing of 20 bands was 4 mm. The distance of the eye-piece from the slit was 100 cm. When a lens was mounted on the axis at a distance of 20 cm from the slit, a well defined image was seen in the eye-piece and the virtual sources was 1.18 cm. Calculate the wavelength of light used. (5900 A.U.)

- [32] Interference fringes are observed with a bi-prism of refracting angle  $1^\circ$  and refractive index 1.5 on a screen 80 cm away from it. If the distance between the source and the bi-prism is 20 cm, calculate the fringe-width when the wavelength of light used is (i)  $6900 \text{ \AA}$  and (ii)  $5890 \text{ \AA}$ . ( $0.01976 \text{ cm}$ ,  $0.01687 \text{ cm}$ )

- [33] In an experiment with Fresnel's bi-prism fringes for light of wavelength  $5 \times 10^{-5} \text{ cm}$  are observed 0.2 mm apart at a distance of 175 cm from the prism. The prism is made of glass of refractive index 1.50 and it is at a distance of 25 cm from the illuminated slit. Calculate the angle at the vertex of the bi-prism. ( $177^\circ 42'$ )

- [34] Calculate the separation between the coherent sources formed by a bi-prism whose inclined faces make angles of 2 degrees with its base, the slit source being 10 cm away from the bi-prism. the refractive index of the material of the bi-prism is 1.5.

- [35] A Fresnel's bi-prism with angle of  $1^\circ 30'$  and refractive index 1.52 is used to obtain interference fringes. Find the width of a fringe with light of wavelength 6563 A.U. when the distance between the source

and the prism is 20 cm and that between the prism and the screen is 80 cm. (0.1205)

- [36] In a double slit interference arrangement one of the slits is covered by a thin mica sheet whose refractive index is 1.58. The separation between the sources and the distance of the screen from the sources are 1 mm and 50 cm respectively. If the central fringe gets shifted by 2 mm due to the introduction of the mica sheet, determine the thickness of the mica sheet. ( $\approx 6.7 \times 10^{-4}$  cm)
- [37] A thin sheet of transparent material ( $\mu = 1.60$ ) is placed in the path of one of the interfering beams in a bi-prism experiment using sodium light ( $\lambda = 5890 \times 10^{-8}$  cm). The central fringe shifts to a position originally occupied by the 12<sup>th</sup> bright fringe. Calculate the thickness of the sheet. ( $1.178 \times 10^{-3}$  cm)
- [38] When a thin piece of glass  $3.4 \times 10^{-4}$  cm thick is placed in the path of one of the interfering beams in a bi-prism arrangement, the central bright fringe is found to shift through a distance equal to the width of four fringes. Find the refractive index of the material of the bi-prism. Wavelength of light used is  $5.46 \times 10^{-5}$  cm. (1.6424)
- [39] On placing a thin film of mica of thickness  $8 \times 10^{-5}$  cm in the path of one of the interfering beams in a bi-prism arrangement, it is found that the central fringe shifts through a distance equal to the spacing between two successive bright fringes. If  $\lambda = 6 \times 10^{-5}$  cm, find the refractive index of mica. (1.75)
- [40] Newton's rings are observed with a plano-convex lens in contact with a glass plate. The radius of the first bright ring is 1 mm. If the radius of the convex surface is 4 metres, what is the wavelength of light used? (5000 A.U.)
- [41] Newton's rings formed with sodium light between a flat glass plate and a convex lens are viewed normally. What will be the order of the dark ring which will have double the diameter of that of the 40<sup>th</sup> dark ring. (160).
- [42] Newton's rings are formed by light reflected normally from a plano-convex lens and a plane glass plate with a liquid between them. The diameter of the  $n^{\text{th}}$  ring is 2.18 mm and that of the  $(n+10)^{\text{th}}$  ring is 4.51 mm. Calculate the refractive index of the liquid, given that the radius of curvature of the lens is 90 cm and the wavelength of light used is  $5893 \times 10^{-8}$  cm. (1.7)
- [43] In a Newton's rings experiment the diameter of the 15<sup>th</sup> ring was found to be 0.590 cm and that of the 5<sup>th</sup> ring was 0.336 cm. If the



radius of the plano-convex lens is 100 cm, calculate the wavelength of light used. ( $5880 \text{ \AA}$ ).

- [44] Newton's rings are formed by reflection in the air film between a plane surface and a spherical surface. If the diameters of the 3<sup>rd</sup> and 23<sup>rd</sup> bright rings are 0.181 cm and 0.501 cm respectively, what is the radius of curvature of the spherical surface. Wavelength of light used is 5456 A.U. ( $\approx 50 \text{ cm}$ ).
- [45] Newton's rings are formed by light of  $\lambda = 5890 \text{ A.U.}$  between a double convex lens and a glass plate with a liquid between them. The diameters of the 5<sup>th</sup> and 15<sup>th</sup> ring in the reflected system are 2.78 mm and 4.51 mm respectively. If the radius of curvature of the lens is 80 cm, calculate the refractive index of the liquid. (1.5 approx.)
- [46] In a Newton's rings experiment the radius of the 10<sup>th</sup> dark ring obtained by reflected light is found to be 0.6 cm. If the wavelength of light used is  $6000 \text{ \AA}$ , calculate the radius of curvature of the face of the lens in contact with the glass plate. (6 m).
- [47] In a Newton's rings arrangement, if a drop of water ( $\mu = 4/3$ ) be placed in between the lens and the plate, the diameter of the 10<sup>th</sup> ring is found to be 0.6 cm. Obtain the radius of curvature of the face of the lens in contact with the plate. The wavelength of light used is  $6000 \text{ \AA}$ .
- [48] In a Newton's rings experiment the diameter of the 10<sup>th</sup> bright ring changes from 1.40 to 1.27 cm as a liquid is introduced between the lens and the plate. Find the index of refraction of the liquid. (1.215).
- [49] In a Newton's ring arrangement, if air is replaced by a liquid of  $\mu = 1.3$ , in what proportion would the diameters of the ring change?
- [50] If the diameter of a bright ring is 0.65 cm and that of the 10<sup>th</sup> ring beyond is 0.95 cm, calculate the radius of curvature of the face of the convex lens in contact with the glass plate.  $\lambda = 5890 \text{ A.U.}$
- [51] In moving one mirror in a Michelson interferometer through a distance of 0.1474 mm, 500 fringes cross the centre of the field of view. What is the wavelength of light? ( $5896 \text{ \AA}$ ).
- [52] Fringes of equal inclination are observed in Michelson interferometer. As one of the mirror is moved back by 1 mm, 3663 fringes move out from the centre of the pattern. Calculate  $\lambda$ . ( $5460 \text{ \AA}$ ).



- [53] Calculate the distance between the two successive positions of a moveable mirror of a Michelson interferometer giving best fringes in the case of sodium light having lines of wavelength  $5890 \text{ \AA}$  and  $5896 \text{ \AA}$ . (0.2894 mm).
- [54] The Michelson interferometer experiment is performed with a source which consists of two wavelengths  $4882 \text{ \AA}$  and  $4886 \text{ \AA}$ . Through what distance does the mirror have to be moved between two positions of the disappearance of the fringes?
- [55] Light of wavelength  $6000 \text{ \AA}$  falls normally on a thin wedge shaped film of refractive index 1.4, forming fringes that are 2 mm apart. Find the angle of the wedge. [ $1.07 \times 10^{-4} \text{ rad}$ ].
- [56] A glass wedge of angle 0.01 radian is illuminated by monochromatic light of  $6000 \text{ \AA}$  falling normally on it. At what distance from the edge of the wedge, will be the  $10^{\text{th}}$  fringe observed by reflected light? [0.30 cm].

## CHAPTER XXVIII

## DIFFRACTION

*Introduction – Fraunhofer diffraction by a single slit – Fraunhofer diffraction at double slit – Fraunhofer diffraction at N-slits – Diffraction grating – Angular dispersive power of the grating – Solved problems – Exercises.*

## 28.1 Introduction

Let us refer to Fig. 28.1 where a plane wave is incident on a long narrow slit of width  $a$ . According to geometrical optics, only the region PQ of the screen which is of the same dimension as the slit and directly opposite to it, will be illuminated. The rest of the screen will remain absolutely dark and is known as the geometrical shadow. However, on careful observation it will be found that if the width of the slit is not very large compared to the wavelength of light

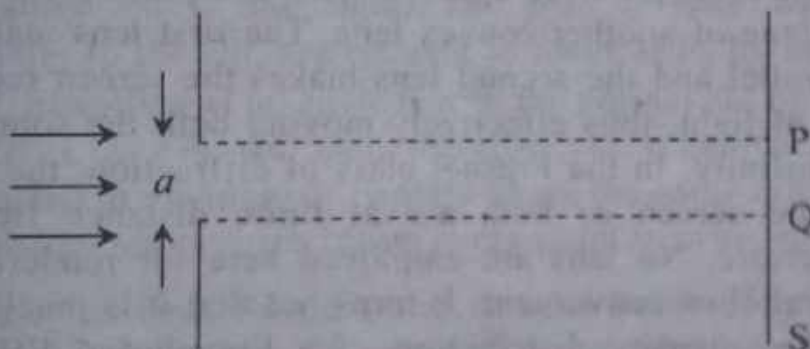


Fig. 28.1

used, some light energy will encroach into the region of geometrical shadow. As the width of the slit is made smaller and smaller, this encroachment of light into the geometrical shadow becomes larger and larger. Not only that the illumination outside the geometrical shadow is not uniform, it shows rhythmic variations in intensity. *These rhythmic variations in intensity and the bending of light around the corners of an obstacle or the encroachment of light into the region of geometrical shadow constitute a class of phenomena known as the diffraction of light.*



Ordinarily as the wavelength of light is extremely short, the amount of bending is extremely small and could only be observed on very careful observations. This proved to be a great obstacle to the early proponents of wave theory, as it was argued, quite naturally, that if light is to be a wave motion, then it should be able to encroach into the region of geometrical shadow *i.e.*, bend round an obstacle. It was only in 1815 that Fresnel, employing Huygens' wave theory of light, successfully demonstrated that light does bend round obstacles. He proved that in the case of ordinary obstacles, which are usually million times greater than the wavelength of light, the amount of bending is almost non-existent with the result that light appears to travel in straight lines – thus appearing to conform corpuscular theory rather than the wave theory of light.

The diffraction phenomena are usually divided into two categories known for historical reasons as (i) *Fraunhofer diffraction* and (ii) *Fresnel diffraction*. In the Fraunhofer class of diffraction, the source of light and the screen are at infinite distances from the diffracting aperture. This is very conveniently achieved by placing the source at the focal plane of a convex lens and placing the screen on the focal plane of another convex lens. The first lens makes the light beam parallel and the second lens makes the screen receive a parallel beam of light, thus effectively moving both the source and the screen to infinity. In the Fresnel class of diffraction, the source of light, or the screen or both are at finite distance from the diffracting aperture. No lens are employed here for rendering the light beam parallel or convergent. It turns out that it is much easier to calculate the intensity distribution of a Fraunhofer diffraction pattern. Further, it is not difficult to observe the Fraunhofer diffraction pattern. An ordinary laboratory spectrometer is all that one needs for observing this pattern; the collimator renders the incident light beam parallel and the telescope receives parallel beam of light on its focal plane. The diffracting aperture is placed on the prism table.

Only diffraction pattern of the Fraunhofer type will be discussed here.

### 28.2. Fraunhofer diffraction by a single slit

Let a beam of parallel monochromatic light be incident normally



on an opaque plate in which there is a slit. *A slit is a rectangular aperture of length large compared to its breadth.* The beam, transmitted through the slit, spreads out perpendicularly to the length of the slit. When this beam is brought to focus on a screen by a lens, a diffraction pattern is obtained. The pattern consists of a central band, much wider than the slit width, situated directly opposite to the slit and bordered by dark and bright bands of decreasing intensity. The central bright band is extremely intense and its width is twice as great as that of fainter side bands.

The origin of the pattern can be understood on the basis of interference of secondary wavelets. According to Huygens' principle, these wavelets can be thought of as sent out by every point of the wavefront at the instant it occupies the plane of the slit. A section of the slit, illuminated with parallel light from left, is represented in Fig. 28.2. Let  $a$  be the width of the slit and  $ds$  be an element of width of the wavefront in the plane of the slit, at a distance  $s$  from the centre  $O$ , which shall be referred to as the origin. Each secondary wavelet can be regarded as a spherical wave spreading out to the right. The part of each wavelet travelling normally to the slit, are brought to focus at  $P_0$  by the lens while the parts, travelling at an angle  $\theta$  with the normal are brought to focus at  $P$ .  $AB$ ,  $CD$ ,  $EF$ , etc. can, therefore, be regarded as parallel rays diffracted at an angle  $\theta$ . Exactly along the same argument it follows that diffracted rays start from every point in every direction.

Let us first consider the wavelet emitted by the element  $ds$  situated at the origin. In spherical waves, the intensity decreases as the inverse square of the distance from the source. This follows directly from the fact that, provided there is no conversion of the energy into other forms, the same amount must pass through any sphere with the source as its centre. Since the area of a sphere increases as the square of its radius, the energy per unit area at a distance  $r$  from the source, or the intensity will vary as  $\frac{1}{r^2}$ . The amplitude must then vary as  $\frac{1}{r}$  and one may write the equation of a spherical wave as

$$y = \frac{a}{r} \sin 2\pi \left( \frac{t}{T} - \frac{x}{\lambda} \right)$$

where  $a$  is the amplitude at unit distance from the source. Thus the wavelet emitted by the element  $ds$  will produce an infinitesimal displacement  $dy$  at  $P_0$ , which may be represented by the equation of a spherical wave in the form

$$dy = \frac{r.ds}{x} \sin 2\pi \left( \frac{t}{T} - \frac{x}{\lambda} \right) \quad (28.1)$$

where  $r.ds$  is the amplitude of the wave at unit distance from the origin.

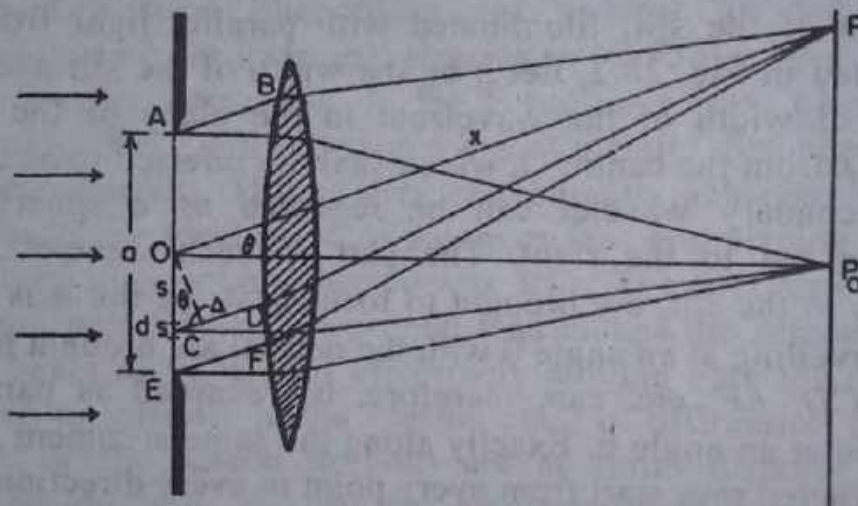


Fig. 28.2

The part of this wavelet reaching  $P$  traverses an optical path  $x_n$  and produces a vibration at  $P$  given by

$$dy_o = \frac{rds}{x} \sin 2\pi \left( \frac{t}{T} - \frac{x_n}{\lambda} \right) \quad (28.2)$$

As the position of  $ds$  is varied, the displacement it produces will vary in phase because of the different path length to  $P$ . When  $ds$  is at a distance  $s$  below the origin, the contribution can be seen from the figure to be given by

$$dy_s = \frac{rds}{x} \sin 2\pi \left( \frac{t}{T} - \frac{x_n + \Delta x}{\lambda} \right)$$



$$= \frac{rds}{x} \sin 2\pi \left( \frac{t}{T} - \frac{x_n}{\lambda} - \frac{s \sin \theta}{\lambda} \right) \quad (28.3)$$

where  $\Delta x = s \sin \theta$  is the additional distance covered.

We shall now sum up the contributions of all elements  $ds$  from one edge of the slit to the other to get the resultant displacement at  $P$ . This sum is obtained by integrating the expression over the width of the slit, *i.e.*, between the limits  $s = -\frac{a}{2}$  and  $s = +\frac{a}{2}$ . During this summing up process the amplitude factor  $r/x$  may be dropped, since it is practically the same for all wavelets and one is interested only in *relative* intensities on the screen.

$$y = \int_{-\frac{a}{2}}^{+\frac{a}{2}} dy_s$$

$$= \int_{-a/2}^{+a/2} \sin 2\pi \left( \frac{t}{T} - \frac{x_n}{\lambda} - \frac{s \sin \theta}{\lambda} \right) ds \quad (28.4)$$

Assuming  $\phi = 2\pi \left( \frac{t}{T} - \frac{x_n}{\lambda} \right)$  and  $\psi = 2\pi s \frac{\sin \theta}{\lambda}$ , we have

$$y = \int_{-a/2}^{+a/2} \sin (\phi - \psi) ds$$

$$= \int_{-a/2}^{+a/2} (\sin \phi \cos \psi - \cos \phi \sin \psi) ds$$

$$= \int_{-a/2}^{+a/2} \left[ \sin \phi \cos \frac{2\pi \sin \theta}{\lambda} \cdot s - \cos \phi \sin \frac{2\pi \sin \theta}{\lambda} s \right] ds$$

$$\text{now } \int \sin mx \, dx = -\cos \frac{mx}{m} \text{ and } \int \cos mx \, dx = \frac{\sin mx}{m}$$

Hence



$$y = \left[ \sin \phi \frac{\frac{2\pi \sin \theta}{\lambda} s}{2\pi \sin \theta} + \cos \phi \frac{\frac{\cos 2\pi \sin \theta}{\lambda} s}{2\pi \sin \theta} \right]_{-a/2}^{+a/2}$$

$$= \left[ \frac{s \sin \psi \sin \phi}{\psi} + \frac{s \cos \psi \cos \phi}{\psi} \right]_{-a/2}^{+a/2}$$

$$\text{when } s = +a/2 \quad \psi = \frac{\pi a \sin \theta}{\lambda}$$

$$s = -a/2 \quad \psi = -\frac{\pi a \sin \theta}{\lambda}$$

$$\therefore y = \left[ \frac{a}{2} \frac{\sin (\pi a \sin \theta) / \lambda}{(\pi a \sin \theta) / \lambda} \sin \phi + \frac{a}{2} \frac{\sin (\pi a \sin \theta) / \lambda}{(\pi a \sin \theta) / \lambda} \sin \phi \right. \\ \left. + \frac{a}{2} \frac{\cos (\pi a \sin \theta) / \lambda}{(\pi a \sin \theta) / \lambda} \cos \phi - \frac{a}{2} \frac{\cos (\pi a \sin \theta) / \lambda}{(\pi a \sin \theta) / \lambda} \cos \phi \right] \\ = a \frac{\sin (\pi a \sin \theta) / \lambda}{(\pi a \sin \theta) / \lambda} \sin \phi \\ = a \frac{\sin (\pi a \sin \theta) / \lambda}{(\pi a \sin \theta) / \lambda} \sin 2\pi \left( \frac{t}{T} - \frac{x_n}{\lambda} \right) \quad (28.5)$$

The above equation represents the resultant vibration at the point  $P$ . This represents a new simple periodic motion which has the same frequency as the source with a phase angle the same as the central ray (ray coming from the element  $ds$  located at the centre) and of amplitude

$$R = (a \sin \beta) / \beta \quad \text{where } \beta = (\pi / \lambda) a \sin \theta$$

Due to the neglect of the factor  $r/x$  during the summing up process, this expression for  $R$  differs by some arbitrary constant from the true amplitude. It will, therefore, be more correct to write

$$R = R_0 \frac{\sin \beta}{\beta} \quad (28.6)$$

The significance of the constant  $R_0$  will appear below. The quantity  $\beta$  is a convenient variable, and can be expressed as

$$\beta = \frac{1}{2} \left( \frac{2\pi}{\lambda} \cdot a \sin \theta \right) \quad (28.7)$$

Now  $a \sin \theta$  is the path difference of two rays originating from opposite edges of the slit. Hence  $\beta$  signifies one-half the phase difference in radians between the contributions from opposite edges of the slit. It determines the intensity  $I$  by the relation,

$$I = R^2 = R_0^2 \frac{\sin^2 \beta}{\beta^2} \quad (28.8)$$

### Position of principal maxima

On writing the expansion of  $\sin \beta$  in ascending powers of  $\beta$ , the expression for the resultant amplitude  $R$  becomes

$$\begin{aligned} R &= \frac{R_0}{\beta} \left( \beta - \frac{\beta^3}{3!} + \frac{\beta^5}{5!} - \frac{\beta^7}{7!} + \dots \right) \\ &= R_0 \left( 1 - \frac{\beta^2}{3!} + \frac{\beta^4}{5!} - \frac{\beta^6}{7!} + \dots \right) \end{aligned}$$

It is obvious that  $R$  will be maximum, if the negative terms vanish. This is only possible when

$$\beta = \frac{\pi a \sin \theta}{\lambda} = 0; \quad \text{or, } \theta = 0.$$

For  $\beta = 0$ , the quotient  $\sin \beta / \beta$  becomes indeterminate but it should be remembered that  $\sin \theta$  approaches  $\theta$  for small angles and is equal to it when  $\theta$  vanishes. Hence for  $\beta = 0$  (i.e.,  $\theta = 0$ ),  $R = R_0$ . Thus the maximum value of  $R$  is  $R_0$ ; it represents the amplitude when all the wavelets arrive in phase. The condition  $\theta = 0$  simply means that this maximum is formed by parts of secondary wavelets which travel normally to the slit.  $R_0^2$  is then the value of the maximum intensity which is designated by  $I_0$ . The position of this maximum is, therefore, directly opposite to the slit and is at the



centre of the pattern. This maximum is known as the *principal maximum* and as explained below, it is bordered symmetrically by the dark and the bright bands.

### Position of minimum intensity

The intensity will be zero in the diffraction pattern at the positions where  $\sin \beta = 0$ . The values of  $\beta$  which satisfies the above condition are  $\beta = m \pi$  where  $m = \pm 1, \pm 2, \pm 3, \dots$  which gives on substituting the value of  $\beta$

$$\frac{a \pi \sin \theta}{\lambda} = \pm m \pi$$

$$\text{or, } a \sin \theta = \pm m \lambda.$$

where  $m = 1, 2, 3,$

The value  $m = 0$  is not admissible, because the value  $\theta = 0$  corresponds to the principal maximum.

The first minimum occurs at

$$\theta = \pm \sin^{-1} \left( \frac{\lambda}{a} \right), \text{ the second minimum at}$$

$$\theta = \pm \sin^{-1} \left( \frac{2\lambda}{a} \right), \text{ etc. In words, the rays diffracted at an angle } \theta$$

interfere destructively if the path difference between the extreme diffracted rays is an integral multiple of  $\lambda$ . Since  $\sin \theta$  cannot exceed unity, the maximum value of  $m$  is the integer which is less than (and closest to) —

### Position of secondary maxima

In addition to the principal maxima at  $\theta = 0$ , there are weak secondary maxima between the equally spaced minima. The exact positions of these maxima can be obtained by differentiating the expression for intensity  $I = (I_0 \sin^2 \beta) / \beta^2$  with respect to  $\beta$  and equating it to zero. Therefore, for maxima and minima, we have the condition,



$$\frac{dI}{d\beta} = 2I_0 \left[ \frac{\sin \beta \cos \beta}{\beta^2} - \frac{\sin^2 \beta}{\beta^3} \right] = 0$$

$$\text{or, } \sin \beta [\beta - \tan \beta] = 0$$

Hence, either

$$\sin \beta = 0 \quad \text{or, } \beta - \tan \beta = 0.$$

Now  $\sin \beta = 0$  gives the values of  $\beta$  (except zero) for which the intensity is zero (positions of minimum intensity). The conditions of maxima are, therefore, given by the roots of the equation

$$\beta - \tan \beta = 0; \quad \text{or, } \beta = \tan \beta.$$

The above equation is a *transcendental* equation. The values of  $\beta$  satisfying this equation are given by the points of intersection of the curves

$$y = \beta \quad \text{and} \quad y = \tan \beta, \text{ plotted on the same graph.}$$

The curve  $y = \beta$  is simply a straight line through the origin and inclined at  $45^\circ$  to the axis of  $\beta$ . To plot the curve  $y = \tan \beta$ , the following characteristics of the curve were derived mathematically.

(i) The curve cuts the axis of  $\beta$  at points where  $y = \tan \beta = 0$ , that is, at points given by  $\beta = 0, \pm \pi, \pm 2\pi, \pm 3\pi$ , etc.

(ii) At points where it cuts the axis of  $\beta$ , the curve is inclined at  $45^\circ$  to the axis of  $\beta$  at those points.

(iii) Asymptotes of the curve are parallel straight lines defined by  $\beta = \pm \pi/2, \pm 3\pi/2, \pm 5\pi/2$ , etc.

The equation  $y = \tan \beta$ , therefore, represents a family of curves having the characteristics mentioned above. The curves are plotted in Fig. 28.3 along with the curve  $y = \beta$ . As can be seen from the

figure, the intersections occur at  $\beta = 0, \pm 1.43\pi (\simeq \frac{3\pi}{2}), \pm 2.46\pi (\simeq \frac{5\pi}{2}), \pm 3.47\pi (\simeq \frac{7\pi}{2}), \pm 4.48\pi (\simeq \frac{9\pi}{2})$ , etc. The value  $\beta = 0$ , of course, gives the position of principal maximum. The other values namely  $\beta = 1.43\pi, 2.46\pi, 3.47\pi, 4.48\pi$ , etc. give the respective

positions of the first, the second, the third, the fourth secondary maxima, etc. As can be seen, they are displaced somewhat towards the middle of two consecutive minima and this displacement increases with the order of the secondary maxima.

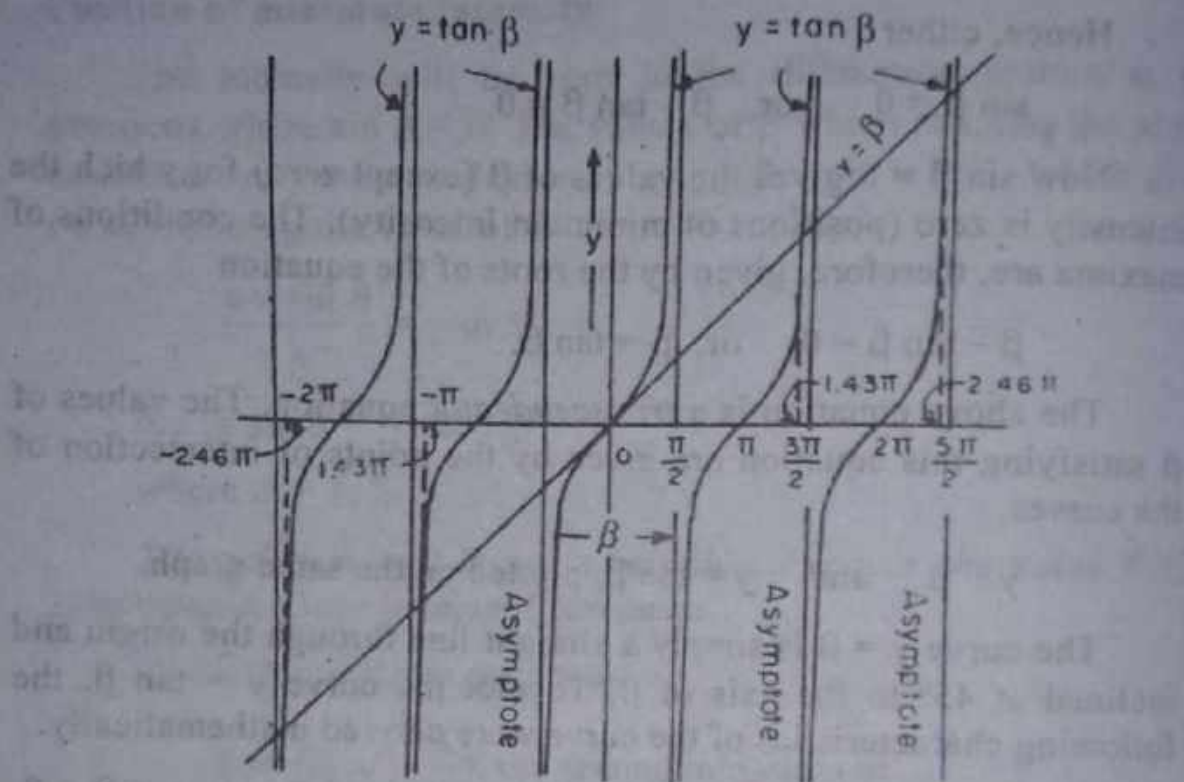


Fig. 28.3

We can now form an idea regarding the relative intensities of the maxima. For simplicity of argument let  $R_0 = 1$ . For the principal maximum,  $\beta = 0$  and  $I_0 = 1$ . For the first secondary maximum  $\beta = 1.43\pi$ . Therefore,

$$I_1 = R_0^2 \left[ \frac{\sin 1.43\pi}{1.43\pi} \right]^2 = 0.0496.$$

The intensity of the first secondary maximum is about 4.96% of the principal maximum. The relative intensities of the other secondary maximum can be similarly worked out. The exact value of  $\beta$  of the principal maximum and the first six secondary maxima are given below together with their relative intensities, assuming the intensity of the principal maximum to be unity.

Order of maxima	$\beta$	$I$
0	0	1
1	$1.430\pi$	0.0496
2	$2.459\pi$	0.0168
3	$3.471\pi$	0.0083
4	$4.477\pi$	0.0050
5	$5.482\pi$	0.0034
6	$6.484\pi$	0.0024

The intensity distribution curve for the diffraction pattern of Fraunhofer diffraction by single slit can now be plotted and is shown in Fig. 27.4. The principal maximum occurs at  $\beta = 0$  and from this the intensity diminishes to 0 at  $\pm \pi$ , then passes through several secondary maxima at  $\beta = 1.43 \pi$ ,  $2.46 \pi$ , etc., with evenly spaced points of zero intensity at  $\beta = \pm \pi$ ,  $\pm 2\pi$ ,  $\pm 3\pi$ , etc.

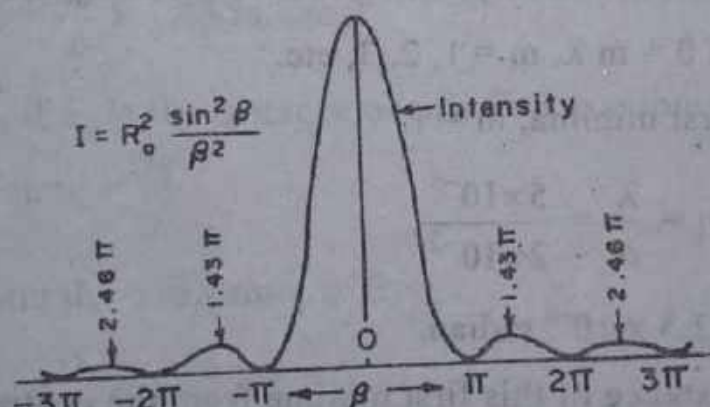


Fig. 28.4

It should be pointed out that the spread of the principal maximum is inversely proportional to the width of the slit. Moreover, if the width of the slit is equal to the wavelength of light used i.e.,  $a = \lambda$ , then for  $m = 1$ , equation  $a \sin \theta = \pm m \lambda$  gives  $\theta = \pm \pi/2$ . This means the first minimum occurs at  $90^\circ$  with the normal. Under this condition, there is no maxima and minima on the screen. The light after traversing the slit spreads out in all directions, with an



intensity which decreases steadily as  $\theta$  increases. This is in agreement with Huygens' principle of secondary waves, for the slit has effectively now become the source of secondary waves.

Finally, if  $a < \lambda$ , then from the original condition  $a \sin \theta = m \lambda$ , we get  $\sin \theta > 1$  for the first minimum ( $m = 1$ ). This is impossible. Therefore, the expression for intensity (eqn. 28.4) does not hold when the width of the slit becomes less than one wavelength. The treatment breaks down.

**Example 28.1.** A parallel beam of light is incident normally on a narrow slit of width 0.2 mm. The Fraunhofer diffraction pattern is observed on a screen, placed at the focal plane of a convex lens of focal length 20 cm. Calculate the distance between the first two minima and the first two maxima on the screen. Assume  $\lambda = 5 \times 10^{-5}$  cm and that the lens is placed very close to the slit.

**Soln.**

In the Fraunhofer diffraction pattern due to a single slit, the angles of diffraction corresponding to zero intensity are given by

$$a \sin \theta = m \lambda, \quad m = 1, 2, 3, \text{ etc.}$$

For the first minima,  $m = 1$

$$\text{or, } \sin \theta_1 = \frac{\lambda}{a} = \frac{5 \times 10^{-5}}{2 \times 10^{-2}}$$

$$\text{or, } \theta_1 = 2.5 \times 10^{-3} \text{ radian.}$$

Let the distance of this first minima from the central maxima be  $x_1$ ; since the lens is very close to the slit, then, very approximately,

$$\sin \theta_1 = \frac{x_1}{f} \quad \text{or, } \theta_1 = \frac{x_1}{f} \text{ radians.}$$

Equating the two values of  $\theta_1$ , we get

$$\frac{\lambda}{a} = \frac{x_1}{f}; \quad \text{or, } x_1 = \frac{\lambda}{a} \cdot f$$

$$= 2.5 \times 10^{-3} \times 20 \text{ cm.}$$

Similarly, if  $\theta_2$  and  $x_2$  are the angles of diffraction for the second minima ( $m = 2$ ) and the distance of the second minima from the central maxima respectively, we have

$$\theta_2 = \frac{2\lambda}{a} = \frac{x_2}{f}; \quad \text{or, } x_2 = 2 \cdot \frac{\lambda}{a} \cdot f$$

$$= 2 \times 2.5 \times 10^{-3} \times 20 \text{ cm.}$$

Hence the distance of separation between the first two minima

$$= (5.0 - 2.5) \times 10^{-3} \times 20 \text{ cm} = \mathbf{0.05 \text{ cm.}}$$

The first maxima occur at  $\beta = 1.43 \pi (\simeq \frac{3}{2} \pi)$

$$\text{where } \beta = \frac{\pi a \sin \theta}{\lambda}$$

$$\text{or, } a \sin \theta_1 = 1.43 \lambda (\simeq \frac{3\lambda}{2})$$

$$\text{or, } \theta_1 = \frac{\lambda}{a} \times 1.43 \text{ radians.}$$

Again, if  $x_1$  is the distance of this first maxima from the central maxima, then  $\theta_1 = \frac{x_1}{f}$

Equating the two values of  $\theta_1$

$$x_1 = \frac{\lambda}{a} \times 1.43 \times f.$$

Similarly, the distance  $x_2$  of the second maxima from the central maxima is given by

$$x_2 = \frac{\lambda}{a} \times 2.46 \times f.$$

Hence, the distance of separation between the first two maxima

$$= \frac{\lambda}{a} \times (2.46 - 1.43) \times f$$

$$= 2.5 \times 10^{-3} \times (2.46 - 1.43) \times 20$$

$$= 0.05 \text{ cm.}$$

**Example 28.2.** A rectangular aperture of width 0.0025 cm is illuminated with parallel beam of monochromatic light of wavelength 500 mμ. A converging lens of focal length 50 cm placed close to the aperture collects the light passing through it producing a diffraction pattern on a screen in its focal plane. Calculate the width of the central maxima on the screen.

**Soln.**

Let  $x$  be the distance of the first minima from the central maxima.

Then

$$x = \frac{\lambda}{a} f \text{ (see example 28.1)}$$

where  $\lambda = 500 \text{ m}\mu = 5 \times 10^{-5} \text{ cm}$ . ( $1 \text{ m}\mu = 1 \times 10^{-3} \times 10^{-4} \text{ cm}$ )

$a = \text{width of the slit} = 25 \times 10^{-4} \text{ cm}$ .

$f = 50 \text{ cm}$ .

$$\therefore x = \frac{50 \times 5 \times 10^{-5}}{25 \times 10^{-4}} = 1 \text{ cm}$$

Now  $x$  is also the half linear width of the central maxima. Hence the linear width of the central maxima

$$= 2 \times 1 = 2 \text{ cm.}$$

**Example 28.3.** A lens whose focal length is 40 cm forms a Fraunhofer diffraction pattern of slit 0.3 mμ wide. Calculate the distance of the first dark band and of the next bright band from the axis. (wavelength of light used is 5890 Å)

**Soln.**

The angles of diffraction corresponding to zero intensity in the Fraunhofer diffraction pattern due to single slit is given by

$$a \sin \theta = m\lambda \text{ where } m = 1, 2, 3, \text{ etc.}$$



For the first dark band  $m = 1$ ,

$$\text{or, } \sin \theta = \frac{\lambda}{a} = \frac{5890 \times 10^{-8}}{0.3 \times 10^{-1}}$$

$$\text{or, } \theta = \frac{1}{3} \times 589 \times 10^{-5} \text{ radian (for small angles } \sin \theta \approx \theta).$$

If the distance of the first dark band from the axis be  $x$  cm, then the angle of diffraction  $\theta$ , is also expressed by the relation.

$$\theta = x/f = x/40 \text{ radian.}$$

Equating the two values of  $\theta$ ,

$$x = \frac{1}{3} \times 40 \times 589 \times 10^{-5} \text{ cm} = 0.785 \text{ mm.}$$

The angle of diffraction corresponding to first bright band on either side of the central band is approximately given by

$$a \sin \theta' = (3/2)\lambda$$

$$\text{or, } \theta' = 3\lambda/2a$$

Hence, the distance  $x'$  of the first bright band from the axis is

$$\begin{aligned} x' &= \theta' \times f = (3 \times 5890 \times 10^{-8} \times 40) / 2 \times 0.3 \times 10^{-1} \\ &= 1.178 \text{ mm.} \end{aligned}$$

**Example 28.4.** Find the half angular width of the central bright maximum in the Fraunhofer diffraction pattern due to a slit of width  $12 \times 10^{-4}$  mm when the slit is illuminated by monochromatic light of wavelength  $6 \times 10^{-5}$  cm.

**Soln.**

In the relation  $a \sin \theta = \lambda$ ,  $\theta$  is the half angular width of the central maximum.

$$\text{Hence, } \sin \theta = \frac{\lambda}{a} = \frac{6 \times 10^{-5}}{12 \times 10^{-5}} = 0.50$$

$$\text{or, } \theta = 30^\circ$$

**Example 28.5.** In a Fraunhofer diffraction due to a narrow slit of width 0.02 cm, the screen is placed 2 metre away from the lens used to obtain the pattern. If the first minima lie 5 mm on either side of the central maxima, find the wavelength of light used.

**Soln.**

The angles of diffraction corresponding to zero intensity are given by

$$a \sin \theta = n\lambda$$

$$\text{Here } n = 1, \text{ hence } \sin \theta = \frac{\lambda}{a}$$

Again, if the screen is far away from the lens,

$$\sin \theta = \frac{x}{f} \text{ where } f \text{ is the focal length of the lens or the distance}$$

between the screen and the lens.  $x$  is the distance of the first minima from the axis.

$$\text{Then, } \frac{x}{f} = \frac{\lambda}{a}$$

$$\text{or, } \lambda = \frac{ax}{f} = \frac{0.02 \times 0.5}{200} \text{ cm}$$

$$= 5 \times 10^{-5} \text{ cm}$$

$$= 5000 \text{ \AA}.$$

### 28.3 Fraunhofer diffraction at double slit

We shall now study the Fraunhofer diffraction pattern produced by two parallel slits. The arrangement is similar to that of Young's double slit experiment which helped to demonstrate the wave nature of light. In that experiment it was assumed that the width of the slit was comparable with the wavelength of light and that they were close together. The assumption was essential so that the central maximum in the diffraction patterns of each slit was wide enough to occupy the same large area of the screen.



It was then possible to consider the resultant displacement at any point of the screen, as simply due to superposition of two interfering beams of light. However, when the width of each slit is increased to such an extent that it becomes comparable with the width of the opaque space separating them, diffraction of light at each slit begins to play its part in modifying the interference pattern due to two slits. What is now obtained on the screen is a pattern characterized by a number of equally spaced interference maxima and minima in the region normally occupied by the central maximum in the single slit diffraction. This pattern is called the *Fraunhofer diffraction* at a double slit. We shall show that the central interference maximum of this pattern has an intensity four times greater than that of the central maximum in the single slit diffraction and is bordered by other interference maxima of gradually decreasing intensity. In the region of the secondary maxima (of single slit diffraction), faint secondary maxima are now observed.

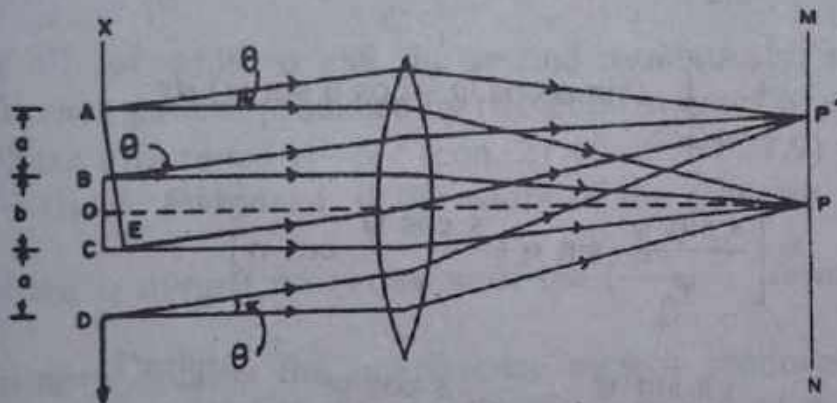


Fig. 28.5

In Fig. 28.5  $AB$  and  $CD$  are two rectangular slits parallel to one another and perpendicular to the plane of the paper. Let  $a$  be the width of each slit and  $b$  be the width of the opaque portion separating them.  $P$  is a point on the screen  $MN$  such that  $OP$  is perpendicular to the screen. When a plane wavefront is incident on the surface of the slits, the transmitted rays are brought to focus at different points on the screen by the convex lens  $L$ . All secondary waves travelling in a direction parallel to  $OP$  will be brought to focus at  $P$ , which, therefore, corresponds to the position of the central bright maximum.



We shall now derive an equation for the intensity in a double slit diffraction. To do that, we shall follow the same procedure as that followed in the single slit diffraction. It will merely be necessary to change the limits of integration in eqn. (28.4) to include the two portions of the wavelength transmitted by the double slit. Thus, if the origin is chosen as the centre of one of the slits, then the integration is to extend from  $s = -(a/2)$  to  $+(a/2)$  and from

$(d - \frac{a}{2})$  to  $(d + \frac{a}{2})$ , where  $d = a + b$ . We therefore, have

$$y = \int_{-a/2}^{+a/2} \sin(\phi - \psi) ds + \int_{d-a/2}^{d+a/2} \sin(\phi - \psi) ds \quad (28.9)$$

$$\text{where } \phi = 2\pi \left( \frac{t}{T} - \frac{x}{\lambda} \right) \quad \text{and } \psi = 2\pi s \frac{\sin \theta}{\lambda}$$

$$\begin{aligned} y &= \int_{-a/2}^{+a/2} (\sin \phi \cos \psi - \cos \phi \sin \psi) ds \\ &+ \int_{d-a/2}^{d+a/2} (\sin \phi \cos \psi - \cos \phi \sin \psi) ds \\ &= \left[ \frac{s \sin \psi}{\psi} \sin \phi + \frac{s \cos \psi}{\psi} \cos \phi \right]_{-a/2}^{+a/2} \\ &+ \left[ \frac{s \sin \psi}{\psi} \sin \phi + \frac{s \cos \psi}{\psi} \cos \phi \right]_{d-a/2}^{d+a/2} \end{aligned}$$

Substituting in the limits and combining the terms, we obtain

$$y = 2a \frac{\sin(\pi a \sin \theta) / \lambda}{(\pi a \sin \theta) / \lambda} \cos(\pi d \sin \theta) / \lambda$$

$$\times \sin 2\pi \left( \frac{t}{T} - \frac{x}{\lambda} - \frac{d \sin \theta}{2\lambda} \right)$$

$$= 2a \frac{\sin \beta}{\beta} \cos \gamma \sin 2\pi \left( \frac{t}{T} - \frac{x}{\lambda} - \frac{d \sin \theta}{2\lambda} \right)$$

where, as before,

$$\beta = \pi a \sin \theta / \lambda$$

and where  $\gamma = \frac{\pi d \sin \theta}{\lambda}$

Replacing  $a$  by  $R_0$  as before,

$$y = 2R_0 \frac{\sin \beta}{\beta} \cos \gamma \sin 2\pi \left( \frac{t}{T} - \frac{x}{\lambda} - \frac{d \sin \theta}{2\lambda} \right) \quad (28.10)$$

The intensity is proportional to the square of the amplitude in the above equation. Hence,

$$I = 4R_0^2 \frac{\sin^2 \beta}{\beta^2} \cos^2 \gamma \quad (28.11)$$

As can be seen, the intensity is the product of two terms: the first term  $\left( \frac{\sin^2 \beta}{\beta^2} \right)$  represents the diffraction pattern produced by a single slit of width  $a$  and the second term  $(\cos^2 \gamma)$  represents the interference pattern produced by two point sources of equal intensity and phase difference  $\gamma = \delta/2$  (eqn. 27.10 of Art 27.9) and separated by a distance  $d$ . Indeed, if the width of the slits are very small so that there is almost no variation of the  $\left( \frac{\sin^2 \beta}{\beta^2} \right)$  term with  $\theta$ , then one simply obtains the interference pattern produced in Young's double-slit experiment. *Thus each slit produces a diffracted beam in which the intensity distribution depends upon the slit-width; these diffracted beams then interfere with each other to produce the final diffraction pattern.*

### Positions of maxima and minima

Eqn. (28.11) tells us that the resultant intensity will be zero when either of the two terms is zero. For the first term this will occur when

$$\beta = \pi, 2\pi, 3\pi, \dots$$



and for the second term, when

$$\gamma = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2},$$

Since by definition  $\gamma = (\pi/\lambda) d \sin \theta$ , and

$$\beta = (\pi/\lambda) a \sin \theta,$$

the corresponding angles of diffraction are given by the following relations

$$a \sin \theta = \lambda, 2\lambda, 3\lambda, \dots = m\lambda \quad \text{minima}$$

where  $m = 1, 2, 3, \dots$

$$\text{and } d \sin \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2},$$

$$= (n + \frac{1}{2}) \lambda \quad \text{minima}$$

where  $n = 0, 1, 2, 3$ , etc. This equation simply expresses the condition that for minima, the path difference between the parallel rays diffracted from any pair of corresponding points in the two slits should be odd multiple of  $\lambda/2$  on reaching the focal plane of the focussing lens.

The exact positions of the maxima are not given by any simple relation. However, when the slits are very narrow and when the maxima near the centre of the pattern are considered, it may be justifiably assumed that the approximate positions of the maxima

may be found by neglecting the variation of the factor  $\frac{\sin^2 \beta}{\beta^2}$ . The positions of the maxima will then be determined solely by the  $\cos^2 \gamma$  factor. The interference maxima then occur

$$\text{when } \gamma = 0, \pi, 2\pi$$

$$\text{or, when } d \sin \theta = 0, \lambda, 2\lambda, 3\lambda, \dots n\lambda \quad \text{maxima}$$

The whole number  $n$  represents physically the number of wavelengths in the path difference from corresponding points in the two slits and represents the order of interference. *Points separated by a distance  $(a + b)$  in the two slits are known as corresponding points.*



### Missing orders in double-slit diffraction pattern

If the separation between the slits  $b$  is varied, keeping the width of the individual slits  $a$  constant, the scale of the interference pattern varies, but that of the diffraction pattern remains the same. A study of the resultant diffraction pattern reveals that certain orders of interference maxima are missing.

Let us suppose that for certain values of  $\theta$ , the following relations simultaneously hold

$$a \sin \theta = \pm m \lambda$$

$$(a + b) \sin \theta = d \sin \theta = \pm n \lambda$$

But the first relation expresses the condition of zero intensity in the diffraction pattern while the second equation expresses the condition of maximum intensity in the interference pattern. Thus for certain values of  $\theta$  the positions of certain interference maxima correspond to the diffraction minima at the same position on the screen and consequently will be missing (or at least reduced to two maxima of very low intensity).

From the above equations we have

$$\frac{d}{a} = \frac{n}{m}$$

Since  $n$  and  $m$  are both integers,  $d/a$  must be in the ratio of two integers in order to have missing orders. This ratio determines the orders which are missing.

For example,

- (i) Let  $d/a = 2 = n/m$ . But  $d = a + b$ . This means  $b = a$ . Hence  $n = 2m$ . Giving  $m$  integral values, we get the corresponding values of missing orders  $n$ . If  $m = 1, 2, 3, 4$ , etc.  $n = 2, 4, 6, 8$ , etc. Thus the orders 2, 4, 6, of the interference maxima will be missing in the diffraction pattern and there will be three interference maxima (the zero order and the two first order maxima) within the central diffraction maximum (Fig. 27.6)

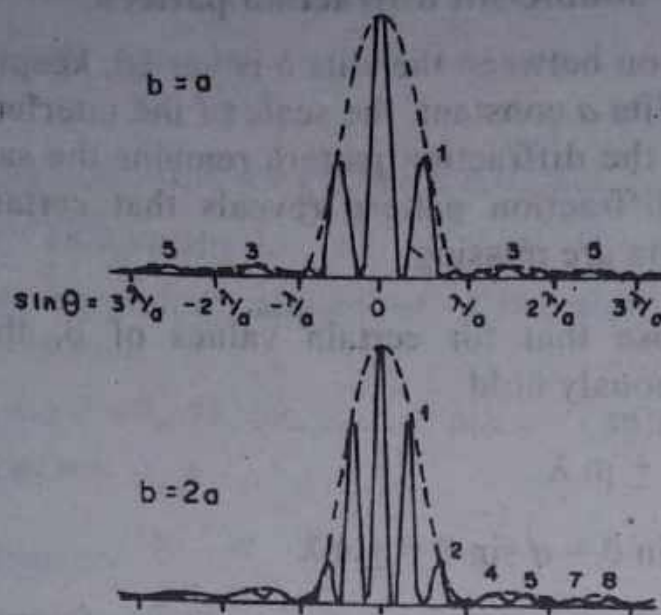


Fig. 28.6

(ii) Let  $d/a = 3$ , that is,  $b = 2a$ , then  $n = 3m$ .

Therefore when  $m = 1, 2, 3, 4$  etc.,

the missing orders are  $n = 3, 6, 9, 12$  etc.

Thus there will now be five interference maxima (zero order, two first order and two second order maxima) within the central diffraction maximum.

In this way as the ratio  $d/a$  increases, the number of interference maximum within the central diffraction maximum also increases. When  $d/a = 1$  the two slits exactly join and all orders should be missing. However, the diffraction patterns then observed on the screen is similar to that due to a single slit of width equal to  $2a$ .

The physical picture of the cause of missing orders is as follows. Considering for example the missing order  $n = 3$ , the point on the screen, where this order is expected is just three wavelengths further from the centre of one slit than from the centre of the other. Hence the waves from the two slits might be expected to arrive in phase and produce a maximum. But since  $AB = BC$ ; this point is at the same time one wavelength further from one edge than the other of the same slit. Consequently the addition of the secondary wavelengths from this slit produces zero intensity at this point. The



same holds for the second slit, so that although we may add the contributions from the two slits, both contributions are zero and must therefore give zero resultant intensity at this point. This means the second order maximum will be absent.

### Intensity curve

We can now plot the intensity curve for the double slit diffraction pattern. The upper curve in Fig. 28.7 is a plot of  $\cos^2 \gamma$  against  $\gamma$  and the

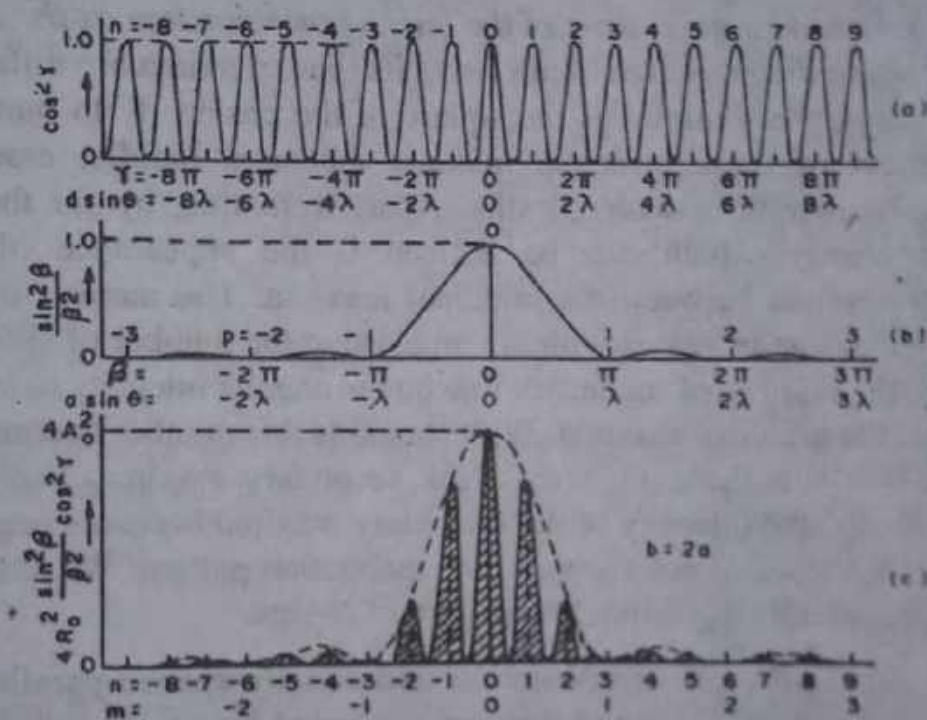


Fig. 28.7

values of the order, of the half-phase difference  $\gamma = \frac{\delta}{2}$  and of the path difference are indicated for the various maxima. All these maxima are of equal intensity and equidistant on a scale of  $d \sin \theta$  or practically on a scale of  $\theta$ , since when  $\theta$  is small  $\sin \theta \approx \theta$  and the maxima occur at angles  $\theta = 0, \lambda/d, 2\lambda/d, \dots$ . The variation of the factor  $\frac{\sin^2 \beta}{\beta^2}$  must be taken into account in the diffraction produced

by a single slit of finite width  $a$ . The single slit diffraction pattern is given by the curve in the middle. The complete double slit pattern as



given by eqn. (28.11) is the product of these two factors. This is obtained by multiplying the ordinates of the upper curve by those of the middle curve and the constant  $4R_0^2$ . The resultant pattern is shown in the lowest curve of the figure.

#### 28.4 Fraunhofer diffraction at N-slits

We shall next consider the diffraction produced by  $N$  parallel slits. The diffraction pattern suffers a modification as the number of slits is increased beyond two. *The most striking modification in the pattern consists of a gradual narrowing of the interference maxima as the number of slits is gradually increased.* With two slits these maxima are diffuse, the intensity varying essentially as the square of the cosine. With more slits, the sharpness of these principal maxima increases rapidly, essentially becoming narrow lines with 20 slits. Apart from this, by far the most important change which can be noticed is the appearance of weak secondary maxima between the principal maxima. The number of these secondary maxima increases with the increase in the number of slits. With three slits, the number of secondary maxima is one, its intensity being 11.1 per cent of the principal maxima. With four slits this number becomes two and with five slits there are three weak secondary maxima. With more number of slits, the intensity of the secondary maxima becomes negligibly small so that these are not visible in the diffraction pattern. We shall now discuss theoretically the diffraction pattern of  $N$ -slits.

Let Fig. 28.8 represent  $N$  similar and equally spaced parallel slits. The width of each slit is  $a$  and they are separated by an opaque space of width  $b$ . The distance between the centres of two consecutive slits is  $d = a + b$ . The method used in Arts. 28.2 and 28.3 to derive the intensity function for the single and double slit respectively, may be extended here by increasing the limits of integration to cover more than two slits. For  $N$  number of slits the integral now becomes a series of  $N$  terms, each of which must be integrated within limits determined by the dimension and the position of the corresponding slits. The integral then becomes

$$y = \int \sin(\theta - \psi) ds$$

$$= \left[ \frac{s \sin \psi}{\psi} \sin \theta + \frac{s \cos \psi}{\psi} \cos \theta \right]_{-a/2}^{+a/2}$$

$$\begin{aligned}
& + \left[ \frac{s \sin \psi}{\psi} \sin \varnothing + \frac{s \cos \psi}{\psi} \cos \varnothing \right]_{d-a/2}^{d+a/2} \\
& + \left[ \frac{s \sin \psi}{\psi} \sin \varnothing + \frac{s \cos \psi}{\psi} \cos \varnothing \right]_{2d-a/2}^{2d+a/2} \\
& + \left[ \frac{s \sin \psi}{\psi} \sin \varnothing + \frac{s \cos \psi}{\psi} \cos \varnothing \right]_{3d-a/2}^{3d+a/2} \\
& + \dots + \left[ \frac{s \sin \psi}{\psi} \sin \varnothing + \frac{s \cos \psi}{\psi} \cos \varnothing \right]_{(N-1)d-a/2}^{(N-1)d+a/2} \quad (28.12)
\end{aligned}$$

where  $\varnothing = 2\pi \left( \frac{t}{T} - \frac{x}{\lambda} \right)$  and  $\psi = \frac{2\pi s \sin \theta}{\lambda}$

Each term in the above equation represents the contribution of the  $k^{\text{th}}$  slit where  $k = 0, 1, 2, 3, \dots, N-1$ , and has the form

$$a \frac{\sin \beta}{\beta} [\sin (\varnothing - k \delta)]$$

where  $\beta = \frac{\pi a \sin \theta}{\lambda}$  and  $\delta = \frac{2\pi d \sin \theta}{\lambda}$

The sum of the series is obtained from the trigonometric formula

$$\begin{aligned}
& \sum_{k=0}^{N-1} \sin (\varnothing - k \delta) \\
& = \frac{\sin \frac{N \delta}{2}}{\sin (\delta/2)} \sin \left( \varnothing - \frac{N-1}{2} \delta \right) \quad (28.13)
\end{aligned}$$

Substituting in the limits, each term of eqn. (28.13) will be found to contain the common factor  $\frac{a \sin \beta}{\beta}$ , and the remaining factors can be grouped in a series whose sum, as mentioned above, is



$$\frac{\sin N\gamma}{\sin \gamma} \sin 2\pi \left( \frac{t}{T} - \frac{x}{\lambda} - \frac{N-1}{2} \frac{d \sin \theta}{\lambda} \right) \quad (28.14)$$

The expression for  $y$ , therefore, becomes

$$y = a \frac{\sin \beta}{\beta} \frac{\sin N\gamma}{\sin \gamma} \sin 2\pi \left( \frac{t}{T} - \frac{x}{\lambda} - \frac{N-1}{2} \frac{d \sin \theta}{\lambda} \right) \quad (28.15)$$

In finding the intensity we are interested only in the resultant amplitude of eqn. (28.15), the square of which gives the intensity. The expression for intensity therefore, becomes

$$I = R_o^2 \frac{\sin^2 \beta}{\beta^2} \cdot \frac{\sin^2 N\gamma}{\sin^2 \gamma} \quad (28.16)$$

$\beta$  and  $\gamma$  have the same physical significance as in Arts. 28.2 and 24.3;  $\beta$  is one-half the phase difference from opposite edges of any one slit, and  $\gamma = \delta/2$  is one-half the phase difference from corresponding points in any two adjacent slits.

The expression for intensity as obtained above (eqn. 28.16) should hold for any number of slits. If  $N = 1$ ,  $I = R_o^2 (\sin^2 \beta) / \beta^2$  which is the same as that obtained for a single slit (eqn. 28.8).

When  $N = 2$ , we have

$$\begin{aligned} I &= R_o^2 \frac{\sin^2 \beta}{\beta^2} \cdot \frac{\sin^2 2\gamma}{\sin^2 \gamma} \\ &= R_o^2 \frac{\sin^2 \beta}{\beta^2} \cdot \frac{(2 \sin \gamma \cos \gamma)^2}{\sin^2 \gamma} \\ &= 4 R_o^2 \frac{\sin^2 \beta}{\beta^2} \cos^2 \gamma \end{aligned}$$

and is the same as that obtained for the double slit (eqn. 28.11). It can be seen from eqn. (28.16), that the intensity distribution for  $N$  slits is the product of two terms; the first term  $\frac{\sin^2 \beta}{\beta^2}$  represents the diffraction pattern produced by a single slit and the second term



$\frac{\sin^2 N\gamma}{\sin^2 \gamma}$  represents the interference pattern produced by  $N$  equally spaced point sources.

### Position of the principal maxima

The condition for the principal maxima is  $\sin \gamma = 0$ , or,  $\gamma = \pm n\pi$  where  $n = 0, 1, 2, 3, \dots$ . But for these values of  $\gamma$ , both the numerator and denominator of the term  $\frac{\sin^2 N\gamma}{\sin^2 \gamma}$  become equal to zero, i.e., the term becomes indeterminate. However, by applying the method of calculus, it can easily be seen that

$$\begin{aligned} \lim_{\gamma \rightarrow \pm n\pi} \frac{\sin N\gamma}{\sin \gamma} &= \lim_{\gamma \rightarrow \pm n\pi} \frac{\frac{d}{d\gamma} \sin N\gamma}{\frac{d}{d\gamma} \sin \gamma} \\ &= \lim_{\gamma \rightarrow \pm n\pi} \frac{N \cos N\gamma}{\cos \gamma} \\ &= \frac{N \cos N(\pm n\pi)}{\cos(\pm n\pi)} = \pm N \end{aligned}$$

Thus, the intensity of the principal maximum is proportional to  $N^2$  and, therefore, increases with the increase of the number of slits. The intensities of the principal maxima are, however, modulated by the single slit diffraction pattern. The actual intensity is given by the product of  $N^2$  and  $R_o^2 \frac{\sin^2 \beta}{\beta^2}$ , i.e.,

$$I = N^2 R_o^2 (\sin^2 \beta / \beta^2) \quad (28.17)$$

$$\text{where } \beta = \frac{\pi a \sin \theta}{\lambda}$$

from  $\gamma = \pm n\pi$  we have

$$\gamma = \frac{\delta}{2} = \frac{\pi d \sin \theta}{\lambda} = \pm n\pi$$

$$\text{or, } d \sin \theta = \pm n\lambda \quad (n = 0, 1, 2, 3, \dots) \quad (28.18)$$

If the width of the slit is  $a$  and the width of the opaque space is  $b$ , then

$$d = (a + b)$$

$$\text{or, } (a + b) \sin \theta = n\lambda$$

$$\text{or, } \sin \theta = \frac{n\lambda}{(a + b)} = \frac{n\lambda}{d}$$

Eqn. (28.18) simply expresses the condition that if the difference of paths of the parallel diffracted rays from the corresponding points of two consecutive slits is an integral multiple of  $\lambda$ , all the parallel diffracted rays in that direction reinforce each other, so as to form the principal maximum in the focal plane of the lens. Putting  $n = 0, 1, 2, 3, \dots$  the directions of the principal maxima  $\theta_0, \theta_1, \theta_2, \theta_3, \dots$  can be determined. The whole number  $n$  in eqn. (28.18) represents the order of the interference maximum. *Physically it gives the number of wavelengths in the difference of paths of the diffracted rays from any pair of corresponding points.*

### Position of minima

To find the minima, we note that the numerator of the function  $\frac{\sin^2 N\lambda}{\sin^2 \gamma}$  becomes zero to give zero intensity when  $N\gamma = 0, \pi, 2\pi, \dots = \pm m\pi$ . However, in the special cases when  $m = 0, N, 2N, \dots$   $\gamma$  becomes equal to  $0, \pi, 2\pi, 3\pi$ , etc. and the denominator also becomes equal to zero and we have the principal maxima described above. Hence the condition for a minimum is  $N\gamma = \pm m\pi$  (or,  $\gamma = \pm \frac{m}{N}\pi$ ) but  $\gamma \neq \pm m\pi$ . Thus the condition for a minimum excludes those values of  $m$  for which  $m = nN$ ,  $n = 0, 1, 2, 3, 4$  etc., being the order of the principal maximum.

Substituting the value of  $\gamma$ , we obtain the condition for a minimum in terms of the path difference between the parallel diffracted rays from any pair of corresponding points.

$$\begin{aligned}(a + b) \sin \theta = d \sin \theta &= \pm \frac{m\lambda}{N} \\ &= \pm \frac{\lambda}{N}, \pm \frac{2\lambda}{N}, \pm \frac{3\lambda}{N}, \dots \pm \frac{(N-1)\lambda}{N}, \\ &\quad \pm \frac{(N+1)\lambda}{N}, \pm \frac{(N+2)\lambda}{N} \quad \text{minima}\end{aligned}$$

omitting the values

$$d \sin \theta = 0, \pm \frac{N}{N}\lambda, \pm \frac{2N\lambda}{N}, \pm \frac{3N\lambda}{N}$$

which represent the position of principal maxima.

### Positions of secondary maxima

From the expressions for the principal maxima and minima, it becomes clear that between two principal maxima we have at least  $(N - 1)$  minima. This is also true for minima points between any two adjacent principal maxima. Between two such consecutive minima the intensity has to have a maximum. Hence there will be  $(N - 2)$  other maxima. These are referred to as *secondary maxima*. Thus for 3 slits there is only one, for 4 slits there are two and for 5 slits there are three secondary maxima.

The positions of the secondary maxima can be obtained by differentiating eqn. (28.16) with respect to  $\gamma$  and then equating it to zero.

Thus

$$\frac{dI}{d\gamma} = \frac{R_o^2 \sin^2 \beta}{\beta^2} 2 \left[ \frac{\sin N\gamma}{\sin \gamma} \right] \times$$

$$\frac{N \cos N\gamma \sin \gamma - \sin N\gamma \cos \gamma}{\sin^2 \gamma} = 0$$

$$\text{or, } N \cos N\gamma \sin \gamma - \sin N\gamma \cos \gamma = 0$$

$$\text{or, } N \tan \gamma = \tan N\gamma$$



The roots of this equation other than  $\gamma = \pm m\pi$  give the positions of secondary maxima. The intensity of these secondary maxima decreases gradually as one goes out on either side of the principal maxima, and, as in the case of single slit pattern, are slightly shifted towards the adjacent principal maxima. These features of the secondary maxima are evident in Fig. 28.9, which is a plot of the intensity distribution of the diffraction pattern by 4 slits. When  $N$  is very large the principal maxima will be much more intense in comparison to the secondary maxima. Two points may be mentioned here:

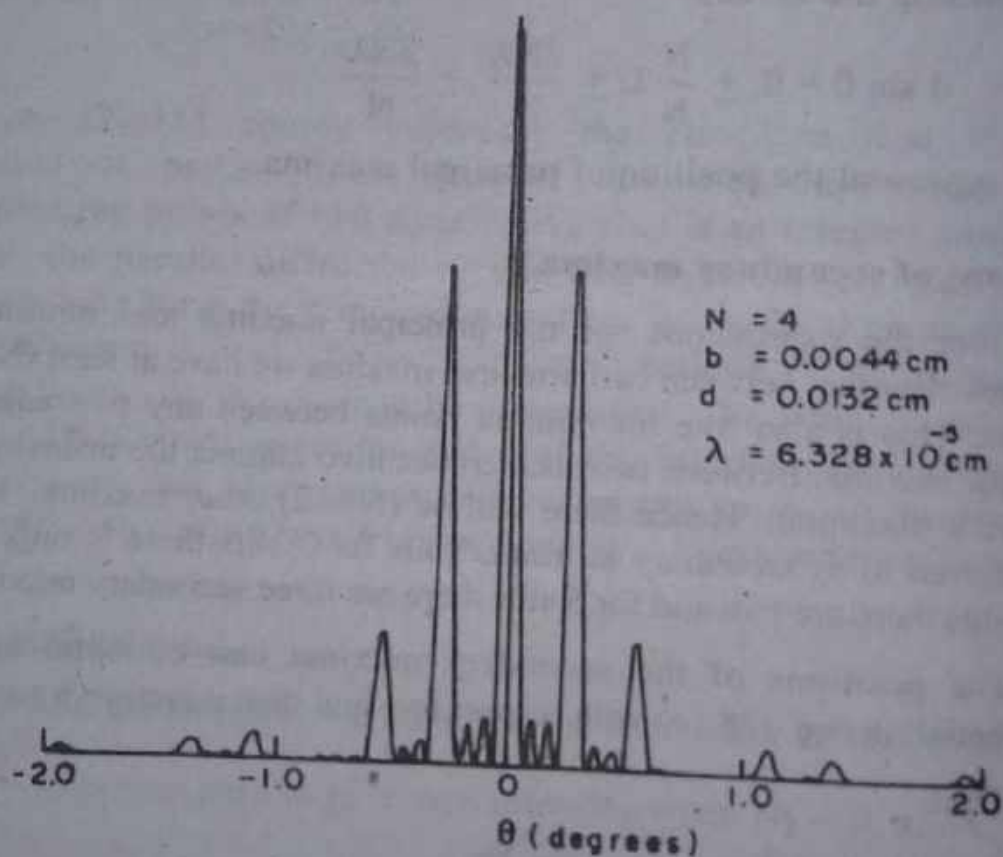


Fig. 28.9

(i) A particular principal maximum may be absent if it corresponds to the angle which also determines the minimum of the single slit diffraction pattern. This happens when the relations  $d \sin \theta = m\lambda$  and  $a \sin \theta = \lambda, 2\lambda, 3\lambda, \dots$  are satisfied simultaneously for the same value of  $\theta$ . This is usually referred to as an *absent order*. Even

if eqn.  $a \sin \theta = \lambda, 2\lambda, \dots$  does not hold exactly (i.e., if  $a \sin \theta$  is close to an integral multiple of  $\lambda$ ), the intensity of the corresponding principal maximum will be very weak (for example, the maximum around  $\theta \simeq 0.8^\circ$  in Fig. 28.9).

(ii) In addition to the minima predicted by eqn. ( $N\gamma = m\pi$ ,  $m \neq N, 2N$ ), we will also have the diffraction minima given by eqn.  $a \sin \theta = n\lambda$ ,  $n = 1, 2, 3, \dots$ ; however, when  $N$  is very large, the number of such minima will be very small.

### Width of the principal maxima

As shown above, the  $m^{\text{th}}$  order of the principal maximum in the diffraction pattern produced by  $N$ -slits occurs at

$$d \sin \theta_m = m\lambda, m = 1, 2, 3, \dots$$

$$\text{or, } N d \sin \theta_m = Nm\lambda \quad (28.19)$$

while the minima occur at the angles given by

$$(a + b) \sin \theta = d \sin \theta = \pm \frac{\lambda}{N}, \pm \frac{2\lambda}{N}, \dots$$

$$\text{or, } Nd \sin \theta = \pm \lambda, \pm 2\lambda, \dots, (N-1)\lambda \quad (28.20)$$

But the values

$$N d \sin \theta = 0, N\lambda, 2N\lambda, \dots \quad (28.21)$$

give respectively 0, 1<sup>st</sup>, 2<sup>nd</sup>, 3<sup>rd</sup>, ... order principal maximum.

Let  $\theta_m + \Delta\theta_{1m}$  and  $\theta_m - \Delta\theta_{2m}$  represent the angles of diffraction of the first minimum on both sides of the principal maximum of the  $m^{\text{th}}$  order,  $\theta_m$  being the angle of diffraction for the  $m^{\text{th}}$  principal maximum. The term  $\frac{1}{2} (\Delta\theta_{1m} + \Delta\theta_{2m})$  is known as the *angular half-width* of the principal maximum of the  $m^{\text{th}}$  order. For a large value of  $N$ ,  $\Delta\theta_{1m} \simeq \Delta\theta_{2m}$  which we write as  $\Delta\theta_m$ ; or, the angular half-width of the  $m^{\text{th}}$  principal maximum is  $\Delta\theta_m$ .

From eqns. (28.20) and (28.21) it is clear that the first minimum on either side of the principal maximum of the  $m^{\text{th}}$  order is given by the eqn.

$$Nd \sin(\theta_m \pm \Delta\theta_m) = (mN\lambda \pm \lambda) \quad (28.22)$$

Dividing eqn. (28.22) by (28.19), we get

$$\frac{\sin(\theta_m \pm \Delta\theta_m)}{\sin \theta_m} = \frac{(mN\lambda \pm \lambda)}{mN\lambda} \quad (28.23)$$

$$\begin{aligned} \text{But } \sin(\theta_m \pm \Delta\theta_m) &= \sin \theta_m \cos \Delta\theta_m \pm \cos \theta_m \sin \Delta\theta_m \\ &= \sin \theta_m \pm \Delta\theta_m \cos \theta_m \end{aligned} \quad (28.24)$$

The width of a principal maximum is the angular separation between the first two minima lying on either side of the principal maximum.

Hence, eqn. (28.23) becomes

$$\frac{\sin \theta_m \pm \Delta\theta_m \cos \theta_m}{\sin \theta_m} = 1 \pm \frac{1}{mN}$$

$$\text{or, } 1 \pm \Delta\theta_m \cot \theta_m = 1 \pm \frac{1}{mN}$$

$$\text{or, } \Delta\theta_m = \frac{1}{Nm \cot \theta_m} \quad (28.25)$$

For  $m = 1$ ,  $\theta_m = 20^\circ$  and  $\cot 20^\circ = 2.745$ . For a typical value of  $N = 14000$  lines per inch, the value of  $\Delta\theta_1 = 1/38430$  and therefore, is of the order of  $10^{-4}$  radian. This shows that the principal maximum becomes sharper as  $N$  increases. In practice, the principal maximum appears as a sharp line.

If the number of slits is  $N$ , then the separation between the first and the last slits is  $(N - 1)$ .  $(a + b) = (N - 1).d$ . When  $N$  is large, this can be approximately written as  $N.d$ . Now substituting the value of  $m$  from eqn. (28.18), we get

$$\Delta\theta_m = \frac{\lambda}{N.d. \cos \theta_m} \quad (28.26)$$

Eqn. (28.26) shows that the sharpness of the principal maximum depends upon the total width *i.e.*, the separation between the first and the last slits.



## 28.5 The diffraction grating

Any arrangement which is equivalent in its action to a large number of parallel equidistant slits of the same width is known as a *diffraction grating*, the corresponding diffraction pattern is known as the *grating spectrum*. Since the exact positions of the principal maxima in the diffraction pattern depend on the wavelength of light, the principal maxima corresponding to different wavelengths (spectral lines) associated with a source will correspond to different angles of diffraction. The grating is, therefore, a very powerful instrument for the study of spectral lines and provides us with an easily obtainable experimental setup for the determination of wavelengths. Since the principal maximum becomes sharper as the number of slits increases, a good quality grating requires a large number of slits – typically of the order of 14000 per inch. The first grating was made by Joseph Fraunhofer which consisted of a large number of parallel fine stretched wires placed very closely side by side at regular intervals on a frame. The diameters of these wires were of the order of 0.05 mm and their spacings varied from 0.053 mm to 0.687 mm. Now, gratings are made by ruling equidistant parallel grooves with a diamond point on an optically transparent sheet of material. After each groove is ruled, the machine lifts the diamond points and moves the sheet forward for the ruling of the next groove. As the distance between two consecutive grooves is very small, the movement of the sheet is obtained by driving the carriage carrying the sheet with the help of the rotation of a screw. Moreover, the pitch of the screw must be constant so that the grooves should be as equally spaced as possible, which is an important requirement of a good quality grating. It was not until 1882, when Rowland manufactured a nearly perfect screw, that the problem of construction of gratings was successfully solved. Rowland's arrangement gave 14438 lines per inch, corresponding to  $d = 2.54/14438 = 1.693 \times 10^{-4}$  cm. For such a grating, the maximum value of  $m$  would be 2 when the wavelength of light used is  $6 \times 10^{-5}$  cm. This means that only the first two orders of the spectrum will be observed. If  $\lambda = 5 \times 10^{-5}$  cm, then the third order spectrum will also be visible.

The diffraction gratings are of two types – transmission type and the reflection type. The grooves, mentioned above, act as opaque spaces and the space between any two consecutive grooves

is transparent to light. Such surfaces act as transmission gratings. If on the other hand, the lines are ruled on a silvered plane or concave surface, then light is reflected from the positions of the mirror in between any two lines. Such surfaces act as reflection gratings.

Gratings used for the study of the visible region of the spectrum contain about 10,000 lines per inch. Gratings with originally ruled surfaces are only few. For practical purposes, replicas of the original grating are prepared. These are made by taking the cast of an actual grating on a transparent film like that of cellulose acetate. A solution of cellulose acetate of appropriate strength is poured on the ruled surface and allowed to dry to form a strong thin film, detachable from the parent grating. These impressions of the grating are preserved by mounting the film between two glass sheets. This serves as a plane transmission grating. A large number of replicas can be prepared in this way from a single original ruled surface.

### Formation of spectra by a grating

The formation of secondary maxima discussed in Art. 28.4 are of little importance in the formation of a spectra by grating. The principal maxima are usually referred to as spectral lines, because when the primary source of light is a narrow slit they become sharp, bright lines on the grating.

If the slit and the rulings are parallel to each other, then the spectral lines will appear on the screen parallel to the rulings. In Art. 28.4 it has been shown that positions of the principal maxima are given by

$$d \sin \theta = m\lambda, \quad m = 0, 1, 2, \quad (28.27)$$

This relation is also known as the grating equation and can be used to study the dependence of the angle of diffraction  $\theta$  on the wavelength  $\lambda$ .

The principal maximum of the zeroth order occurs at  $\theta = 0$  irrespective of the wavelength. Thus if a polychromatic light (e.g. white light) is used as the source of light, the central maximum (principal maximum for  $\theta = 0$ ) for all the wavelengths will unfold one above the other and will appear to be of the same colour as the source. However, for  $m \neq 0$ , the angles of diffraction will be different for different wavelength and, therefore, the various spectral



lines appear at different positions. Thus by measuring the angles of diffraction for various colours, the values of the corresponding wavelengths can be determined by using eqn. (28.27) provided the order of diffraction *i.e.*, the value of  $m$  is known.  $d = (a + b)$  in eqn. (28.25) is called the grating element or grating space.

### Determination of wavelength

The diffraction grating is often used in the laboratory for the measurement of wavelength of light. Fig. 28.10 shows the schematic diagram for this measurement. After all the initial adjustments, the spectrometer is focussed for parallel rays. The grating is then mounted

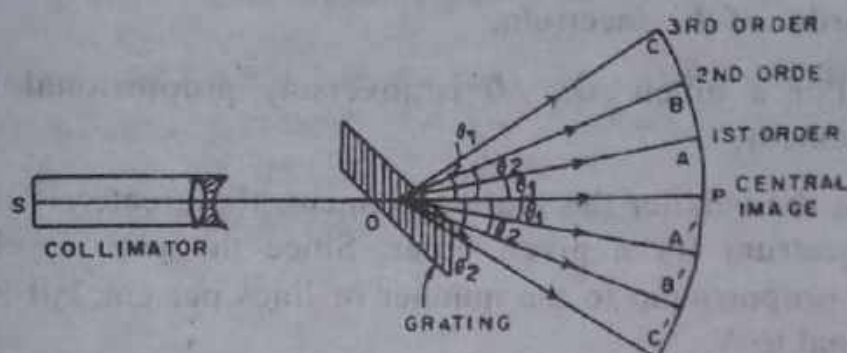


Fig. 28.10

on the spectrometer table. The light from the source under test is rendered parallel by the collimator and falls normally on the diffraction grating. The transmitted light is brought to focus by the telescope objective at the cross-wires. When the telescope is in line with the collimator, the image of the slit gives the position of the central maxima. The telescope is then turned around so as to receive the first principal maxima on both sides of the central maxima. The angle between these positions is  $2\theta_1$  where  $\theta_1$  is the angle of diffraction for the first order spectrum. Then eqn. (28.27) gives us  $d \sin \theta_1 = \lambda$ . Thus we can calculate  $\lambda$  from this relation provided the grating element is known.

### 28.6 Angular dispersive power of the grating

If we differentiate eqn. (28.27) with respect to  $\lambda$ , keeping  $m$  constant, we get



$$\frac{\Delta\theta}{\Delta\lambda} = \frac{m}{d \cos\theta} \quad (28.28)$$

The ratio  $\frac{\Delta\theta}{\Delta\lambda}$  or the rate of change of the angle of diffraction with the change in wavelength is called the *angular dispersive power* of the grating. A study of the expression (28.28) leads to the following conclusion:

(i)  $\frac{\Delta\theta}{\Delta\lambda}$  is directly proportional to  $m$ , the order of the spectrum. This means that the separation between the principal maxima corresponding to the wavelengths  $\lambda$  and  $\lambda + \Delta\lambda$  increases with the order of the spectrum.

(ii) For a given  $\Delta\lambda$ ,  $\Delta\theta$  is inversely proportional to  $d$ , the grating element.

Thus, the smaller the grating element, the greater is the spread of the spectrum for a given order. Since the grating element is inversely proportional to the number of lines per cm,  $\Delta\theta$  is directly proportional to  $N$ .

(iii) Assuming  $\theta$  to be very small (*i.e.*,  $\cos \theta \simeq 1$ ), we can see that for a given  $\Delta\lambda$ ,  $\Delta\theta$  is directly proportional to the order of the spectrum  $m$ , so that for a given  $m$ ,  $\frac{\Delta\theta}{\Delta\lambda}$  is constant. The angular dispersion, therefore, increases with the wavelength. Thus a grating spectra is much more spread near the red end than at the blue end of the spectrum.

---

**Example 28.6.** Deduce the missing orders for a double slit Fraunhofer diffraction pattern if the widths are  $8.8 \times 10^{-3}$  cm and they are  $4.4 \times 10^{-3}$  cm apart.

**Soln.**

The directions of the diffraction minima are given by

$$a \sin \theta = \pm m\lambda$$

and the directions of interference maxima are given by

$$(a + b) \sin \theta = d \sin \theta = \pm n\lambda$$

$$\therefore \frac{(a + b)}{a} = \frac{8.8 \times 10^{-3} + 4.4 \times 10^{-2}}{8.8 \times 10^{-3}}$$

$$= \frac{n}{m} = 6$$

or,  $n = 6m$ .

For values of  $m = 1, 2, 3$ , etc.

$$n = 6, 12, 18, \text{ etc.}$$

Thus the orders of 6, 12, 18, ..... of the interference maxima will be missing in the diffraction pattern.

**Example 28.7.** The limits of the visible spectrum are approximately 4000 Å.U. and 7000 Å.U. Find the angular breadth of the first order visible spectrum produced by a plane diffraction grating having 15000 lines per inch when light is incident normally on the grating.

**Soln.**

The grating element

$$d = (a + b) = \frac{2.54}{15000} = 1.69 \times 10^{-4} \text{ cm.}$$

The angular deviation of the violet in the first order is

$$\sin \theta = \frac{4000 \times 10^{-8}}{1.69 \times 10^{-4}} = 0.237$$

$$\text{or, } \theta = 13^\circ 40'$$

The angular deviation of the red is

$$\sin \theta = \frac{7000 \times 10^{-8}}{1.69 \times 10^{-4}} = 0.414$$

$$\text{or, } \theta = 24^\circ 30'$$

Hence the angular breadth of the first order visible spectrum is

$$24^\circ 30' - 13^\circ 40' = 10^\circ 50'$$

**Example 28.8.** Prove that the violet ( $\lambda \approx 4000 \text{ \AA}$ ) of the third order visible spectrum overlaps the red ( $\lambda \approx 7000 \text{ \AA}$ ) of the second order.

**Soln.**

From the equation of the grating  $(a + b) \sin \theta = n\lambda$ , the angular deviation of the third order violet is

$$\sin \theta = \frac{3 \times 4000 \times 10^{-8}}{(a + b)}$$

and of the second order red is

$$\sin \theta' = \frac{2 \times 7000 \times 10^{-8}}{(a + b)}$$

$\theta$  is smaller than  $\theta'$ , whatever may be the value of  $(a + b)$ . Therefore, the third order violet will always overlap the second order red.

**Example 28.9.** How many orders will be visible if the wavelength of incident radiation be 5000 A.U. and the number of lines on the grating be 14000 an inch?

**Soln.**

$$\text{Grating constant} = \frac{2.54}{14000} \text{ cm.}$$

The maximum observable angle of diffraction is  $90^\circ$ .

Hence, from  $(a + b) \sin \theta = m\lambda$ , we have

$$\begin{aligned} m &= \frac{(a + b) \sin \theta}{\lambda} = \frac{2.54 \times \sin 90^\circ}{14000 \times 5000 \times 10^{-8}} \\ &= \frac{2.54}{70 \times 10^{-2}} = 3.63 \end{aligned}$$

**3 orders will be visible in the grating spectra.**



**Example 28.10.** A diffraction grating used at normal incidence gives a green line ( $\lambda = 5400 \text{ A.U.}$ ) in a certain order superimposed on the violet line ( $\lambda = 4050 \text{ A.U.}$ ) of the next higher order. If the angle of diffraction is  $30^\circ$ , how many lines are there to the centimetre in the grating?

**Soln.**

Suppose the green line in the  $n^{\text{th}}$  order is superimposed on the violet line in the  $(n + 1)^{\text{th}}$  order. Hence we can write

$$(a + b) \sin \theta = n \times 5400 \times 10^{-8}$$

$$= (n + 1) \times 4050 \times 10^{-8}$$

$$\text{or, } n = 3$$

$$\text{or, } (a + b) = \frac{3 \times 5400 \times 10^{-8}}{\sin 30^\circ}$$

$$= 324 \times 10^{-8}$$

Therefore, the number of lines per cm in the grating

$$= \frac{1}{(a + b)} = \frac{1}{324 \times 10^{-8}}$$

$$= 3086 \text{ lines per cm.}$$

**Example 28.11.** Light from sodium lamp is incident normally on a grating  $0.5 \text{ cm}$  wide with 2500 lines. (i) Will the two sodium lines be resolved in the first order spectrum? (ii) Calculate the angular separation between the two sodium lines.  $\lambda_1 = 5890 \text{ A.U.}$  and  $\lambda_2 = 5896 \text{ A.U.}$

**Soln.**

Width of the grating =  $0.5 \text{ cm}$ .

Total number of lines =  $N = 2500$  lines.

(i) The condition for just resolution is

$$\frac{\lambda}{\Delta\lambda} = m N, \text{ (see Art. 29.8)}$$

Here  $\lambda = 5890 \times 10^{-8} \text{ cm}$

$$m = 1, N = ?$$

$$\Delta\lambda = 6 \times 10^{-8} \text{ cm}$$

$$\text{or, } \frac{5890 \times 10^{-8}}{6 \times 10^{-8}} = 1 \times N$$

$$\therefore N = 982$$

As the total number of lines in the grating is 2500, the two lines will appear well resolved.

$$\begin{aligned} \text{(ii) Number of lines per cm} &= N = \frac{2500}{0.5} \\ &= 5000 \text{ lines / cm.} \end{aligned}$$

$$\text{Grating element } (a + b) = \frac{1}{N} = \frac{1}{5000} \text{ cm.}$$

$$\text{(a) For } \lambda_1 = 5890 \times 10^{-8} \text{ cm and } m = 1$$

$$(a + b) \sin \theta_1 = 1 \times \lambda_1$$

$$\frac{1}{5000} \sin \theta_1 = 5890 \times 10^{-8}$$

$$\sin \theta_1 = 0.2945$$

$$\text{or, } \theta_1 = 17.1^\circ = 17^\circ 6'$$

$$\text{(b) For } \lambda_2 = 5896 \times 10^{-8} \text{ cm and } m = 1$$

$$(a + b) \sin \theta'_1 = 1 \times \lambda_2$$

$$\frac{1}{5000} \sin \theta'_1 = 5896 \times 10^{-8}$$

$$\text{or, } \theta'_1 = 17.2^\circ = 17^\circ 12'$$

The angular separation  $\theta'_1 - \theta_1 = 6$  minutes of an arc.

**Example 28.12.** Calculate the least width of a plane diffraction grating having 500 lines/cm, which will just resolve the sodium lines of  $5890 \text{ \AA}$  and  $5896 \text{ \AA}$  in the second order spectrum.

**Soln.**

Let the required number of lines in the grating be  $N$ .

Then  $\frac{\lambda}{\Delta\lambda} = mN$ , (see Art. 28.8)

$$\lambda = 5890 \times 10^{-8} \text{ cm}$$

$$\Delta\lambda = 6 \times 10^{-8} \text{ cm}$$

$$\text{or, } N = \frac{\lambda}{m \cdot \Delta\lambda} \quad m = 1, N = ?$$

$$= \frac{5890 \times 10^{-8}}{2 \times 6 \times 10^{-8}} = 982$$

Therefore, the least width of the grating

$$= \frac{491}{500} = 0.982 \text{ cm.}$$

**Example 28.13.** A diffraction grating having 4000 lines to a cm is used at normal incidence. Calculate the dispersive power of the grating in the third order spectrum in the wavelength region  $5000 \text{ \AA}$ .

**Soln.**

The dispersive power of the grating is given by

$$\frac{\Delta\theta}{\Delta\lambda} = \frac{m}{d \cos\theta}$$

$$\text{Now } d \sin \theta_m = m\lambda \quad m = 3 \quad \lambda = 5000 \times 10^{-8} \text{ cm}$$

$$d = (a + b) \frac{1}{N} = \frac{1}{4000}$$

$$\begin{aligned} \therefore \sin \theta_m &= \frac{m\lambda}{d} \\ &= \frac{3 \times 5000 \times 10^{-8} \times 4000}{1} = 0.6 \end{aligned}$$

$$\text{Therefore, } \cos \theta_m = \sqrt{1 - \sin^2 \theta_m} = 0.8$$

$$\begin{aligned} \text{Hence } \frac{\Delta\theta}{\Delta\lambda} &= \frac{m}{d \cos\theta} = \frac{3 \times 4000}{0.8} \\ &= 15000. \end{aligned}$$



**Example 28.14.** Light of wavelength  $\lambda$  is incident normally on a plane diffraction grating having  $N$  lines per unit length. How many orders of diffracted images can be observed?

**Soln.**

The grating equation is

$$(a + b) \sin \theta_m = m\lambda$$

$$\text{or, } \sin \theta_m = \frac{m\lambda}{(a + b)}$$

Now, the maximum angle of diffraction is  $90^\circ$ . Hence  $\sin \theta_m = 1$

$$\text{Thus } 1 = \frac{m\lambda}{(a + b)} = m\lambda N$$

$$\text{or, } m = \frac{1}{N\lambda}$$

This gives the maximum order of the image which could be seen.

**Example 28.15.** A monochromatic beam of light on passing through a slit 1.6 mm wide falls on a screen held close to the slit. The screen is then gradually moved away and the middle of the patch of light on it becomes dark when the screen is 50 cm from the slit. Calculate the wavelength of light.

**Soln.**

Width of the slit = 1.6 mm = 0.16 cm

Half-width =  $r = 0.08$  cm

Distance between the slit and the screen

$$= b = 50 \text{ cm}$$

$$\text{Here } (b^2 + r^2) = (b + \lambda)^2$$

$$\therefore 50^2 + (0.08)^2 = (50 + \lambda)^2$$

$$\text{or, } \lambda = \frac{(0.08)^2}{2 \times 50} \text{ app. (neglecting } \lambda^2)$$

$$= 0.000064 \text{ cm} = 6400 \text{ \AA}$$

**Example 28.16.** Calculate the angle between the lamp filament and its first diffracted image produced by a fabric with 160 threads/cm.  $\lambda = 6000 \text{ \AA}$

**Soln.**

$$(a + b) = \frac{1}{160}, n = 1, \lambda = 6000 \times 10^{-8} \text{ cm}$$

From  $(a + b) \sin \theta = n\lambda$ , we have

$$\begin{aligned} \sin \theta &= \frac{n\lambda}{a + b} = \frac{1 \times 6000 \times 10^{-8}}{1/160} \\ &= 160 \times 6 \times 10^{-5} \\ &= 96 \times 10^{-4} = 0.0096 \end{aligned}$$

$$\therefore \theta = \sin^{-1} 0.0096 = 33 \text{ min.}$$

### EXERCISES

- [1] Distinguish between the Fresnel and Fraunhofer classes of diffraction. Describe and explain the Fraunhofer diffraction pattern produced by a single slit illuminated by monochromatic light.
- [2] Discuss the Fraunhofer diffraction at a single slit. Draw the intensity distribution for the diffraction pattern.
- [3] Discuss Fraunhofer diffraction of monochromatic light from a single slit and show that the intensity of the first secondary maximum is roughly 4.5% of that of the principal maximum.
- [4] Describe the features of double slit Fraunhofer's diffraction pattern. How does it change if the distance between the slit centres is varied, keeping the slit-width constant?
- [5] Give the construction and theory of plane transmission grating and show how you would use it to find the wavelength of light.
- [6] Find the half angular width of the central bright maximum in the Fraunhofer diffraction pattern of a slit of width  $10^{-4} \text{ cm}$  when the slit is illuminated by monochromatic light of wavelength  $5000 \text{ \AA}$ .

- [7] A single slit Fraunhofer diffraction pattern is formed using white light. For what wavelength of light does the second minimum coincide with the third minimum for the wavelength 4000 Å?
- [8] Light of wavelength 5000 Å is incident normally on a slit. The first minimum of the diffraction pattern is observed to be at a distance of 5 mm from the central maximum of a screen placed at a distance of 2 metres from the slit. Calculate the width of the slit.
- [9] A slit of width 0.0002 cm is placed immediately in front of a lens of focal length 60 cm. It is illuminated normally by a parallel beam of light of wavelength  $5 \times 10^{-5}$  cm. Calculate the distance between the centre and the first dark band of the diffraction pattern on a screen placed at 60 cm from the lens.
- [10] For a plane diffraction grating with 5000 lines per cm, used at normal incidence, answer the following :
- What is the longest wavelength of light for which a spectrum can be observed?
  - For the wavelength 6000 Å, what is the highest order of spectrum which may be observed?
  - If the opaque spaces are twice as wide as the transparent spaces, which orders of the spectra will be absent?
- [11] A parallel beam of light is normally incident on a plane transmission grating having 10,600 lines per inch. A second order spectrum line is observed at an angle of  $30^\circ$ . Calculate the wavelength of the line.
- [12] A grating with 8000 rulings per inch is illuminated with white light at perpendicular incidence. If the wavelength of the light extends from 4000 Å to 7000 Å, show that only the first order spectrum is isolated but the second and third order spectra overlap.
- [13] Monochromatic light of wavelength 6560 Å falls normally on a grating 2 cm wide. The first order spectrum is produced at an angle of  $18^\circ 41'$  from the normal. Calculate the total number of lines on the grating. [9,522]
- [14] A grating has 6,000 lines per cm. Find the angular separation of two yellow lines of mercury of wavelengths 5,770 Å and 5,791 Å in the second order.
- [15] Calculate the maximum value of the grating element  $d$  so that the first order and the second order spectra do not overlap when the grating is used for studying a light beam containing wavelength components from 4000 Å to 7000 Å.



- [16] Deduce the missing orders for a double slit Fraunhofer diffraction pattern if the width of the slit and the opaque space are 0.16 mm and 0.8 mm respectively. (6, 12, 18, etc.)
- [17] A plane transmission grating having 6000 lines per cm is used to obtain a spectrum of light from a sodium lamp in the second order. Calculate the angular separation between the two sodium lines whose wavelengths are  $5890 \text{ \AA}$  and  $5895 \text{ \AA}$  (3 minutes of an arc).
- [18] A plane transmission grating has 15000 lines per inch. Find the angular separation of the  $5048 \text{ \AA}$  and  $5016 \text{ \AA}$  lines of helium in the second order spectrum ( $16'$ ).
- [19] What is the highest order spectrum, which may be seen with monochromatic light of wavelength 6000 by a means of diffraction grating with 5000 lines/cm? (3).
- [20] Examine if two spectral lines of wavelengths  $5890 \text{ \AA}$  and  $5896 \text{ \AA}$  can be clearly resolved in the (i) first order and (ii) second order by a diffraction grating 2 cm wide and having 425 lines/cm.  
[The lines will appear resolved in the second order but not in the first order]
- [21] Calculate the minimum number of lines per cm in a 2.5 cm wide grating which will just resolve the sodium lines ( $5890 \text{ \AA}$  and  $5896 \text{ \AA}$ ) in the second order spectrum (196.4)
- [22] In a plane transmission grating the angle of diffraction for the second order principal maximum for the wavelength  $5 \times 10^{-5} \text{ cm}$  is  $30^\circ$ . Calculate the number of lines in one cm of the grating surface. (1000).
- [23] A monochromatic beam of light on passing through a slit of width 2 mm falls on a screen close to the slit. The screen is then gradually moved away and the middle of the path of light on it becomes dark when the screen is 80 cm from the slit. Calculate the wavelength of light. ( $6250 \text{ \AA}$ ).
- [24] A grating is ruled over a width of 0.1 meter and the number of lines on the grating is 5000 lines per cm. Find the smallest wavelength difference that could be resolved in the region of  $5000 \text{ \AA}$  in the first order. ( $0.1 \text{ \AA}$ ).

## CHAPTER XXIX

## RESOLVING POWER

*Fraunhofer diffraction at a circular aperture – Resolving power – Rayleigh's criterion for limit of resolution – Resolving power of a telescope – Relationship between the magnifying power and resolving power of a telescope – Resolving power of a microscope – Relationship between the magnifying power and resolving power of a microscope – Resolving power of a plane diffraction grating – Solved problems – Exercises.*

## 29.1 Fraunhofer diffraction at a circular aperture

Let AB be a section of the circular aperture of diameter  $d$ .  $C$  is the centre of the aperture and  $P$  is a point on the screen (Fig. 29.1).  $CP$  is perpendicular to the screen while the screen is perpendicular to the plane of the paper. Let a beam of parallel monochromatic rays be incident normally on the aperture. Light waves from every point in the plane of the aperture are diffracted in all directions. The rays travelling along the normal to the aperture *i.e.*, along  $CO$ , are brought to focus by the lens at its principal focus  $P$ . All the secondary waves emanating from points equidistant from  $O$  travel the same distance before reaching  $P$  and hence they all reinforce one another.  $P$ , therefore, corresponds to the position of central maxima. Light waves diffracted at an angle  $\theta$  with the normal is brought to focus at  $P_1$ , a distance  $x$  from  $P$  in the focal plane. Let  $AD$  and  $BE$  be the extreme rays in this diffracted beam emanating from the points  $A$  and  $B$  (extremities of a diameter). Then the path difference between them is

$$AD = AB \sin \theta = d \sin \theta$$

Following the same arguments as in the case of single slit diffraction pattern, the point  $P_1$  will be of zero intensity if this path difference is an integral multiple of  $\lambda$ , *i.e.*,

$$d \sin \theta = n\lambda$$

(minima)

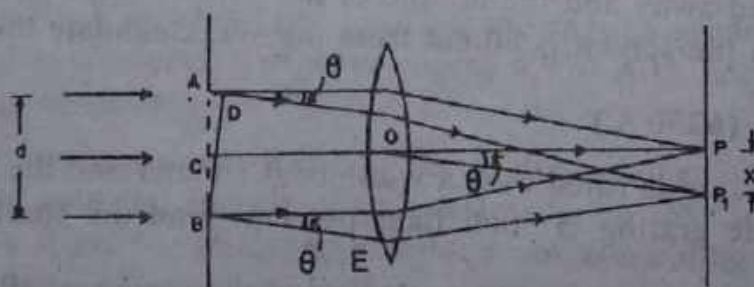


Fig. 29.1



If this path difference is an odd multiple of  $\lambda/2$ , there will be maximum intensity at  $P_1$ , i.e.,

$$d \sin \theta = (2n + \frac{1}{2}) \lambda/2 \quad (\text{maxima})$$

where  $n = 1, 2, 3, 4$ , etc. Of course for  $n = 0$ , we have central maximum at P.

If  $P_1$  is a point of minimum intensity, then all points at the same distance from P as  $P_1$  and lying on a circle of radius  $x$  will be of minimum intensity. Thus the diffraction pattern due to a circular aperture is characterised by a central bright disc, called *Airy's disc*, surrounded by alternate bright and dark concentric rings called the *Airy's rings*. The intensity of the dark ring is zero and that of the bright rings decreases gradually outwards from P.

It is obvious from the figure that  $\angle P_1OP = \theta$ . Then, if the lens is very near the aperture.

$$\begin{aligned} \sin \theta &= \theta = PP_1/OP_1 \\ &= PP_1/OP \text{ approx.} = x/f \end{aligned}$$

where  $f$  is the focal length of the lens. Therefore, the condition for the minima becomes.

$$d \cdot x/f = n\lambda$$

$$\text{or, } x = \frac{nf\lambda}{d} \quad (\text{dark rings}) \quad (29.1)$$

Similarly, for maxima

$$d \cdot x/f = (2n + \frac{1}{2}) \lambda/2$$

$$\text{or, } x = (2n + \frac{1}{2}) \lambda f/2d \quad (\text{bright rings}) \quad (29.2)$$

The radius of the first dark ring is obtained by putting  $n = 1$  in eqn. (29.1).

Thus

$$x_1 = f\lambda/d$$



This dark ring bounds the central maxima. *Therefore,  $x_1$  may also be taken as the radius of the central bright disc or Airy's disc.* The more exact theory by Airy indicates that the radius of the first dark ring is given by the condition that the difference of paths between the extreme diffracted rays should be  $1.22 \lambda$ . Then

$$d \sin \theta = 1.22 \lambda$$

The radius of the first dark ring *i.e.*, the central bright disc becomes

$$x_1 = 1.22 f \lambda / d$$

The intensity distribution of the diffraction pattern is similar to the diffraction pattern for a rectangular slit. About 84% of the transmitted light energy is concentrated in the central bright disc (Airy's disc). It may be emphasized here that the result is not due to any manufacturing defects in a lens but is a consequence of the nature of light itself. If  $D$  is the diameter of the lens filling the aperture, then

$$x_1 = 1.22 f \lambda / D$$

Thus the radius of the Airy's disc or the radius of the first dark ring depends.

- (i) inversely upon the diameter of the lens
- (ii) directly upon the wavelength of light.

The Fraunhofer diffraction pattern for a circular aperture is of fundamental importance in the discussion of the resolving power of telescope and microscope, in view of the circular form of the lenses employed as objectives and eye-pieces in these instruments.

## 29.2 Resolving power

The purpose of most of the optical instruments, like telescopes and microscopes, is not only to give a magnified image of the objects but also to reveal greater amount of detail or *grain* in them. The magnifying power of a microscope or a telescope, depends only on the focal lengths of the lenses forming the objective and the eye-piece of the instrument. By a proper choice of these focal lengths, it is possible to increase the size of the image, *i.e.*, the image subtends

a large angle in the eye. But it must be remembered that *increase in the size of the image, beyond a certain limit does not necessarily mean gain in detail*. The upper limit to the useful magnification is set by the fact that the image of the point source formed by the lens, even if it is free from all aberrations, chromatic or monochromatic, is not a point image. But in reality, it is the diffraction pattern that sets a theoretical upper limit to the useful magnification. When parallel light passes through any aperture it cannot be focussed to a point image, but instead, a diffraction pattern is formed. The aperture involved here is usually that of the objective lens of the microscope or the telescope and is therefore, circular. The diffraction pattern is, therefore, characterised by the central bright disc bordered by the dark and bright concentric circular rings of rapidly diminishing intensity. If the lens diameter or the size of the aperture is large, the diffraction pattern of a point source is small. For two nearby point sources, the diffraction discs of the two patterns may overlap and the two images may not be distinguished. An optical system is said to be able to resolve the two point sources if the corresponding diffraction patterns are sufficiently small or sufficiently well separated from one another so that in both cases, the two images can be distinguished as separate image patterns. *The ability of an optical instrument, expressed in numerical measure, to distinguish between the images of two nearby objects is termed as its resolving power.*

In the case of a prism spectrograph or a diffraction grating whose purpose is to produce a spectrum, the term resolving power is referred to as its ability to resolve two spectral lines whose wavelengths are very close to each other so that the two lines can be viewed or photographed as separate lines. The less the difference in wavelengths between two spectral lines which can just be distinguished as separate, the greater is the resolving power.

### 29.3 Rayleigh's criterion for limit of resolution

Let us now consider what stands in the way of two close objects being seen as separate. Based on the assumption that point objects always give rise to point images in accordance with the laws of geometrical optics and rectilinear propagation of light, two point objects will always give rise to two distinctly separate point images,



however close they may be. But in reality light from a distant point source and diffracted by a circular opening (lens aperture), is focussed by a lens not as a geometrical point but as a disc of finite radius (Airy's disc) surrounded by one or two faint bright and dark rings. If light from two point sources situated very close to each other is focused by a lens then two such similar diffraction patterns are produced. If the two point sources are separated wide enough, there will be no overlapping of the two diffraction patterns; the images will be clearly distinguishable as separate *i.e.*, they will be well resolved. But if the sources are too close, then the discs of the two diffraction patterns may overlap and the images may become indistinguishable. The question to be decided is how far apart must these diffraction patterns be so as to be distinguished as separate.

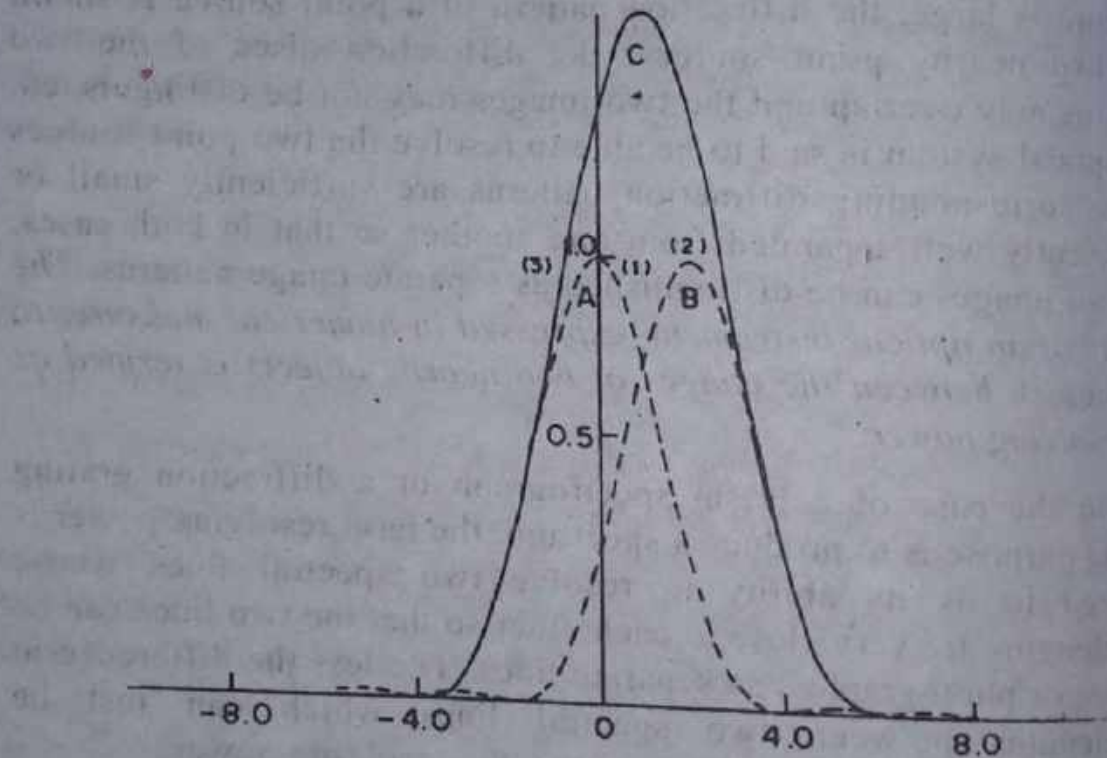


Fig. 29.2

Fig. 29.2 shows the diffraction pattern of two spectral lines *A* and *B* whose wavelengths  $\lambda$  and  $\lambda + d\lambda$  are very close. The angle of diffraction of the first minimum of *A* is greater than the angle of diffraction of the central maximum of *B*. the two wavelengths must have been emitted by different sources so that the disturbances are incoherent. Consequently, the resultant intensity at any point in the spectrum is simply the sum of the intensities for each wavelength.



The intensity distribution as shown in the figure shows only one peak at  $C$ , thereby indicating as if it is a diffraction pattern of only one spectral line. Thus under this condition, the two spectral lines are not resolved.

Fig. 29.3 shows the diffraction pattern of two spectral lines whose wavelengths are  $\lambda$  and  $\lambda_1$ . Let the difference in wavelength be such that the central maximum of the diffraction pattern of one falls over the second minima in the diffraction pattern of the other. The resultant intensity distribution as obtained by summing the intensities due to individual patterns is depicted in the figure. As can be seen there is a marked decrease in the resultant intensity in between

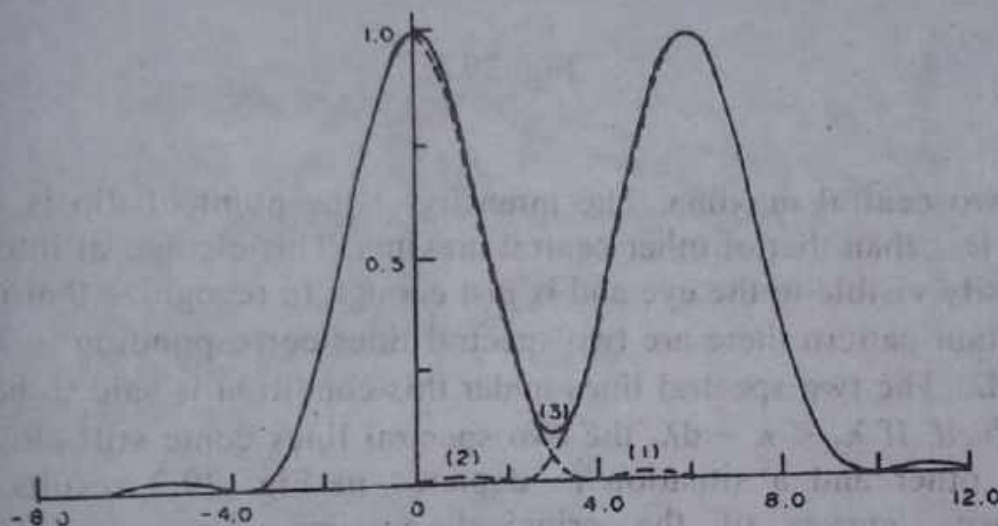


Fig. 29.3

the centres of the two patterns. Thus the two spectral lines are distinctly separated and are said to be *well resolved*. However, if the difference in wavelength is smaller, the two spectral lines with their central maxima will be closer. In the limit let the values of  $\lambda$  and  $\lambda_1$  ( $= \lambda + d\lambda$ ) be such that the central maximum in the diffraction pattern of one falls exactly on the first point of minimum intensity in the diffraction pattern of the other. Therefore, the intensities of the maxima in the resultant intensity pattern are equal to those of the original maxima. The resultant intensity as shown in Fig. 29.4 is a double humped curve with a distinct dip at a point half way between

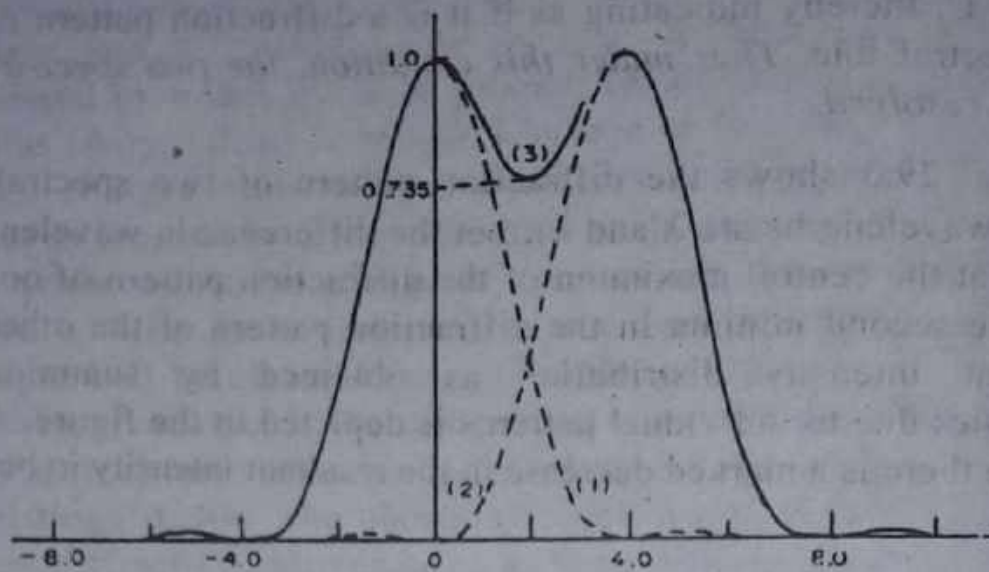


Fig. 29.4

the two central maxima. The intensity at the point of dip is about 20% less than that of other central maxima. This change of intensity is easily visible to the eye and is just enough to recognize that in the resultant pattern there are two spectral lines corresponding to  $\lambda$  and  $\lambda + d\lambda$ . The two spectral lines under this condition is said to be *just resolved*. If  $\lambda_1 < \lambda + d\lambda$ , the two spectral lines come still closer to each other and a situation as depicted in Fig. 29.2 results. The intensity curves of the principal maxima show considerable overlapping and the resultant intensity curves exhibits no dip but only one maximum in the centre, thereby indicating as if it is a diffraction pattern of only one spectral line. Thus, under this condition, the two spectral lines are not resolved in their diffraction pattern. This prompted Lord Rayleigh to suggest a practical criterion for the limit of resolution of two close diffraction patterns and is known as *Rayleigh's criterion* for resolution. According to this criterion, *two point sources are just resolved if the central maximum of the diffraction pattern of one source just falls on the first minimum of the diffraction pattern of the other source and vice versa*. Alternately, the angular separation between the central maxima of the two spectral lines in a given order, should be equal to half angular width of either principal maxima.



### 29.4 Resolving power of a telescope

The telescope is used to view distant objects and therefore the amount of detail or grain which it reveals depends on the angle that two point objects subtend at the objective rather than on the linear separation between the objects.

Let a point source of monochromatic light of wavelength  $\lambda$  be situated at a given distance from the telescope objective AB. A beam of parallel rays, shown by solid lines, from this object is incident normally on the objective (Fig. 29.5). The diffraction pattern consists of a central bright disc (Airy's disc) surrounded by concentric dark and bright rings, the common centre being  $P_1$  where

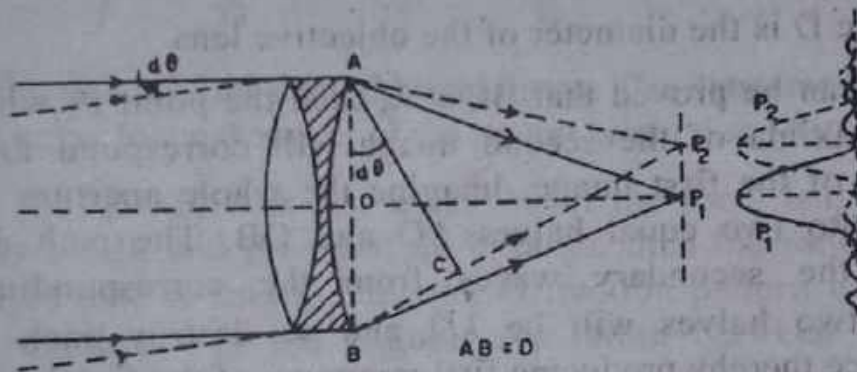


Fig. 29.5

the point image would have occurred in the absence of the diffraction pattern. The path difference between the secondary waves travelling in the direction  $AP_1$  and  $BP_1$  is zero and hence they reinforce one another at  $P_1$ . Similarly all the secondary waves from the corresponding points between A and B will have zero path difference. Thus,  $P_1$  corresponds to the position of central maximum of the first image.

Let us consider another point source situated very close to the first one. Parallel rays from this source are indicated by the broken lines. The image of this point is also a diffraction pattern similar to that of the first.  $P_2$  is the position of the central maximum of this second diffraction pattern. Obviously, these patterns will lie very close to each other with a considerable amount of overlapping. If the



overlapping is too much, the telescope may not be able to distinguish them as separate. According to Rayleigh's criterion, the patterns will be just resolved if the central maxima of one just falls on the first minima of the other and *vice versa*, i.e., the separation  $P_1P_2$  between the centres of the two central maxima is at least equal to the radius of either one.

The secondary waves travelling in the directions  $AP_2$  and  $BP_2$  will meet at  $P_2$  on the screen. Let the angle  $P_2AP_1$  be  $d\theta$  which is same as that subtended by the two point objects at the objective lens. The path difference between the secondary waves traveling in the directions  $BP_2$  and  $AP_2$  is equal to  $BC$ .

From  $\triangle ABC$ ,

$$BC = AB \sin d\theta = AB \cdot d\theta \text{ (for small angles)}$$

$$\text{or, } BC = D \cdot d\theta$$

where  $D$  is the diameter of the objective lens.

If it can be proved that  $BC = \lambda$ , then the point  $P_2$  which is the central maxima of the second image will correspond to the first minimum of the first image. Imagine the whole aperture  $AB$  to be divided into two equal halves  $AO$  and  $OB$ . The path difference between the secondary waves from the corresponding points between two halves will be  $\lambda/2$  and so destroy each other by interference thereby producing first minimum of the first image at  $P_2$ .

The condition of resolution, therefore, becomes

$$BC = D \cdot d\theta = \lambda ; \quad \text{or, } d\theta = \frac{\lambda}{D} \quad (29.3)$$

Eqn. (25.3) above, holds good for a rectangular aperture. According to theory of Fraunhofer diffraction at a circular aperture, the path difference  $BC$  should be equal to  $1.22\lambda$ .

Hence

$$D \cdot d\theta = 1.22\lambda ; \quad \text{or, } d\theta = \frac{1.22\lambda}{D} \quad (29.4)$$

The aperture  $D$  is equal to the diameter of the metal ring in which the objective lens is mounted.  $d\theta$  in eqn. (29.4) refers to the limit of resolution of the telescope. The reciprocal of  $d\theta$  measures the

resolving power of the telescope.

$$\frac{1}{d\theta} = \frac{D}{1.22\lambda} \quad (29.5)$$

Thus resolving power of a telescope may be defined as the inverse of the least angle subtended at the objective by two distant point objects which can be just distinguished as separate in its focal plane.

From eqn. (29.5), it is clear that a telescope with large diameter of the objective has higher resolving power.

If  $f$  is the focal length of the telescope objective, then

$$d\theta = \frac{r}{f} = \frac{1.22\lambda}{D}; \quad \text{or, } r = \frac{1.22f\lambda}{D} \quad (29.6)$$

where  $r$  is the radius of the central bright image. The diameter of the first dark ring is equal to the diameter of the central bright disc (Airy's disc).

From eqn. (29.6) it is also clear that if the focal length of the objective is small, and the aperture is large, then the radius of the central bright disc is small. Thus the diffraction pattern is sharper, or put in another way the angular separation between two just resolved point sources will be smaller. Correspondingly the resolving power of the telescope will be higher.

Let two distant stars subtend an angle of one second of an arc at the objective of the telescope. Let the wavelength of light be  $5500 \text{ \AA}$ . Since 1 second of an arc  $= 4.85 \times 10^{-6}$  radian, the diameter of the objective required for the stars to be just resolved

$$\begin{aligned} d\theta &= \frac{1.22\lambda}{D} \\ \text{or, } D &= \frac{1.22\lambda}{d\theta} = \frac{1.22 \times 5500 \times 10^{-6}}{4.85 \times 10^{-8}} \\ &= 13.9 \text{ cm approximately.} \end{aligned}$$

Thus the resolving power of a telescope increases directly with the increase in the diameter of the objective lens. It means that two stars which could not be resolved by a telescope with a smaller



objective could be resolved with one with a large objective, even though both may have the same magnifying power. This explains the use of large-diameter objective – quite apart from the desirability of collecting more light from the fainter stars and other celestial bodies which might otherwise not be visible at all. It can further be noticed from eqn. (29.5) that the resolving power is inversely proportional to the wavelength of light.

### 29.5 Relationship between the magnifying power and resolving power of a telescope

One point of great practical significance regarding the resolving power of a telescope may be considered here. Human eye has a poor resolving power. According to Helmholtz, the angular limit of resolution of human eye lies between  $1'$  and  $2'$ . This limit is determined by the diameter of the pupil. Taking diameter of the pupil to be 2 mm in bright sunlight, and  $\lambda = 5500 \text{ \AA}$ , the angular limit is given by

$$\theta = \frac{1.22\lambda}{D} = \frac{1.22 \times 5500 \times 10^{-8}}{0.2} = 3.5 \times 10^{-4} \text{ rad}$$

$$\approx 1 \text{ minute.}$$

It means that if the two objects subtend an angle less than  $1'$  at the eye, then it would not be able to resolve them, *i.e.*, see them as separate objects. Thus if a telescope is to be used visually, *i.e.*, in conjunction with the eye, then it is useless for its resolving power to be greater than that of the eye. Therefore, the finite limit of resolution of the eye has to be considered in designing optical instruments which are to be used in conjunction with the eye. The resolving power of the telescope is established by the diameter of the objective lens and its overall magnification is adjusted by choosing a proper eye-piece so that the final image is *blown up* to a size where its resolvable details just match the resolving power of the eye. The magnification affecting this proper matching is called the *normal magnification* of the instrument. Larger magnification which makes the image large and perhaps easier to look at but reveals no further details is known as *empty magnification*.

For example, consider the refracting telescope at Yorkes observatory



in America. The telescope has an objective of diameter 102 cm. Taking green light of  $\lambda = 5500 \text{ \AA}$ , the angular limit of resolution is

$$\begin{aligned} d\theta &= \frac{1.22\lambda}{D} = \frac{1.22 \times 5500 \times 10^{-8}}{102} \\ &= 6 \times 10^{-7} \text{ rad} = \frac{1}{8} \text{ second.} \end{aligned}$$

This is the angle made at the objective of the telescope by two objects 2.5 cm apart, and lying at a distance of nearly 30 km. Or, in other words the telescope is capable of resolving two objects separated by 2.5 cm at a distance of 30 km from its objective. But since the resolving power of the eye is, say 1 minute, the image formed by the objective of the telescope will have to be magnified  $\frac{1 \times 60}{1/8} = 480$  times by the eye-piece. Otherwise, even though the two objects will be actually resolved by the telescope, they may not be seen as separate objects by the eye.

The magnifying power of the telescope is given by  $M = D/d$  where  $D$  is the diameter of the objective and  $d$  that of the exit pupil of the instrument. If the focal lengths of the eye-piece and the objective are so related that the diameter  $d$  of the exit pupil becomes  $D_e$ , the diameter of the entrance pupil of the eye (eye-lens), then magnification is said to be normal. Hence,

$$\text{normal magnification} = D/D_e$$

The limit of resolution of the telescope objective of diameter  $D$  is given by

$$d\theta = 1.22\lambda/D \quad (29.7)$$

and the limit of resolution of the unaided eye is given by

$$d\theta' = 1.22\lambda/D_e \quad (29.8)$$

Combining eqns. (29.7) and (29.8)

$$\frac{d\theta'}{d\theta} = \frac{\text{limit of resolution of the eye}}{\text{limit of resolution of the telescope}}$$

$$= \frac{1.22\lambda}{D_e} / \frac{1.22\lambda}{D} = \frac{D}{D_e} = \text{Normal magnifying power of the telescope.}$$

*Thus, the product of the normal magnifying power of a telescope and its limit of resolution is equal to the limit of resolution of the unaided eye.*

In the example mentioned above, if the normal magnifying power of the telescope is 480, full advantage of the high resolving power of the telescope can be taken.

## 29.6 Resolving power of a microscope

The treatment of resolving power of a microscope is somewhat different from that for the telescope. In the case of a telescope the interest centres on the smallest permissible angular separation between two distant objects which determines the limit of resolution when the images appear just resolved. But in the case of a microscope, the object is very near the objective of the microscope (just beyond the focus of the objective) and subtends a large angle at the objective. The primary function of the microscope is not to magnify an object but to reveal those finer details in the object which are invisible to unaided eye. The extent to which these finer details can be revealed depends not on the magnifying power but on the resolving power of the microscope. *The limit of resolution of a microscope can be conveniently defined as the least permissible linear distance between two point objects so that the two images appear just resolved, i.e., their diffraction patterns formed by the microscope objective are just distinguishable.* The resolving power is given by the reciprocal of this limit of resolution.

The objective of the microscope is represented in Fig. 29.6 (a) by a simple convex lens whose aperture is  $AB$ .  $O$  and  $O'$  are two point objects (assumed self-luminous) lying very close to each other at a small distance from the microscope objective  $L$ . Waves of light from  $O$  and  $O'$ , assumed to be of the same wavelength  $\lambda$ , are diffracted by the circular periphery of the objective lens. A diffraction pattern corresponding to each object, consisting of a central bright disc bordered by alternate concentric circular dark and



bright rings, is formed at the focal plane of the objective lens. The centre of central bright disc in the diffraction pattern of  $O$  is at  $I$ , while for  $O'$  it is at  $I'$ .

According to Rayleigh's criterion, the two images will be just resolved if the central maxima of one pattern falls on the first minima of the other and *vice versa*. This is equivalent to saying that the distance  $II'$  between the centres of the two central maxima should be at least equal to the radius of the either central bright disc (Airy's disc).

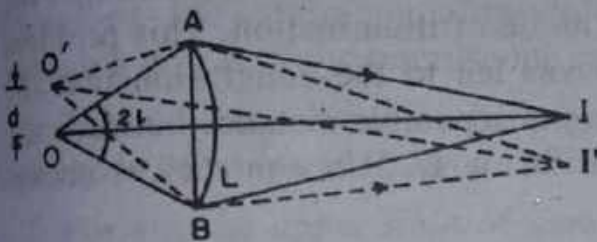


Fig. 29.6 (a)

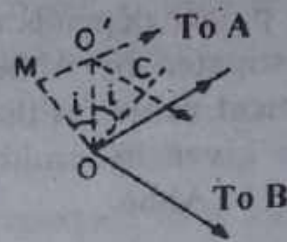


Fig. 29.6 (b)

Let us now find the condition under which the first minima of the diffraction pattern due to  $O'$  lies at the central maxima of diffraction pattern due to  $O$ . According to the theory of Fraunhofer diffraction at a circular aperture, this is possible only if the path difference between the extreme rays  $O'BI$  and  $O'AI$  is equal to  $\lambda$  (or more strictly speaking  $1.22 \lambda$ ).

The magnified view of  $OO'$  and the rays starting from them is shown in Fig. 29.6(b). Let us now find the difference in path of the rays  $O'BI$  and  $O'AI$ . As can be seen from Fig. 29.6 (b),  $O'B$  is longer than  $O'A$  by a distance  $O'C$  which is equal to  $OO' \sin i = d \sin i$  where  $i$  is half the angle subtended at the axial point object  $O$  by the rim of the microscope objective and  $d$  is the linear distance  $OO'$  between the point objects. On the other hand,  $O'A$  is smaller than  $OB$  (which is equal to  $OB$ ) by an amount  $O'M = OO' \sin i = d \sin i$ . Hence

$$O'BI - O'AI = O'C + O'M = 2d \sin i$$

For the first minima of the diffraction pattern due to  $O'$  to lie at the central maxima of the diffraction pattern due to  $O$ ,

$$O'BI - O'AI = 2d \sin i = 1.22 \lambda$$



Hence the condition for the images of  $O$  and  $O'$  to be just resolved is given by

$$d = \frac{1.22 \lambda}{2 \sin i} \quad (29.9)$$

It may be noted that eqn. (29.9) which gives the linear distance between the two just resolvable objects has been derived on the assumption that the point objects  $O$  and  $O'$  are independently self-luminous and so are not coherent sources. But actually, the objects viewed with a microscope are not self-luminous, but are illuminated with light from a condenser or a sub-stage mirror. It is found that the resolving power depends on the mode of illumination. This problem was investigated by Abbe who was led to the conclusion that the best practical value for the minimum separable distance between the objects is given by omitting the factor 1.22 in eqn. (29.9) above. According to Abbe,

$$d = \lambda / 2 \sin i \quad (29.10)$$

where  $\lambda$  is the wavelength of light in vacuum. But the entire difference in path under consideration, *i.e.*,  $(O'C + O'M)$  is in the medium between the object and the objective. If there is a liquid of refractive index  $\mu$  between the object and the objective,  $\lambda$  changes to  $\lambda/\mu$ . Hence

$$d = \frac{\lambda/\mu}{2 \sin i} = \frac{\lambda}{2 \mu \sin i} \quad (29.11)$$

The space between the object and the objective is filled with cedar weed oil in microscopes of high resolving power. This is referred to as oil immersion objective. To the expression  $\mu \sin i$  in eqn. (29.11) above, Abbe gave the name *numerical aperture* (abbreviated as N.A.) of the objective of the microscope and is a characteristic of the particular objective used. Microscope objective lenses are usually specified in terms of N.A.

Hence, the limit of resolution for a microscope, with non-self-luminous objects is given by

$$d = \frac{\lambda}{2 \mu \sin i} = \frac{\lambda}{2 (\text{N.A.})} \quad (29.12)$$

Since resolving power is defined as the reciprocal of the limit of resolution,

$$R.P = \frac{2\mu \sin i}{\lambda} = \frac{2 (N.A.)}{\lambda} \quad (29.13)$$

The advantage of using oil immersion objective is two folds. Firstly the loss of light by reflection at the first lens surface is decreased and since the numerical aperture of an oil immersion objective is higher than an ordinary objective, the resolving power of the microscope is increased. The resolving power of a microscope can be considerably increased by decreasing the value of  $\lambda$ . Therefore, by employing ultra-violet light and quartz lenses, the resolving power of the microscope can be further increased.

The increase in the resolving power of the microscope with the increase in the numerical aperture of its objective is illustrated below.

For air, the upper limit of numerical aperture of a microscope objective is 0.95. The least resolvable distance in air with white light of effective wavelength  $5500 \times 10^{-8}$  cm is, therefore,

$$d_{air} = \frac{5500 \times 10^{-8}}{2 \times 0.95} \text{ cm} = 2.894 \times 10^{-5} \text{ cm}$$

But by filling the space between the object and the objective by oil, the numerical aperture may be increased to 1.6. Hence the least resolvable distance becomes,

$$d_{oil} = \frac{5500 \times 10^{-8}}{2 \times 1.6} \text{ cm} = 1.72 \times 10^{-5} \text{ cm}$$

Thus the resolving power of the oil immersion type objective is approximately twice that of the ordinary objective. The resolving power can be further increased by using light of shorter wavelength as in the case of ultra-violet microscope which uses radiation of shorter wavelength in the range of 4000 A.U. to 2000 A.U.

### 29.7 Relationship between the magnifying power and resolving power of a microscope

From the stand-point of the resolving power of microscope objective, the overall magnification  $M$  of a microscope can be expressed in a slightly different form, as the product of the linear



magnification  $m$  of the objective and the angular magnification  $\beta$  of the eye-piece.

$$M = m \beta \quad (29.14)$$

In an actual microscope the medium between the eye-piece and the objective is air ( $\mu' = 1.0$ ). Let a ray of light be incident on the objective lens (Fig. 29.7). Then Abbe's sine condition gives

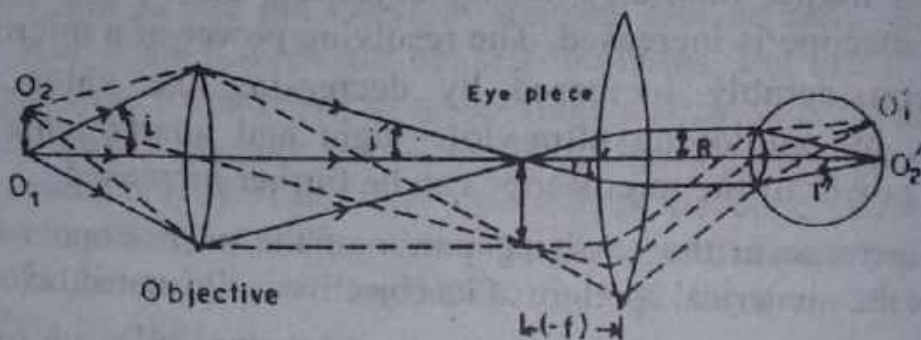


Fig. 29.7

$$d \mu \sin i = d' \mu' \sin i'$$

where

$d$  = distance of separation between two point objects in the incident (oil) medium,

$d'$  = distance of separation between the two images in the air medium between the objective and the eye-piece,

$\mu$  = refractive index of oil,

$\mu'$  = refractive index of air,

$i$  = angle of inclination of the incident ray with the axis,

and  $i'$  = angle of inclination of the refracted ray with the axis.

Then the linear magnification  $m$  of the objective is

$$m = \frac{d'}{d} = \frac{\mu \sin i}{\sin i'} \quad (29.15)$$

The angle  $i'$  is always small and therefore from the geometry of Fig. 29.7, it is clear that



$$\sin i' = R/f \text{ approximately} \quad (29.16)$$

where  $f$  is the focal length of the eye-piece and  $R$  is the radius of the exit pupil.

The angular magnification produced by the eye-piece is

$$\beta = 25/f \quad (29.17)$$

where  $f$  is in centimetres and the least distance of distinct vision is 25 centimetres.

Combining eqns. (29.15), (29.16) and (29.17) with eqn. (29.14) we get,

$$M = \frac{\mu \sin i}{R/25} \quad (29.18)$$

$R$  is in centimeters.

At normal magnification, the full apertures of both the objective and the eye are filled with light (Fig. 29.7). Then  $\mu \sin i$  is the maximum numerical aperture (NA) of the objective while  $R/25$  is that of the eye. Hence

$$\text{normal magnification} = \frac{\text{max. NA of objective}}{\text{max. NA of eye}} \quad (29.19)$$

Now the least resolvable distance by the compound microscope objective is

$$d = \frac{1.22\lambda}{2\mu \sin i} = \frac{0.61\lambda}{\text{max. NA of objective}}$$

and the least resolvable distance by an unaided eye is

$$d_1 = \frac{1.22\lambda}{2\sin i'} = \frac{0.61\lambda}{\text{max. NA of eye}}$$

Then

$$\frac{d_1}{d} = \frac{\text{max. NA of objective}}{\text{max. NA of eye}} \quad (29.20)$$

Combining eqns. (29.19) and (29.20), we get

normal magnification of the microscope

$$= \frac{d_1}{d} \quad (29.21)$$

*Thus the limit of resolution of a microscope multiplied by its normal magnifying power, is equal to the limit of resolution of the eye.*

If the radius of the pupil of the eye-lens is 1 mm, and if an object is placed in air at the least distance of distinct vision (25 cm), then the maximum numerical aperture of the unaided eye is 0.004. The maximum numerical aperture of the microscope objective is 1.6, which is  $1.6/0.004 = 400$  times as great as that of the unaided eye. Then the least distance between two point objects that can be just resolved by the microscope objective of N.A. = 1.6 as obtained from eqn. (29.20) is given by

$$d = d_1 / 400 \quad (29.22)$$

which is  $1/400^{\text{th}}$  as great as for the unaided eye. Actually, the resolving power of the microscope objective as well as of the eye is defined as the reciprocal of the least distance between two just resolvable point objects. By combining eqns. (29.21) and (29.22) it can be seen that the magnifying power of the microscope should be at least 400 in order to take full advantage of the available resolving power of the objective of numerical aperture 1.6. In other words, if the magnification is normal, then the diffractive discs formed on the retina of the eye would be resolved. As in the case of a telescope, increasing the magnification beyond this value would be useless if the microscope is to be used visually. A lower magnification, however, will not take full advantage of the available resolving power.

## 29.8 Resolving power of a plane diffraction grating

The function of a plane diffraction grating is to resolve two images of the same slit, formed by light waves of slightly different wavelengths, say  $\lambda$  and  $\lambda + d\lambda$ . The images of the slit are referred to as the spectral lines corresponding to wavelengths  $\lambda$  and  $\lambda + d\lambda$ . The limit of resolution or resolving power of a grating is defined as the smallest wavelength difference  $d\lambda$  for which the spectral lines can just be resolved at the wavelength  $\lambda$ . It is expressed mathematically



as  $\lambda/d\lambda$  where  $\lambda$  is the average of the two wavelengths and  $d\lambda$  is their difference. Rayleigh's criterion for resolution will be applied to derive an expression for the resolving power of the plane diffraction grating.

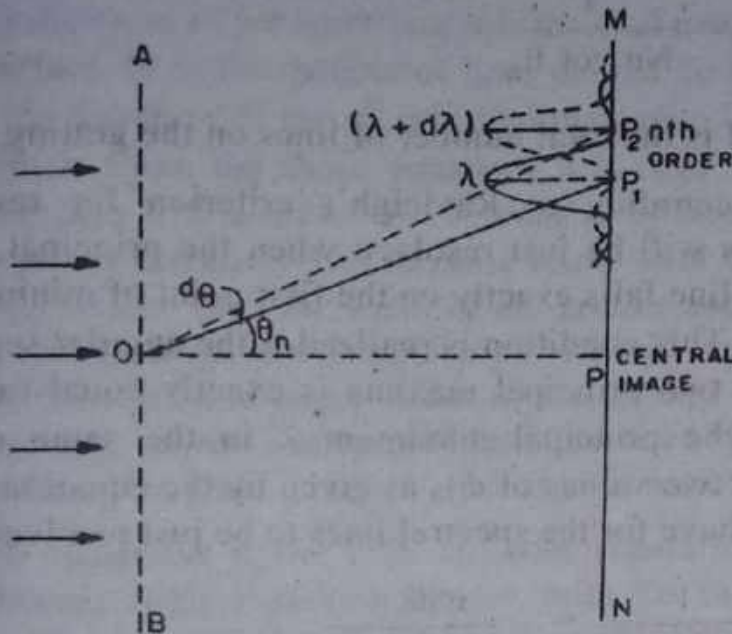


Fig. 29.8

Let a plane wavefront of light to be analyzed be incident normally on the grating surface  $AB$  (Fig. 29.8). A diffraction pattern is formed in the focal plane  $MN$  of the telescope lens, corresponding to each of the wavelengths, say  $\lambda$  and  $\lambda + d\lambda$ , present in the incident light.  $P_1$  is the  $n^{\text{th}}$  primary maximum of a spectral line of wavelength  $\lambda$  at an angle of diffraction  $\theta_n$ .  $P_2$  is the  $n^{\text{th}}$  primary maximum of the second spectral line of wavelength  $\lambda + d\lambda$  at a diffracting angle  $\theta_n + d\theta_n$ .

The angular deviation  $\theta_n$  of the primary maximum in the  $n^{\text{th}}$  order for a wavelength  $\lambda$  is given by

$$(a + b) \sin \theta_n = n\lambda \quad (29.23)$$

where  $(a + b)$  is the grating element.

Differentiating both sides of eqn. (29.23), we get



$$d\theta_n = \frac{n d\lambda}{(a+b)\cos\theta_n} \quad (29.24)$$

The half-angular breadth of the principal maximum of  $\lambda$  in the  $n^{\text{th}}$  order is given by (Art. 29.4)

$$d\theta_n = \frac{1}{Nn \cot \theta_n} \quad (29.25)$$

where  $N$  is the total number of lines on the grating surface.

Now, according to Rayleigh's criterion for resolution, two spectral lines will be just resolved when the principal maximum of one spectral line falls exactly on the first point of minimum intensity of the other. This condition is realized if the angular separation,  $d\theta_n$ , between the two principal maxima is exactly equal to half-angular breadth of the principal maximum  $\lambda$  in the same order. Hence equating the two values of  $d\theta_n$  as given by the equations (29.24) and (29.25), we have for the spectral lines to be just resolved,

$$\frac{1}{Nn \cot \theta_n} = \frac{n d\lambda}{(a+b)\cos\theta_n}$$

or, 
$$\frac{1}{Nn} = \frac{n d\lambda}{(a+b)\cos\theta_n} \quad (29.26)$$

Combining eqns. (29.23) and (29.26), we have

$$Nn = \lambda/d\lambda \quad (29.27)$$

The quantity  $\lambda/d\lambda = Nn$  measures the resolving power of a grating. As will be observed, *the resolving power of a grating is directly proportional to the order  $n$ , and in the given order it is proportional to the total number of lines on the grating surface.* Of course, the total number of lines will be effective, if the diameters of the collimating and telescope lenses are great enough so as to illuminate the entire ruled surface of the grating with the incident beam of light. It would further be observed that the resolving power is independent of the spacing  $(a+b)$  of the lines.

The dispersive power of a grating is given by

$$\frac{d\theta}{d\lambda} = \frac{n}{(a+b) \cos \theta} = \frac{nN'}{\cos \theta} \quad (29.27)$$

and the resolving power of a grating is given by

$$\lambda/d\lambda = nN$$

where  $n$  is the order of the spectrum,  $N$  is the total number of lines on the grating surface,  $N'$  is the number of lines per cm on the grating surface and  $\theta$  is the direction of the  $n^{\text{th}}$  principal maximum corresponding to a wavelength  $\lambda$ . From the above equations it is clear that *the dispersive power increases with increase in the number of lines per cm and the resolving power increases with increase in the total number of lines on the grating surface (i.e., the width of the grating surface)*. If  $N'$  is the same for two gratings, the dispersive powers will be same in both the cases but one with the larger width of the grating surface will have higher resolving power. The spectral lines produced by a grating having a large width of the grating surface are sharp and narrow.

High dispersive power refers to wide separation of the spectral lines whereas high resolving power refer to the ability of the instrument to show nearby spectral lines as separate ones.

**Example 29.1.** The Mount Wilson Observatory telescope has an objective of diameter 2.54 metre. Assuming the wavelength of light to be  $5.5 \times 10^{-5}$  cm, estimate the smallest angular separation of two stars which can be resolved by it.

**Soln.**

The angular limit of resolution of the objective lens is given by

$$d\theta = \frac{1.22 \lambda}{D} \quad D = 2.54 \text{ m} = 254 \text{ cm}$$

$$= \frac{1.22 \times 5.5 \times 10^{-5}}{254} \quad \lambda = 5.5 \times 10^{-5} \text{ cm}$$

$$= 2.642 \times 10^{-7} \text{ rad}$$

$$= 0.0544 \text{ second of an arc.}$$



**Example 29.2.** Two equally bright stars subtend an angle of one second at the objective of a telescope. Assuming the mean wavelength of light to be 5500 Å, find (i) the smallest diameter of a telescope objective lens which will be able to distinguish these stars as separate, (ii) the normal magnification of the telescope if the pupil of the eye has a diameter of 5 mm and (iii) the focal length of the eye-piece which should be used with the telescope.

**Soln.**

$$(i) \quad d\theta = \frac{1.22 \lambda}{D}$$

$$1 \text{ second} = \frac{1}{3600} \times \frac{2\pi}{360} \\ = 4.85 \times 10^{-6} \text{ rad}$$

$$\therefore 4.85 \times 10^{-6} = \frac{1.22 \times 5500 \times 10^{-8}}{D}$$

$$\text{or, } D = \frac{1.22 \times 5500 \times 10^{-8}}{4.85 \times 10^{-6}} = 13.85 \text{ cm}$$

(ii) Normal magnification

$$= \frac{D}{D_e} = \frac{13.85}{0.5} = 27.7$$

(iii) Magnification of the telescope =  $F/f$

$$\therefore 27.7 = \frac{13.85}{f}$$

$$\text{or, } f = \frac{13.85}{27.7} = 0.5 \text{ cm.}$$

**Example 29.3.** Calculate the minimum magnifying power required for a telescope whose objective has an aperture of 5 metres, in order to make full use of the resolving power.

**Soln.**



In order to make full use of the resolving power of the telescope, the minimum magnifying power required is given by the relation,

$$\text{R.P. of the telescope} \times \text{magnifying power} = \text{R.P. of the eye}$$

$$\text{or, } \frac{1.22 \lambda}{D} \times \text{magnifying power} = \frac{1.22 \lambda}{D_e}$$

$$\therefore \text{Magnifying power}$$

$$= \frac{D}{D_e} = \frac{5 \times 100}{0.5} = 1000$$

**Example 29.4.** Two pin holes 1.5 mm apart are placed in front of a source of light of wavelength  $5.5 \times 10^{-5}$  cm and seen through a telescope with its objective stopped down to a diameter of 0.4 cm. Find the maximum distance from the telescope at which the pin holes can be resolved.

**Soln.**

$$d\theta = \frac{1.22 \lambda}{D}$$

$$\lambda = 5.5 \times 10^{-5} \text{ cm}$$

$$D = 0.4 \text{ cm}$$

$$\text{Again } d\theta = \frac{x}{d}$$

$$x = 1.5 \text{ mm} = 0.15 \text{ cm}$$

$$\therefore \frac{x}{d} = \frac{1.22 \lambda}{D}$$

$$\text{or, } d = \frac{x \cdot D}{1.22 \lambda}$$

$$= \frac{0.15 \times 0.4}{1.22 \times 5.5 \times 10^{-5}} = 894.2 \text{ cm.}$$

**Example 29.5.** The smallest object detail that can be resolved with a certain microscope with light of wavelength 6000 Å is  $3.5 \times 10^{-5}$  cm. Find (i) the numerical aperture of the objective when

used 'dry' and (ii) the numerical aperture of the objective if an immersion oil of refractive index 1.5 is used.

**Soln.**

The linear limit of resolution of microscopes using a condenser is,

$$d = \frac{\lambda}{2 \cdot \text{N.A.}}$$

$$\therefore \text{(i) N.A.} = \frac{\lambda}{2 \times d} = \frac{6000 \times 10^{-8}}{2 \times 3.5 \times 10^{-5}} \\ = 0.86 \text{ (approx.)}$$

$$\text{(ii) Oil-immersion numerical aperture} \\ = \mu \times \text{dry aperture} = 1.5 \times 0.86 = 1.44$$

**Example 29.6.** Calculate the useful magnifying power of a telescope of 10 cm objective, assuming that the limit of resolution of the eye is 2 minutes of an arc. Wavelength of light used is  $6000 \text{ \AA}$ .

**Soln.**

Diameter of the objective  $D = 10 \text{ cm}$ ;

$$\lambda = 6000 \text{ \AA} = 6 \times 10^{-5} \text{ cm}$$

Limit of resolution of the telescope,

$$d\theta = \frac{1.22\lambda}{D} = \frac{1.22 \times 6 \times 10^{-5}}{10} \\ = 7.32 \times 10^{-6} \text{ rad.}$$

Limit of resolution of the eye,

$$d\theta' = 2 \text{ minutes of an arc} \\ = \frac{2}{60} \times \frac{2\pi}{7 \times 180} \text{ rad} = 582 \times 10^{-6} \text{ rad.}$$

Useful magnifying power of the telescope

$$= \frac{d\theta'}{d\theta} = \frac{582 \times 10^{-6}}{7.32 \times 10^{-6}} = 79.5.$$

**Example 29.7.** Calculate the least value of the angular separation of two stars which can be resolved by a telescope of 200 cm aperture. If the aperture of the human eye be 2 mm and if the focal length of the eye-piece be 1 inch, what must be the minimum focal length of the objective if the full resolving power of the telescope is to be utilized?

**Soln.**

The least value of angular separation of two stars which can be resolved by the telescope is given by

$$d\theta = \frac{1.22\lambda}{D}$$

Assuming the mean wavelength of white light to be  $5000 \times 10^{-8}$  cm, we have

$$\begin{aligned} d\theta &= \frac{1.22 \times 5000 \times 10^{-8}}{200} \\ &= 30.5 \times 10^{-8} \text{ rad.} \end{aligned}$$

The minimum magnifying power of the telescope required to make full use of its resolving power is expressed by

$$\text{Magnifying power} = \frac{D}{D_e} = \frac{200}{0.2} = 1000$$

Magnifying power of a telescope is also given by  $f_o/f_e$  where  $f_o$  and  $f_e$  are the focal lengths of the objective and the eye-piece respectively. Thus

$$1000 = \frac{f_o}{1}, \quad \text{or, } f_o = 1000 \text{ inches.}$$

**Example 29.8.** Calculate the minimum number of lines in a grating which just resolve the sodium lines in the first order spectrum. The wavelengths are  $5890 \text{ \AA}$  and  $5896 \text{ \AA}$ .

**Soln.**

$$\text{Resolving power} = \frac{\lambda}{d\lambda} = n/N$$



$$n = 1, \lambda = 5893 \times 10^{-8} \text{ cm},$$

$$d\lambda = (5896 - 5890) \times 10^{-8} \text{ cm}$$

$$\therefore N = \frac{1}{n} \left[ \frac{\lambda}{d\lambda} \right] = \frac{1}{1} \left[ \frac{5893}{6} \right]$$

$$= 982 \text{ approx.}$$

**Example 29.9.** Light is incident normally on a grating of 0.5 cm width with 2500 lines. Find the angular separation of two sodium lines in the first order spectrum. Would you expect the two lines to be distinguishable? The wavelengths of the two lines are 5890 A.U. and 5896 A.U.

**Soln.**

The grating equation is  $(a + b) \sin \theta = n\lambda$

$$(a + b) = 0.5/2500 = 1/5000$$

Considering the first line,

$$\frac{1}{5000} \sin \theta_1 = 1 \times 5890 \times 10^{-8}$$

$$\text{or, } \sin \theta_1 = 0.295, \text{ or, } \theta_1 = 17^\circ 8'$$

Similarly for the second line

$$\frac{1}{5000} \sin \theta_2 = 1 \times 5896 \times 10^{-8}$$

$$\text{or, } \sin \theta_2 = 0.2948, \text{ or, } \theta_2 = 17^\circ 9'$$

Hence, the angular separation

$$= \theta_2 - \theta_1 = 17^\circ 9' - 17^\circ 8' = 1'$$

The minimum resolving power necessary to separate these lines

$$= \frac{\lambda}{d\lambda} = \frac{5893 \times 10^{-8}}{6 \times 10^{-8}} = 982$$

Actual resolving power of the grating in the first-order

$$nN = 1 \times 2500 = 2500$$

Since the actual resolving power is much higher than the required minimum, the two lines would definitely be resolved.

**Example 29.10.** Calculate the number of lines per inch on a plane diffraction grating which when used on a spectrometer with telescope and collimator of aperture  $3/4$  inch, will just resolve the two  $D$  lines in the sodium spectrum of the second order.  $D_1 = 5896 \text{ \AA}$  and  $D_2 = 5890 \text{ \AA}$ .

**Soln.**

Diameter of the telescope objective

$$= 3/4 \text{ inch} = 3/4 \times 2.54 \text{ cm}$$

Resolving power of the telescope is

$$d\theta = \frac{1.22\lambda}{D} = \frac{1.22 \times 5893 \times 10^{-8}}{3/4 \times 2.54}$$

Let the requisite number of lines in the grating be  $x$  per inch or  $x/2.54$  cm for just resolving the  $D$  lines in the second order spectrum.

Hence,

$$(a + b) = 2.54/x \text{ cm}$$

$$\text{Now, } (a + b) \sin \theta = n\lambda$$

Differentiating both sides,

$$(a + b) \cos \theta d\theta = n d\lambda$$

$$\begin{aligned} \text{or, } d\theta &= \frac{n d\lambda}{(a + b) \cos \theta} = \frac{n d\lambda}{(a + b) \sqrt{1 - \sin^2 \theta}} \\ &= \frac{n d\lambda}{(a + b) \sqrt{1 - [n^2 \lambda^2 / (a + b)^2]}} \\ &= \frac{n d\lambda}{(2.54) \sqrt{1 - [n^2 x^2 \lambda^2 / (2.54)^2]}} \end{aligned}$$

$$\text{For } n = 2, d\lambda = 6 \times 10^{-8} \text{ cm,}$$

$$d\theta = 2x \times 6 \times 10^{-8} / 2.54 \text{ approx.}$$

For just resolution of the  $D$  lines in the second order, this angular separation,  $d\theta$ , must be equal to the limit of resolution of the telescope in the spectrometer. Hence



$$\frac{2 \times 6 \times 10^{-8}}{2.54} = \frac{1.22 \times 5893 \times 10^{-8}}{3/4 \times 2.54}$$

or,  $x = 799$  lines per inch.

### EXERCISES

- [1] Explain clearly what is meant by resolving power of an optical instrument. Discuss the usefulness of Rayleigh's criterion for determining the resolving power of optical instruments.
- [2] Distinguish clearly between the resolving power and the magnifying power of a telescope. Derive an expression for the resolving power of a telescope and discuss the relation between the resolving power and the magnifying power.
- [3] Derive an expression for the resolving power of a telescope. Show that the product of the magnifying and resolving powers of a telescope is equal to the resolving power of the eye.
- [4] What is meant by the resolving power of a microscope? How is it related to the numerical aperture of its objective?
- [5] Derive an expression for the resolving power of a microscope. Discuss Abbe's theory of resolution of a microscope.
- [6] Describe in detail the Rayleigh's criterion for the resolution of spectral lines. Obtain an expression for the resolving power of a plane transmission grating.
- [7] Explain clearly the concept of resolving power of an optical instrument. What is Rayleigh's criterion of the limit of resolution? Derive an expression for the resolving power of a plane diffraction grating.
- [8] Discuss the relationship between the resolving power and magnifying power of a microscope. Show that the limit of resolution of a microscope multiplied by its normal magnifying power, is equal to the limit of resolution of the eye.
- [9] Calculate the aperture of the objective of a telescope which may be used to resolve two stars separated by  $4.88 \times 10^{-8}$  radian for light of wavelength  $6000 \text{ \AA}$ . [15 cm]



- [10] The telescope at an observatory has a diameter of 100 inches. Estimate the smallest angle between two stars that can be separated by it. Assume the wavelength of light to be 5080 A.U.
- [11] The angular separation of two stars is 1.5 seconds. Find the minimum aperture of a telescope objective if the two stars are to be distinguished as separate. Assume the wavelength of light to be  $5700 \text{ \AA}$ . [9.55 cm]
- [12] Calculate the magnifying and resolving powers of a microscope from following data
- Numerical aperture = 0.12
- Focal length of the objective as well as of the eye-piece = 2.5 cm
- Distance of the cross-wires from the objective = 15 cm
- Wavelength of light = 6000 A.U.
- [13] The objective of the telescope at St. Palomar is 5.08 m. What is the least distance between the two objects at moon's surface which can be just resolved by this telescope. Take the distance between the earth and moon as nearly 384,000 km and assume the mean wavelength of moon's light as 5890 A.U.
- [14] A plane transmission grating has 40,000 lines per cm. These rulings are studied with an oil immersion microscope with light of wavelength  $4000 \text{ \AA}$ . Calculate the numerical aperture of the microscope required to just resolve the rulings. [0.8]
- [15] A plane transmission grating has 16,000 lines per inch over a length of 5 inches. Find in the wavelength region of 6000 A.U. in the second order.
- (i) the resolving power of the grating
- (ii) the smallest wavelength difference that can be resolved.
- [16] If a plane transmission grating, one cm wide, is required to resolve the D - lines of sodium of wavelengths 5890 and 5896 A.U. respectively in the third order spectrum, how close must the ruling be? [1/327 cm]
- [17] In the second order spectrum of a grating a spectral line appears at  $10^\circ$ ; another of  $5 \times 10^{-9} \text{ cm}$  greater appears at  $3^\circ$  farther. Find the wavelength of the lines and the minimum grating width required to resolve them.

## CHAPTER XXX

### POLARIZATION

*Polarization – Methods of producing plane polarized light – Polarization by reflection – Brewster's law – Intensity of polarized light: Malus' law – Optic axis of a crystal – Principal plane – Polarization by double refraction – Huygens' theory of double refraction – Nicol prism – Retardation plates – Dichroism: polarization by selective absorption – Polaroid – Optical activity – Specific rotation – Laurent's half-shade polarimeter – Photo-elasticity – Solved problems – Exercises.*

#### 30.1 Polarization

The phenomenon of interference and diffraction have been able to establish firmly the wave nature of light; however, both these phenomenon fail to establish whether light waves are longitudinal or transverse.

We know that in the longitudinal wave motion, the particles of the medium execute a to and fro periodic motion about their mean positions along the direction of the propagation of the waves. But in the transverse wave motion, the vibration of the particles is perpendicular to the direction of propagation of the wave. So in case of longitudinal waves, they are symmetrical about their direction of propagation and two waves can be similar in all their properties.

In the case of transverse waves, there can be number of directions in which the particles can execute periodic vibrations perpendicular to the direction of propagation. So two similar transverse waves may differ from one another because of their *different directions of vibrations*.

According to electromagnetic theory, light is an electromagnetic disturbance in which the electric and magnetic field are at right angles to each other and also at right angles to the direction of propagation, the two fields varying continuously and rapidly with time. Experimentally, it has been established that it is the electric field  $E$  – also known as the electric vector, which produces the optical polarization that is being considered here. Thus vibrations of light means the oscillations of the electric vector up and down along a vertical line transverse to the



direction of propagation. Now light is emitted in the form of wave-trains by individual atoms when in an excited state. In each wave-train, the electric vector is confined in one plane at right angles to the direction of propagation of light. A beam of natural light consists of millions of such wave-trains emitted by a very large number of randomly oriented radiating atoms and molecules in the light source. Therefore, light is a random mixture of vibrations in all possible transverse directions. Looking at such a beam end-on, there should be just as many waves vibrating in one plane as there are vibrating in any other as shown in Fig. 30.1(a). In other words, the natural unpolarized light may be looked upon

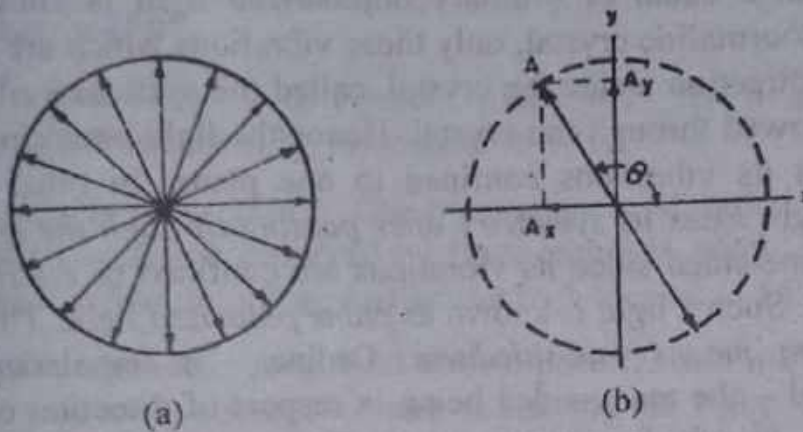


Fig. 30.1

as a mixture of waves in all possible transverse directions. This is referred to as *perfect symmetry*. As light waves are transverse in nature, each vibration of Fig. 30.1 (a) can be resolved into two component vibrations along two planes at right angles to each other and also perpendicular to the direction of propagation of light [Fig. 30.1(b)]. Although these two components may not be equal to each other, the similarly resolved components from all waves will average out to be equal. Thus a beam of ordinary unpolarized light may be regarded as being made up of two kinds of vibrations only – half the vibration vibrating in a vertical plane, say, along the plane of the paper, referred to as *parallel vibrations* and represented by *double headed arrows* as in Fig. 30.2(i); while the other half vibrating perpendicular to the plane of the paper, referred to as *perpendicular vibrations* and represented by *dots* as in Fig. 30.2(ii), Fig. 30.2(iii) will then represent the beam of ordinary unpolarized light.



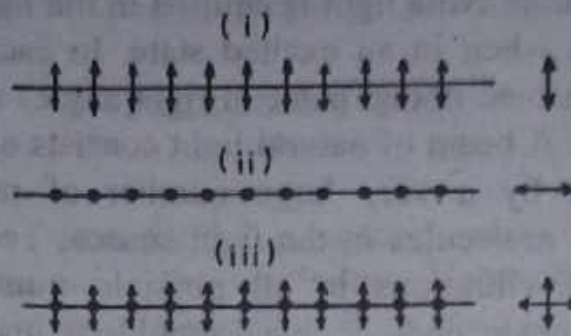


Fig. 30.2

If such a beam of ordinary unpolarized light is allowed to pass through a tourmaline crystal, only those vibrations which are parallel to a particular direction inside the crystal, called the *optic axis* of the crystal, will be allowed through the crystal. Hence the light emerging out of the crystal has its vibrations confined to one plane. In other words, the incident light *loses its symmetry after passing through the crystal* i.e., it becomes one-sided since its vibrations are confined or restricted to one plane only. Such a light is known as *plane polarized light*. Thus the word *polarization means one-sidedness*. Ordinary or unpolarized light is many-sided – the many-sided being in respect of direction of vibration. The plane in which vibrations take place is known as the *plane of vibration*. The plane in which no vibrations occur is the *plane of polarization* and is obviously at right angles to the plane of vibration. In Fig. 30.3 ABCD and EFGH are the planes of polarization and vibration respectively. When this plane polarized light, is allowed to pass through a second tourmaline crystal, it will get through only when the optic axis of the second crystal is parallel to that of the first crystal i.e., the vibrations are parallel to the optic axis. Obviously plane polarized light cannot get through the second crystal if its optic axis is at right angles to that of the first crystal.

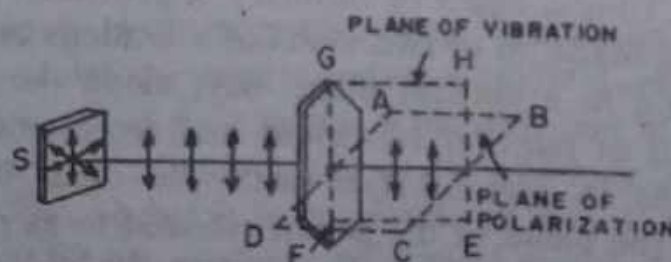


Fig. 30.3

It may just be mentioned here that light can be polarized not only linearly but circularly and elliptically as well. If the particle describes an elliptic orbit of fixed eccentricity and orientation in space, light is said to be elliptically polarised. When the orbit is a circle the beam is said to be circularly polarised, the plane of the orbit in both the cases being perpendicular to the direction of propagation.

### 30.2 Methods of producing plane-polarized light

There are three important methods for producing plane-polarized light. These are :

(i) *by reflection* : Polarization of light by reflection was first noted by Malus in 1808.

(ii) *by refraction* : Brewster discussed thoroughly the phenomenon of polarization by refraction in 1812.

(iii) *by double refraction* : The phenomenon of polarization by double refraction was first observed by Bartholinus in 1669. However, comprehensive investigation were carried out by Huygens in 1690.

Besides the methods mentioned above, polarization may also be produced by selective absorption and scattering of light. Some of these methods will be discussed in detail below.

### 30.3 Polarization by reflection

Polarization of light by reflection from the surface of glass was first discovered by Etienne Louis Malus, a French engineer, in 1808. He observed that when a beam of natural light is incident on a glass surface CD along the path AB, both the reflected light along BE and the transmitted light along BM are partially polarized (Fig. 30.4a). The reflected light consists mostly of dot components along with a few arrow components. The transmitted light mostly consists of arrow components along with a good number of dot components. This is true for all angles of incidence except one particular angle. At this particular angle of incidence, none of the arrow components is reflected; they are all transmitted. Of the dot components about 15% are reflected in the case of the glass surface (the fraction refracted depends upon the refractive index of the reflecting surface), the rest being transmitted along with the arrow components.



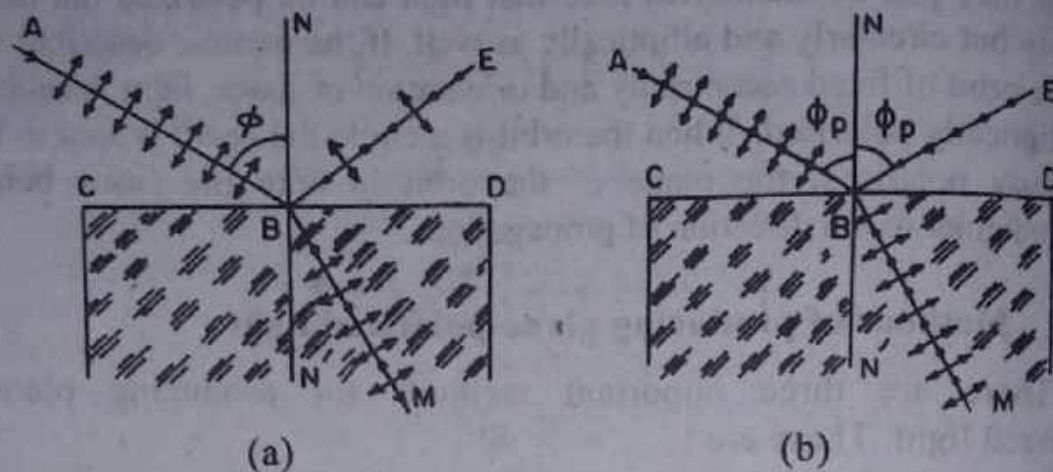


Fig. 30.4

The reflected light, although weak, is completely polarized (Fig. 30.4b). On the other hand, the refracted light is strong but only partially polarized. This particular angle of incidence at which the reflected light becomes completely polarized is known as the *polarizing angle* and is equal to  $57^\circ$  for a glass surface. At any other angle of incidence, the reflected light will not be completely polarized and will always contain a certain amount of arrow components.

### 30.4 Brewster's law

Sir David Brewster, a Scottish physicist, found experimentally in 1812 that the polarizing angle depends upon the refractive indices of the reflecting material and the surrounding medium in which it is placed. According to Brewster, *the tangent of the polarizing angle is equal to the refractive index of the reflecting material with respect to its surroundings*. This is known as *Brewster's law*.

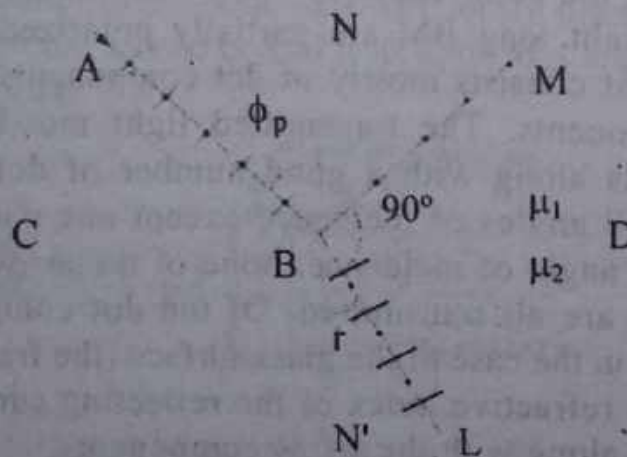


Fig. 30.5



If  $\mu_1$  and  $\mu_2$  are the absolute refractive indices of the surrounding medium and the reflecting material respectively and  $\phi$  the polarizing angle, then Brewster's law can be expressed as

$$\tan \phi = \frac{\mu_2}{\mu_1} ; \quad \text{or, } \tan \phi = {}_1\mu_2$$

where  ${}_1\mu_2$  is the refractive index of the reflecting material with respect to its surrounding medium.

Brewster's law immediately leads to an important deduction which is that *at the polarizing angle of incidence, the reflected and refracted rays are  $90^\circ$  apart, i.e., they are perpendicular to each other.* Referring to Fig. 30.5, Snell's law gives

$$\frac{\sin \phi}{\sin r} = {}_1\mu_2 \quad (30.1)$$

According to Brewster's law

$$\tan \phi = \frac{\sin \phi}{\cos \phi} = {}_1\mu_2 \quad (30.2)$$

From eqns. (30.1) and (30.2),

$$\cos \phi = \sin r$$

$$\text{or, } \sin (90 - \phi) = \sin r$$

$$\text{or, } 90 - \phi = r; \quad \text{or, } r + \phi = 90$$

$$\text{Now } \angle MBN = \angle NBA = \phi$$

$$\text{and } \angle MBN + \angle N'BL = 90^\circ$$

$$\text{Hence, } \angle MBL = \angle MBD + \angle DBL = 90^\circ$$

This proves that the reflected and refracted rays are  $90^\circ$  apart.

### 30.5 Intensity of polarized light : Malus' law

In Art. 30.1, we have seen that light, polarized by a tourmaline crystal, will pass through a second tourmaline crystal if the optic axis of this second crystal is parallel to that of the first crystal. The first crystal is referred to as the *polarizer* while the second crystal is known as the

*analyzer* as it helps to detect polarized light since the eye alone cannot detect if light is polarized. When the optic axes of both the polarizer and the analyzer are parallel to each other, the intensity of the emergent polarized light is maximum. Let this maximum intensity be  $I_0$ . As the analyzer is gradually rotated, the intensity of the transmitted light goes on decreasing until it becomes zero, when the optic axes of the two crystals are at right angles to each other. If the rotation of the analyzer is continued, then light gradually re-appears until the original intensity of the emergent light is restored when the analyzer has turned through  $180^\circ$  with respect to the polarizer *i.e.* the two optic axes again becomes parallel to one another. Consider the case when the analyzer has been rotated through an angle  $\theta$  from its position of maximum intensity. The amplitude  $A$  of the plane polarized light incident on the analyzer can be resolved into two components perpendicular to each other. The component  $A \cos \theta$  is transmitted while the component  $A \sin \theta$  is either rejected or absorbed. Since the intensity is proportional to the square of the amplitude, the intensity  $I$  of the transmitted light at this position is given by

$$I \propto A^2 \cos^2 \theta$$

$$\text{Also } I_0 \propto A^2$$

$$\therefore I = I_0 \cos^2 \theta; \quad \text{or, } \frac{I}{I_0} = \cos^2 \theta \quad (30.3)$$

This is known as *Malus' cosine law* according to which the intensity of the polarized light emerging from the analyzer is proportional to the square of the cosine of the angle between the polarizer and the analyzer.

**Example 30.1.** The refractive index for plastic is 1.25. Calculate the angle of refraction for a ray of light incident at polarizing angle.

**Soln.**

$$\mu = \tan \phi; \quad \text{or, } 1.25 = \tan \phi$$

$$\text{or, } \phi = \tan^{-1}(1.25) = 51.4^\circ$$

$$\text{Now } \phi + r = 90^\circ$$

$$\text{or, } r = 90^\circ - 51.4^\circ = 38.6^\circ$$



**Example 30.2.** How will you orient the polarizer and the analyzer so that a beam of natural light is reduced to (i) 0.125 (ii) 0.25 (iii) 0.5 and (iv) 0.75 of its original intensity?

**Soln.**

From Malus' law  $I = I_0 \cos^2 \theta$

$$\text{or, } \frac{I}{I_0} = \cos^2 \theta$$

$$(i) \cos^2 \theta = I/I_0 = 0.125$$

$$\text{or, } \cos \theta = \sqrt{0.125} = 0.3535$$

$$\therefore \theta = \cos^{-1} (0.3535) \simeq 69^\circ.18'$$

$$(ii) \cos^2 \theta = I/I_0 = 0.25$$

$$\text{or, } \cos \theta = \sqrt{0.25} = 0.5$$

$$\therefore \theta = \cos^{-1} (0.5) \simeq 60^\circ$$

$$(iii) \cos^2 \theta = I/I_0 = 0.5$$

$$\text{or, } \cos \theta = \sqrt{0.5} = 0.707$$

$$\therefore \theta = \cos^{-1} (0.707) \simeq 45^\circ$$

$$(iv) \cos^2 \theta = I/I_0 = 0.75$$

$$\text{or, } \cos \theta = \sqrt{0.75} = 0.8666$$

$$\therefore \theta = \cos^{-1} (0.8666) \simeq 30^\circ$$

**Example 30.3.** The critical angle for certain wavelength of light in the case of a piece of glass is  $40^\circ$ . Find the polarizing angle for glass.

**Soln.**

Critical angle  $C = 40^\circ$

$$\begin{aligned} \text{Refractive index for glass } \mu &= \frac{1}{\sin C} = \frac{1}{\sin 40^\circ} = \frac{1}{0.6427} \\ &= 1.556 \end{aligned}$$



From Brewster's law,

$$\mu = \tan i$$

$$\text{or, } i = \tan^{-1} \mu = \tan^{-1} (1.556) = 57.3^\circ$$

**Example 30.4.** Intensity of light through a polarizer and analyzer is maximum when their principal planes are parallel. Through what angle the analyzing nicol must be rotated so that the intensity gets reduced to  $1/4$  of the maximum value.

**Soln.**

From Malus' law

$$I_1 = I \cos^2 \theta$$

$$I_1 = \frac{1}{4} = I \cos^2 \theta$$

$$\text{or, } \cos^2 \theta = \frac{1}{4}$$

$$\cos \theta = \pm \frac{1}{2}$$

$$\text{or, } \theta = 60^\circ \text{ or } 120^\circ$$

### 30.6 Optic axis of a crystal

Let us consider a calcite crystal which is hydrated calcium carbonate ( $\text{CaCO}_3$ ). It was found in large quantities in times gone by in Iceland and is also known as *Iceland spar*. It occurs in nature in great variety of crystal forms (in the rhombohedral class of the hexagonal system), but breaks readily into simple cleavage rhombohedrons or rhombs. Each of the six faces of the rhombs is a parallelogram whose angles are  $\beta = 78^\circ 7'$  and  $\alpha = 101^\circ 53'$ . At the two opposite corners *A* and *H* (Fig. 30.6) all the three angles are obtuse ( $\alpha$ ) but at the other six corners two of the angles are acute ( $\beta$ ) and the other is obtuse ( $\alpha$ ). If struck a blow with a sharp instrument, each crystal may be made to cleave or break along cleavage planes into two or more similar crystals which always have parallelogram faces with these same angles.

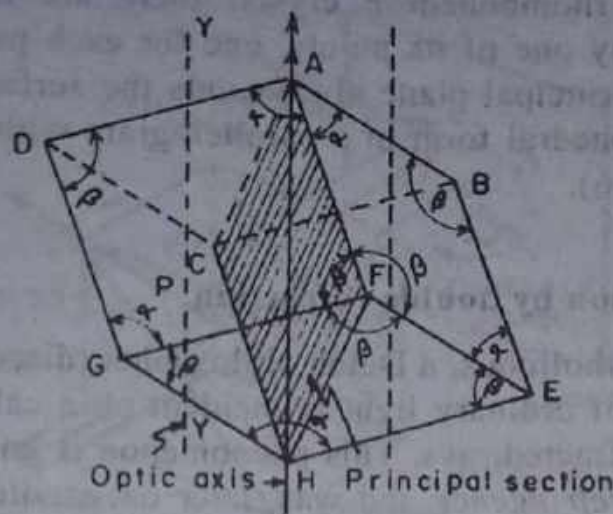


Fig. 30.6

The corners where three obtuse angles meet are known as *blunt corners*. The *optic axis* of a crystal is a direction within the crystal parallel to the straight line through either of the blunt corners and equally inclined to the three edges meeting there. If the edges of the crystals are all equal, then the straight line joining the two blunt corners gives the direction of the optic axis. It may be emphasized here that the *optic axis* is a direction and not a particular line. Hence an optic axis can be drawn through every point of the crystal. However, though any given point, there is one, and only one, line that can be drawn which makes equal angles with the three edges meeting there and this is the optic axis for that point and all other points on that line.

Crystals having one optic axis are called *uniaxial* crystals (e.g. quartz and calcite) and those having two optic axes are called *biaxial* crystals (e.g., mica).

### 30.7 Principal plane

A principal plane is a plane which contains the optic axis and the normal to the crystal surface on which light is incident and is perpendicular to the two opposite faces. Like optic axis, the principal plane is not a definite plane; in fact there may be any number of principal planes parallel to a particular given one.



Obviously, in a rhombohedron crystal there are three principal planes through any one of its points, one for each pair of opposite crystal faces. A principal plane always cuts the surface of a calcite crystal of rhombohedral form in a parallelogram with angles of  $71^\circ$  and  $109^\circ$  (Fig. 30.6).

### 30.8 Polarization by double refraction

Erasmus Bartholinous, a Dutch philosopher, discovered in 1669 that when a ray of ordinary light is incident on a calcite crystal, it splits into two refracted rays. This phenomenon is known as *double refraction* or *bi-refringence* and was, later on, studied in detail by Huygens and Newton. Huygens discovered that both the refracted rays were linearly polarized in mutually perpendicular planes. Nearly all-crystalline substances like quartz, mica, sugar, topaz, selenite, aragonite, ice, etc. are now known to exhibit the phenomenon of double refraction. Calcite and quartz are particularly important because they are used extensively in the manufacture of special instruments.

The phenomenon of double refraction can be easily demonstrated by allowing a narrow beam of unpolarized light to be incident normally on calcite crystal. There will be two refracted beams in place of the usual one as in glass. Since the two opposite faces of a calcite crystal are always parallel and since no double refraction occurs at the second face, the two refracted rays emerge parallel to the incident beam and therefore, parallel to each other. But the two rays are relatively displaced by a distance, which is proportional to the thickness of the crystal, *i.e.*, the path traversed within the crystal. If the rays are received on a screen, there will be two images instead of one. One of the images *O* lies in the direction of the incident ray *i.e.*, passes through the crystal without deviation and, therefore, must have been formed in accordance with the ordinary laws of refraction. The other image *E* is found to be separated from the *O* image, despite its perpendicular incidence and therefore, does not in general, obey the ordinary laws of refraction. If the calcite crystal is rotated around the incident beam as an axis, the *O* image remains stationary while the *E* image revolves in a circular path with its centre at the *O* image (Fig. 30.7).



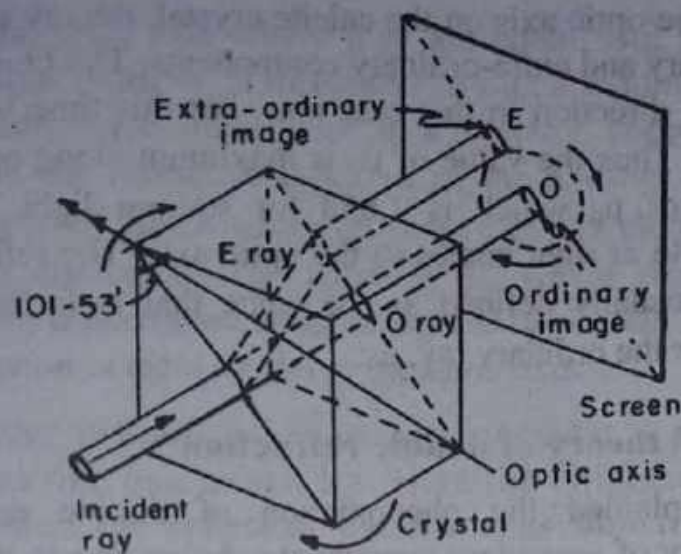


Fig. 30.7

Of the two refracted rays, the one which always obeys ordinary laws of refraction is called the *ordinary ray* or the *O-ray* while the other which does not obey the ordinary laws of refraction and behaves in quite extra-ordinary manner, is called the *extra-ordinary ray* or the *E-ray*. The ratio  $\sin i / \sin r$  physically represents the velocity of light in vacuum to that in the refracting medium. It is, therefore, obvious that the velocity of the *E-ray* is different in different directions whereas the *O-ray* travels with the same velocity in all directions. The *O-ray* consists of dot components *i.e.*, its vibrations are perpendicular to the optic axis *i.e.*, to the principal plane. The *E-ray* on the other hand consists of the arrow components *i.e.*, its vibrations take place in the principal plane.

The refractive index of the crystal with respect to the ordinary light is known as ordinary refractive index,  $\mu_o$ . The refractive index with respect to extra-ordinary light is known as extra-ordinary refractive index,  $\mu_e$ . In the case of calcite crystal  $\mu_o > \mu_e$ . Therefore, the velocity of the light for the ordinary ray inside the crystal will be less compared to the velocity of light for the extra-ordinary ray. Now it has been found that the speed of *E-ray* varies with its direction of travel through the crystal, *i.e.*, the value of  $\mu_e$  varies with the direction of *E-ray* inside the crystal. When the ordinary beam of light is incident on the calcite crystal in a direction parallel to the optic axis of the crystal, the incident light is not split up into ordinary and extra-ordinary rays. In this case, both the *O-ray* and the *E-ray* travel along the same direction with the same velocity. When the ray of light is incident

perpendicular to the optic axis on the calcite crystal, the ray of light is not split up into ordinary and extra-ordinary components. The *O-ray* and *E-ray* travel in the same direction in this case also, but this time with different velocities ( $v_e > v_o$ ). Thus the value of  $\mu_e$  is maximum along optic axis and equals the value of  $\mu_o$  which is 1.658 for sodium light. Its value is minimum *i.e.*, 1.486 at right angles to the optic axis. The refractive index for *E-ray*,  $\mu_e$ , is usually defined as the value that is farthest from the refractive index for the ordinary ray,  $\mu_o$ .

### 30.9 Huygens' theory of double refraction

Huygens explained the phenomenon of double refraction by applying his theory of secondary wavefronts. According to this theory, a point source of light inside an uniaxial doubly refracting medium gives rise to two wavefronts. The wavefront corresponding to the *O-ray* is spherical since its velocity is same in every direction. On the other hand the *E-ray* has different velocities in different directions; hence its wavefront is an ellipsoid of revolution. *Since the velocities of the ordinary and extra-ordinary rays are the same along the optic axis, the two wavefronts touch each other at the points where they are intersected by the optic axis.* The following two cases will now be considered.

(i) *negative crystals* : Let us consider a point source of light *P* inside a crystal like calcite and tourmaline. The spherical wavefront

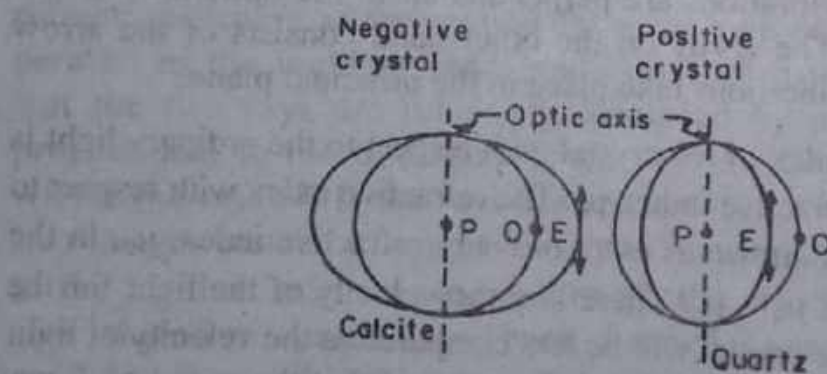


Fig. 30.8

in these crystals is entirely enclosed in the elliptical wavefront as shown in Fig. 30.8. These crystals are referred to as *negative crystals*. In these crystals, the velocity of the *O-wave* which is the same in all directions is less than the velocity of the *E-wave* everywhere



except along the optic axis where they are equal. But the *E-wave* has different velocities in different directions having a minimum value along the optic axis and a maximum value in a direction perpendicular to the optic axis. Thus the refractive index  $\mu_e$  of the *E-wave* is less than the refractive index  $\mu_o$  of the *O-wave* everywhere except along the optic axis where the two are equal. *The value of  $\mu_e$  varies from direction to direction, having a maximum value along the optic axis and a minimum value in a direction at right angles to the optic axis.*

(ii) *positive crystals* : Let us now consider a point source  $P$  inside a crystal like quartz and ice. Here the elliptical wavefront is completely within the spherical wavefront as shown in the figure. These crystals are known as *positive crystals*. Here the velocity of the *O-wave* is greater than that of the *E-wave* everywhere except along the optic axis where they are equal. *Consequently, the refractive index  $\mu_o$  is less than  $\mu_e$  everywhere except along the optic axis where the velocity of *O-wave* is minimum.*

### 30.10 Nicol prism

Nicol prism is an ingenious optical device for producing and analyzing plane polarized light and is based on the phenomenon of double refraction. It may be recalled that when a beam of ordinary light is allowed to pass through a calcite crystal, two completely plane polarized beams, the *O-wave* and the *E-wave*, with vibrations in mutually perpendicular planes are obtained. If by some means one beam can be eliminated, calcite may be used to obtain plane polarized light from natural light. William Nicol, a Scottish physicist and an expert in cutting and polishing gems and crystals, succeeded in around 1828 in eliminating the *O-wave* by using the phenomenon of total internal reflection at a thin film of canada balsam separating two pieces of calcite. His device has come to be known as *Nicol prism*, although in reality it is a *parallelepipedon* – not a prism.

Of the several different forms of Nicol prism, one of the commonest ones is described here. First a calcite rhombohedron of length three times its width is produced by cleavage from a clear natural crystal.  $A'B'CD'MN'P'Q'$  represents such a rhombohedron (Fig. 30.9).  $A'$  and  $P'$  are the blunt corners of the crystal where the three obtuse angles of the three faces meet. An imaginary plane



through the edges  $A'M$  and  $P'C$  contain the optic axis of the crystal; therefore,  $A'M P'C$  is one of the principal section of the crystal with the angle  $\angle A'CP' = \angle P'MA' = 71^\circ$ . The faces  $A'B'CD'$  and  $MN'P'Q'$  are then ground so as to form the new faces  $ABCD$  and  $MNPQ$  in such a way that the angles  $\angle ACP$  and  $\angle PMA$  become equal to  $68^\circ$  instead of  $71^\circ$ . The crystal is then cut along the plane  $AKPL$  passing through  $A$  and  $P$  and perpendicular to the shorter diagonal

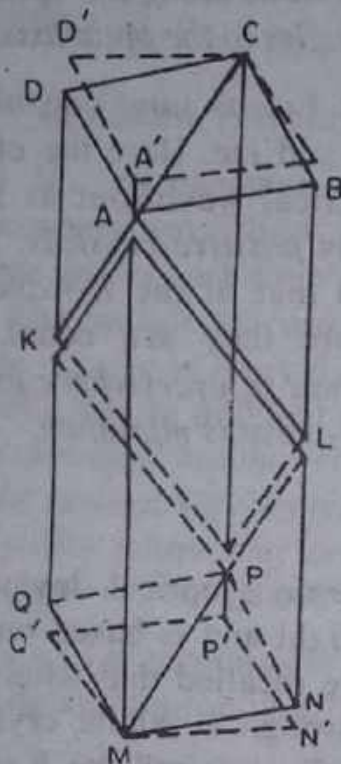


Fig. 30.9

of the end faces  $AC$  and  $PM$ . The two cut surfaces are then ground and polished to optical flatness and cemented together with a thin layer of *canada balsam* which is a transparent, non-doubly refracting plaster with an index of refraction intermediate between the indices of refraction of calcite for the *O-ray* and *E-ray*. For sodium-*D* line (yellow) of mean wavelength  $\lambda = 5893 \text{ A.U.}$ , the values of the indices are

refractive index of calcite for *O-ray*,  $\mu_o = 1.65836$

refractive index of *canada balsam*,  $\mu = 1.55$

refractive index of calcite for *E-ray*,  $\mu_e = 1.48641$

### Action

The action of a Nicol prism can be understood by referring to the Fig. 30.10 where the section  $ACPM$  of the crystal is shown. The diagonal  $AP$  represents the Canada balsam layer in the plane  $ALPK$ . A ray  $IR$  of monochromatic unpolarized light incident on the face  $AC$ , in a direction parallel to  $CP$ , splits into an  $O$ -ray and  $E$ -ray. The  $E$ -ray makes a greater angle with the optic axis than the  $O$ -ray. Since  $\mu_e < \mu_B < \mu_o$ , the Canada balsam acts as a rarer medium for an

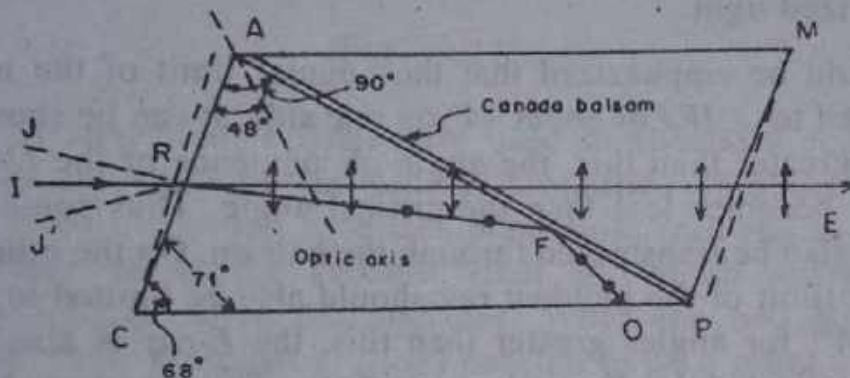


Fig. 30.10

ordinary ray and a denser medium for the extra-ordinary ray. Therefore, at  $F$ , the  $O$ -ray is incident on a rarer medium (balsam) from a denser medium (calcite). When the angle of incidence on the balsam is greater than the *critical angle*, the  $O$ -ray is totally internally reflected and is diverted along the direction  $FO$  where it is effectively absorbed by the blackened side of the prism.

Now the refractive index of the ordinary ray with respect to Canada balsam is

$$\mu = \frac{\mu_o}{\mu_B} = \frac{1.658}{1.550}$$

Hence the critical angle of incidence is obtained from the relation

$$\sin \theta = \frac{1}{\mu} \quad ; \quad \text{or, } \theta = \sin^{-1} \left( \frac{1}{\mu} \right)$$



$$= \sin^{-1} \left( \frac{1.550}{1.658} \right) = 69^\circ$$

The *E-ray*, on the other hand, is passing from a rarer (calcite) into a denser (balsam) medium. It is, therefore, transmitted through the balsam, simply suffering a slight deviation in its path. The *E-ray* finally emerges out of the prism parallel to its original direction but slightly displaced laterally. Thus if the angle of incidence of the *O-ray* on the balsam is greater than the critical angle, a ray of unpolarized light on passing through the Nicol prism becomes plane-polarized light.

It should be emphasized that the angular limit of the incident ray is limited to  $\angle IRJ$  of about  $14^\circ$  on one side; it can be shown that for angles greater than this, the angle of incidence of the *O-ray* on the balsam becomes less than the critical angle. Thus some of the *O-ray* will also be transmitted through the balsam. On the other side, the angular limit of the incident ray should also be limited to  $\angle IRJ'$  of about  $14^\circ$ ; for angles greater than this, the *E-ray* is also totally internally reflected by the balsam surface. This happens because when the *E-ray* travels in a direction at right angles to the optic axis, its refractive index = 1.486. When the *E-ray* travels in the direction of the optic axis, its refractive index becomes 1.658, the same as that of the *O-ray*. Thus, depending upon the direction of propagation of the *E-ray*,  $\mu_e$  lies between 1.486 and 1.658. Therefore, for a particular angle of incidence of the unpolarized light,  $\mu_e$  may be more than 1.55 and the angle of incidence on the balsam layer will be more than the critical angle. Then the *E-ray* will also be totally internally reflected by the balsam layer.

Thus to avoid the transmission of the *O-ray* and the total internal reflection of the *E-ray*, the angle between the extreme rays of light incident on the Nicol prism is limited to about  $28^\circ$ .

### Uses

A Nicol prism can be used both for the production (polarizer) and detection (analyzer) of plane polarized light. Two Nicol prisms are frequently lined up co-axially one behind the other in an arrangement as shown in Fig. 30.17.



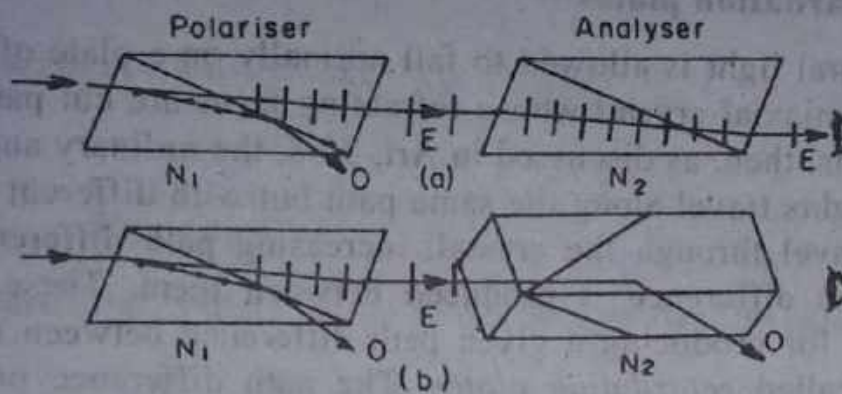


Fig. 30.11

This arrangement is frequently used in specially constructed instruments for studying the optical properties of other crystals. The first Nicol  $N_1$  which produces plane-polarized light, is called the *polarizer* while the second Nicol  $N_2$  which analyzes the incoming light is called the *analyzer*. When the two Nicols are placed with their principal sections parallel to each other as in Fig. 30.11 (a), then the vibrations of the  $E$ -ray which is in the principal section of  $N_1$  is also in the principal section of  $N_2$ . Hence the  $E$ -ray transmitted by  $N_1$  is freely transmitted by  $N_2$ . This setting of the Nicols is known as *parallel position* and the intensity of the transmitted light in this position is maximum. If the second Nicol is gradually rotated, the intensity of the transmitted  $E$ -ray goes on decreasing in accordance with Malus' law till it becomes zero when the principal sections of the two Nicols are at right angles to each other as shown in Fig. 30.11 (b). This is so because the  $E$ -ray coming out from the first Nicol becomes an  $O$ -ray for the second Nicol and is, therefore, totally internally reflected. This position of the Nicols is known as *crossed position*. If  $I_0$  be the intensity of transmitted light in the parallel position of the Nicols and  $I_\theta$  the intensity when the principal sections are inclined at an angle  $\theta$ , then according to Malus' cosine law,

$$I_\theta = I_0 \cos^2 \theta$$

If  $N_2$  is further rotated so that it has turned through an angle  $180^\circ$ , the  $E$ -ray will be freely transmitted again.

### 30.11 Retardation plates

If natural light is allowed to fall normally on a plate of doubly refracting uniaxial crystal whose refracting faces are cut parallel to its optic axis, then, as discussed in Art. 30.8, the ordinary and extraordinary lights travel along the same path but with different speeds. As they travel through the crystal, increasing path difference and hence phase difference is produced between them. These plates, introduced for producing a given path difference between the two rays, are called *retardation plates*. The path difference produced may be deduced in the following manner.

If  $t$  is the thickness of the plate then, within the plate, the optical path for the *E-ray* is  $\mu_e t$  and that for the *O-ray* is  $\mu_o t$  where  $\mu_e$  and  $\mu_o$  are the refractive indices for the extra-ordinary and ordinary rays respectively. The path difference is, therefore,

$$\Delta = (\mu_o - \mu_e) \cdot t \quad \text{where } \mu_o > \mu_e$$

The corresponding phase difference between the two rays is, therefore, given by

$$\delta = \frac{2\pi}{\lambda} (\mu_o - \mu_e) \cdot t.$$

Two such retardation plates are commonly used.

#### (i) quarter-wave plate

If the thickness of the crystal plate is such that a path difference of  $\lambda/4$  (or a phase difference of  $\pi/2$ ) is introduced between the two rays, then the plate is called a *quarter-wave plate*. Let the required thickness of the plate be  $t$ . Then the equivalent optical path of the plate for the two rays are  $\mu_o t$  and  $\mu_e t$ .

For a quarter-wave plate the difference between these paths must be equal to  $\lambda/4$ . For a negative crystal,  $\mu_o > \mu_e$ .

$$\text{Thus the path difference} = (\mu_o - \mu_e) t = \frac{\lambda}{4}$$

$$\text{or, } t = \frac{\lambda}{4(\mu_o - \mu_e)}$$



For a positive crystal,  $\mu_e > \mu_o$

Hence the path difference  $= (\mu_e - \mu_o) t = \frac{\lambda}{4}$

$$\text{or, } t = \frac{\lambda}{4(\mu_e - \mu_o)}$$

In particular, a plate whose thickness satisfies the equation

$$(\mu_o - \mu_e) t = (4n + 1) \frac{\lambda}{4}$$

where  $n = 0, 1, 2, \dots$  behaves like a quarter-wave plate.

When a beam of plane-polarized light is incident on a quarter-wave plate with its plane of vibration inclined at an angle of  $45^\circ$  to the optic axis, the emergent light is *circularly* polarized.

## (ii) half-wave plate

For a half-wave plate, the thickness  $t$  should be such that the optical path difference

$$(\mu_o - \mu_e) t \text{ is } \lambda/2.$$

Thus, for negative crystals,

$$\text{path difference} = (\mu_o - \mu_e) \cdot t = \frac{\lambda}{2}$$

$$\text{or, } t = \frac{\lambda}{2(\mu_o - \mu_e)}$$

and for positive crystals,

$$\text{path difference} = (\mu_e - \mu_o) t = \frac{\lambda}{2}$$

$$\text{or, } t = \frac{\lambda}{2(\mu_e - \mu_o)}$$

In particular, a plate whose thickness satisfies the equation



$$(\mu_o \sim \mu_e) t = (2n + 1) \frac{\lambda}{2}$$

where  $n = 0, 1, 2, \dots$  behaves like a half-wave plate.

When a beam of plane-polarized light is incident on a half-wave plate with its plane of vibration inclined at an angle of  $45^\circ$  to the optic axis, the emergent light is still plane-polarized or linearly polarized, but the direction of polarization of the linear incident light is rotated through  $90^\circ$ . Owing to this property of rotating the plane of vibration of plane polarized light by  $2\theta$ , where  $\theta$  is the angle between the plane of vibration of the incident wave and the principal section of the plate, half-wave plate is employed in Laurent's polarimeter (Art. 30.16) to divide the field of view into two halves presented side by side.

**Example 30.5.** A quartz quarter-wave plate with refractive indices  $\mu_o = 1.544$  and  $\mu_e = 1.553$  is to be used with sodium light ( $\lambda = 5893 \text{ \AA}$ ). Calculate the thickness of the plate.

**Soln.**

$$t = \frac{\lambda}{4(\mu_e - \mu_o)} = \frac{5893 \times 10^{-8}}{4(1.553 - 1.544)} \\ = 16.37 \times 10^{-4} \text{ cm}$$

**Example 30.6** A beam of  $\frac{1}{2}$  nearly polarized light is changed into circularly polarized light by passing it through a slice of crystal  $0.003 \text{ cm}$  thick. Assuming this to be minimum thickness that will produce the effect, calculate the difference in the refractive indices of the two rays in the crystal. The wavelength of light used is  $6 \times 10^{-5} \text{ cm}$ .

**Soln.**

It will be recalled that if the plane of vibration of the incident light is inclined at  $45^\circ$  to the optic axis and if a phase difference of  $\pi/2$  corresponding to a path difference of  $\lambda/4$  is introduced between the  $O$ - and  $E$ -rays, on emergence from the crystal, the  $O$ - and  $E$ -rays

will recombine to produce circularly polarized light. Obviously the crystal used is a quarter wave plate.

$$\therefore t(\mu_o \sim \mu_e) = \frac{\lambda}{4}$$

$$\text{or, } 0.003(\mu_o \sim \mu_e) = \frac{6 \times 10^{-5}}{4}$$

$$\text{or, } (\mu_o \sim \mu_e) = \frac{6 \times 10^{-5}}{0.003 \times 4} = 0.005$$

**Example 30.7.** Plane-polarized light passes through a quartz plate with its optic axis parallel to the faces. Calculate the minimum thickness of the plate for which the emergent beam will be plane polarized. Given,  $\mu_e = 1.553$ ,  $\mu_o = 1.544$  and  $\lambda = 4.95 \times 10^{-5}$  cm.

**Soln.**

For the emergent beam to be plane or linearly polarized, the thickness of the plate should be such that a path difference of  $\lambda/2$  is introduced between the two rays. Hence

$$t(\mu_e \sim \mu_o) = \lambda/2$$

$$\text{or, } t(1.553 - 1.544) = \frac{4.95 \times 10^{-5}}{2}$$

$$\text{or, } t = \frac{4.95 \times 10^{-5}}{2 \times 0.009} = 2.75 \times 10^{-3} \text{ cm.}$$

**Example 30.8.** Calculate the thickness of a quartz crystal, cut parallel to the optic axis, that will produce annulment of the light when placed between a crossed polarizer and analyzer. The refractive indices of the O- and E-rays for the quartz crystal are 1.544 and 1.553 respectively and the wavelength of light used is  $5896 \times 10^{-8}$  cm.

**Soln.**

No light is transmitted when the polarizer and the analyzer are crossed. If the thickness of the quartz crystal is such that a phase



difference of  $2\pi$  is introduced between the  $E$ - and  $O$ -rays in transmission through the crystal, then on emergence they will recombine to form linearly polarized light, the plane of vibration coinciding with that of the incident light. Obviously no light will again be transmitted by the analyzer.

Now a phase difference of  $2\pi$  corresponds to a path difference of  $\lambda$ . Hence

$$t(\mu_e - \mu_o) = \lambda$$

$$\text{or, } t = \frac{\lambda}{(\mu_e - \mu_o)} = \frac{5896 \times 10^{-8}}{1.553 - 1.544}$$

$$= 0.0065 \text{ cm.}$$

**Example 30.9.** Plane-polarized light of wavelength 6000 Å.U. is incident on a thin quartz plate cut with faces parallel to the optic axis. Calculate

(i) the ratio of the intensities of the ordinary and extraordinary lights if the plane of vibration of the incident light makes an angle of  $30^\circ$  with the optic axis.

(ii) the minimum thickness of the plate which introduces a phase difference of  $60^\circ$  between the  $O$ - and  $E$ -rays.

(iii) the minimum thickness of the plate for which the  $O$ - and  $E$ -rays will combine to produce plane-polarized light.

Given  $\mu_o = 1.544$  and  $\mu_e = 1.553$ .

**Soln.**

(i) Let  $A$  be the amplitude of the incident wave. Then the amplitude of the  $E$ -wave is  $A \cos \theta = A \cos 30^\circ = \frac{\sqrt{3}}{2} A$

The amplitude of the  $O$ -wave is

$$A \sin \theta = A \sin 30^\circ = \frac{A}{2}$$

Since the intensity is proportional to the square of the amplitude



$$\frac{I_E}{I_o} = \frac{\left(\frac{\sqrt{3}}{2} A\right)^2}{\left(\frac{A}{2}\right)^2} = 3 : 1$$

Hence the ratio of the intensities is 3 : 1

(ii) phase difference of  $360^\circ$  corresponds to a path difference of  $\lambda$ .

Hence, phase difference of  $60^\circ$

$$= \frac{60^\circ}{360^\circ} \times \lambda = \frac{\lambda}{6}$$

$$\therefore t(\mu_e - \mu_o) = \frac{\lambda}{6}$$

$$\text{or, } t = \frac{\lambda}{6(\mu_e - \mu_o)} = \frac{6000 \times 10^{-8}}{6(1.553 - 1.544)}$$

$$= 0.00111 \text{ cm.}$$

(iii) For the *E*- and *O*-wave to combine to give linearly polarized light, the two-rays must have a phase difference of  $\pi$  radians corresponding to a path difference of  $\lambda/2$ . Hence the minimum thickness of the quartz plate

$$t(\mu_e - \mu_o) = \frac{\lambda}{2}$$

$$\text{or, } t = \frac{\lambda}{2(\mu_e - \mu_o)} = \frac{6000 \times 10^{-8}}{2(1.553 - 1.544)}$$

$$= 0.00333 \text{ cm.}$$

### 30.12 Dichroism – polarization by selective absorption

When ordinary light enters certain crystals and minerals, such as tourmaline, double refraction takes place in much the same way that it does in calcite, but with this difference; the *O*-ray is much

strongly absorbed than the *E-ray*. Hence if such a crystal is cut of the proper thickness, the *O-ray* is completely extinguished by absorption while the *E-ray* is transmitted in appreciable amount as shown in Fig. 30.12. Thus plane polarized light is produced by selective absorption. This phenomenon is known as *dichroism* and the crystals showing this property are said to be *dichroic*. When two such crystals are lined up parallel, with one behind the other, the plane polarized light from the first crystal passes through the second with little loss of intensity. If either crystal is turned at  $90^\circ$  to the other, i.e., in *cross position*, the light is completely absorbed and none passes through. The reason tourmaline is not used in optical instruments in place of Nicol prism is that the crystals are yellowish in colour and do not transmit white light. Due to unequal absorption of the *E-rays* of various wavelengths the transmitted light is greenish in colour.

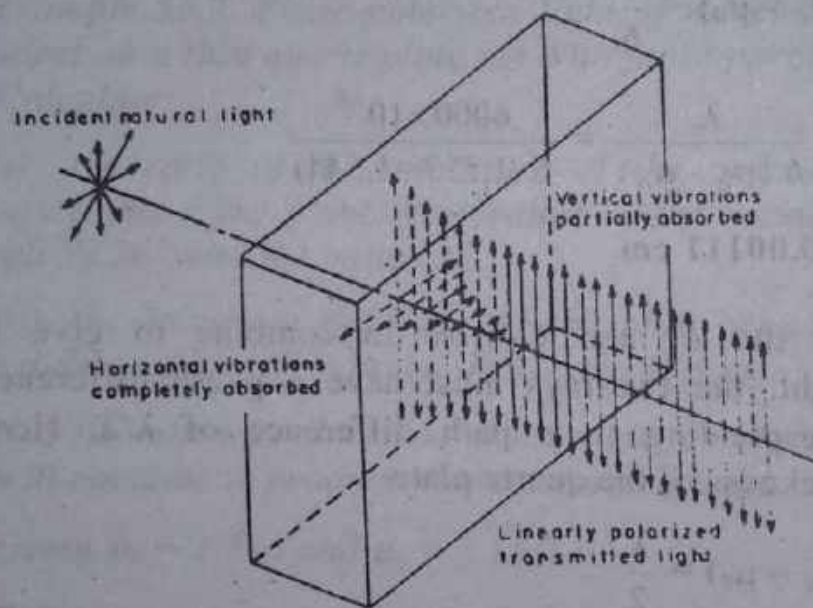


Fig. 30.12

### 30.13 Polaroid

In 1852, W. H. Herapath, an English physician, discovered that small crystals of quinine iodosulphate, called *herapathite* after its discoverer, exhibit strong dichroism – they completely absorb one component of polarization while transmitting the other with transmission close to the ideal 50 per cent for all wavelengths of visible light. But these



crystals are not stable and are affected by slightest strain. They are, therefore, quite useless as such. But the potential usefulness of this material led to extensive research, culminating in the invention by E. H. Land in 1932 of a process which arranges herapathite crystals side by side, oriented with their optic axis all parallel, so that they function as a single crystal of large dimension. This is achieved by preparing a paste of the crystals in nitrocellulose which is then squeezed out through a fine slit. Obviously, only those crystals pass whose axis are parallel to the length of the slit, thereby producing a fine sheet of million of tiny crystals, with their optic axis parallel to each other. This fine sheet, about 0.001 to 0.004 inches thick, is then mounted between two thin sheets of glass, forming what is now called *polaroid*.

More recently polaroid materials have been prepared by aligning molecules rather than the tiny crystals. One of these consists of aligned molecules of polymeric iodine in a sheet of polyvinyl alcohol. Polyvinyl alcohol films are stretched to line up the complex molecules and then are impregnated with iodine. It was discovered by X-ray diffraction studies of these dichroic films that the iodine is present in polymeric form, *i.e.*, as independent long strings of iodine atoms all lying parallel to the fibre axis, with a periodicity in this direction of about  $3.10 \text{ \AA}$ . Films prepared in this way are called *H-polaroid*. It was subsequently found by Land and Rogers that when an oriented transparent film of polyvinyl alcohol is heated in the presence of an active dehydrating catalyst such as hydrogen chloride, the film darkens slightly and becomes strongly dichroic. Such a film becomes very stable and, having no dyestuffs, is not bleached by strong sunlight. This is referred to as *K-polaroid*.

Although the light transmitted by the polaroids is slightly coloured, they find wide applications in everyday life. The most common use of polaroid is in sun-glasses. Polaroids are fitted in motor car head-lights and wind-screen so that the driver can see the light of his own head-light while that from the oncoming car is cut off. They are also used in glass windows of trains and aeroplanes. In aeroplanes one of the polaroid is fixed while the other can be rotated to control the amount of light coming in. polaroid films are used to produce three-dimensional motion pictures.



### 30.14 Optical activity

If two Nicol prisms are placed in crossed position, then no light will emerge out of the analyzer. If now a quartz plate, cut with its faces parallel to the optic axis, is introduced between the two Nicol prisms so that polarized light from the Nicol  $N_1$  falls normally on it, light emerges out of  $N_2$ . What happens is that as polarized light travels through the quartz crystal, its plane of vibration is slowly rotated about its direction of propagation and it emerges out vibrating in some other plane. This action of turning the plane of vibration occurs not on the surface of the crystal but inside the body of the crystal and the amount of rotation depends upon the thickness of the plate. If the thickness is such that it rotates the plane-polarized light through  $90^\circ$ , the vibrations are brought in the principal section of the analyzer which then transmits it freely. If the quartz plate is removed, then no light is transmitted by the analyzer in the crossed position.

*This property of rotating the plane of vibration of plane-polarized light about its direction of propagation possessed by certain crystals or substances is known as optical activity.*

Apart from quartz, many other crystals like sodium chlorate, sodium bromate and cinnabar exhibit the phenomenon of optical activity and such substances are said to be optically active. In addition to these solids, there are certain liquids, e.g., turpentine, sugar solution, alcoholic solution of camphor, etc., are also *optically active*. The optical rotations produced by liquids is, however, very small as compared to that produced by solids. Thus the red region of the spectrum is rotated through an angle of  $18^\circ$  by an one millimetre thick quartz plate whereas the same thickness of turpentine produces a rotation of only twenty five minutes. The rotation in the plane of vibration in a solution depends upon concentration of the optically active substance in the solution. This helps to determine the amount of substance, e.g., cane sugar in a sugar solution.

Optically active substances are of two types. Some of them rotate the plane of vibration to the right and are called right-handed or *dextro-rotatory*. The substances that rotate the plane of vibration to the left are called *left-handed* or *levorotatory*. From the observer's point of view, a right-handed rotation means that when

the observer is looking towards light travelling towards him, the plane of vibration is rotated in a *clock-wise* direction. Similarly a left-handed rotation means an *anti-clockwise* rotation from the point of view of the observer. Cane sugar is dextro-rotatory whereas fruit sugar is laevorotatory. It has been found that some quartz crystals are dextro-rotatory while others are leavo-ratatory. One is the mirror image of the other in their orientation.

### 30.15 Specific rotation

The term specific rotation or specific rotatory power, usually denoted by the letter  $s$ , is used to compare the optical activity of different substances. It is defined differently for solids, liquids and solutions.

In case of solids it is defined by the relation

$$\theta = s \times l$$

where  $\theta$  is the rotation produced (in degrees) by  $l$  millimetre length of the solid. Thus for solids the specific rotation is the angle of rotation produced by 1mm thickness of the solid in the direction of optic axis.

In case of liquids, it has been found that the plane of the linearly polarized light is rotated by liquids containing an optically active substance.

The optical activity of an optically active substance such as sugar is not affected if it is dissolved in an inactive liquid such as water. *For light of given wavelength and at a given temperature, the angle through which the plane polarized light is rotated, depends upon*

- (i) the thickness of the medium (solution)
- (ii) the concentration of the solution or the density of the active substance in the solvent
- and (iii) the nature of the substance

Then

$$\theta \propto l \cdot c$$

$$\text{or, } \theta = s l c$$



where  $l$  = length of the substance in decimetres

$c$  = concentration in  $\text{gm/cm}^3$  of the solution

$s$  = a constant called *specific rotation* which depends upon the nature of the substance.

$$\text{or, } s = \frac{\theta}{l \cdot c}$$

If  $l = 1$  decimetre and  $c = 1 \text{ gm/cm}^3$ , then  $s = \theta$ . *Specific rotation* may, therefore, be defined as the rotation produced by one decimetre (10 cm) length of the solution of unit concentration i.e., containing 1 gm of the optically active substance per  $\text{cm}^3$  of the solution. Thus

*Specific rotation*

$$= \frac{\text{rotation produced by 1 decimetre length of the solution}}{\text{density of the solution in gms. per c.c.}}$$

$$= \frac{\theta}{l/10} \div c = \frac{100}{l \cdot c}$$

$l$  in the above expression is measured in cms

The specific rotation of dextro-rotatory substance is considered *positive* while that of a laevo-rotatory substance is regarded as *negative*.

Instruments used for determining the angle through which the plane of polarization is rotated by the optically active substance are called *polarimeters*. They can be used to find the specific rotation of sugar solution or if specific rotation is known, they can be used to find its concentration. When used for determining the quantity of sugar in a solution, it is known as *saccharimeter*.

### 30.16 Laurent's half-shade polarimeter

The essential parts of a Laurent half-shade polarimeter is shown in Fig. 30.13. It consists of a Nicol prism  $N_1$  which acts as the polarizer while the second Nicol prism  $N_2$  which can be rotated, acts as the analyzer. Behind  $N_1$ , there is a plate QG which is known as the Laurent's half-shade plate and is used for accurately adjusting



the two Nicols in the crossed position.  $T$  is a hollow glass tube having a comparatively larger bore at its middle portion  $T_1$ . When the tube is filled with the solution containing the optically active substance and the two ends are closed with *cover-slips* and *metal covers*, there should not be any air bubble in the path of the light. If there

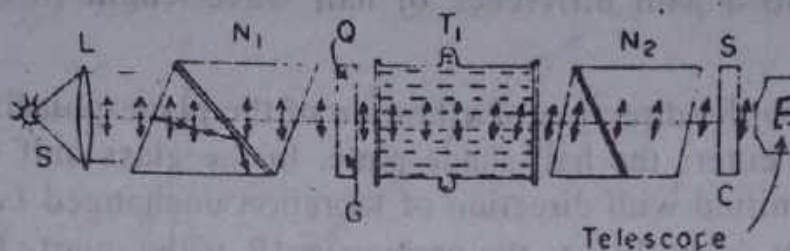


Fig. 30.13

are any air bubbles, they will appear at the upper portion of the wide bore  $T_1$  of the tube. This will ensure that there will not be any air bubble in the path of light.  $SC$  is a circular scale for measuring the angle of rotation of  $N_2$ . The emergent light is viewed by the telescope  $E$ .

Light from a monochromatic source  $S$  (usually a sodium flame) is rendered parallel by the collimating lens  $L$  and falls on the polarizing Nicol  $N_1$ . After passing through  $N_1$ , the beam becomes plane polarized and falls on the Laurent's half-shade plate. By placing this plate in front of the Nicol  $N_2$  the accuracy with which the position of minimum intensity for the analyzing Nicol  $N_2$  can be determined is greatly increased. Laurent's half-shade plate consists of two semi-circular plates, one  $ADB$  of glass and the other  $ACB$  of quartz cemented together along their diameter  $AB$  (Fig. 30.14). The glass

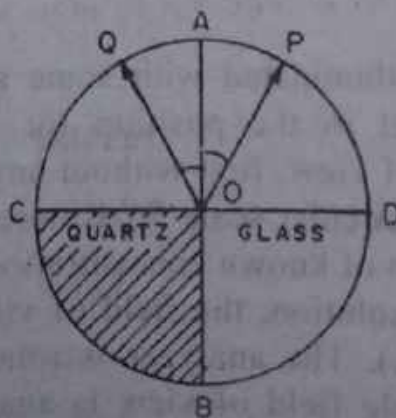


Fig. 30.14

plate is of such thickness that it absorbs and transmits the same amount of light as done by the quartz portion. Half the beam of plane polarized light passes through the quartz plate and the other half passes through the glass plate. Let the optic axis of the quartz plate lie along the line  $AB$  and let its thickness be such that it corresponds to a path difference of half wave-length ( $\lambda/2$ ) of the light used.

Let  $OP$  be the direction of vibration of the plane-polarized light just before it enters the half-shade plate. In the glass half the light will be transmitted with direction of vibration unchanged *i.e.*, along  $OP$  inclined at an angle  $\theta$  to the optic axis  $AB$  of the quartz half. But the plate is so constructed that in the quartz half, the direction of vibration of the emergent beam will be along  $OQ$  making the same angle  $\theta$  with the optic axis as does  $OP$  *i.e.*, inclined at  $2\theta$  to  $OP$ . Now, if the analysing Nicol  $N_2$  is placed with its principal plane perpendicular to  $OP$ , then it will stop all the light in the glass half, but transmit some light in the quartz half. On the other hand, if it is held perpendicular to  $OQ$ , the quartz half will be completely dark but the glass half will transmit some light. The Nicol  $N_2$  is usually adjusted midway between these two contrasts so that both halves are equally bright, and obviously the principal plane of the Nicol is parallel to  $OA$ . If the analyzing Nicol is slightly rotated towards the right then the right side will be much brighter than the left and *vice versa*. Thus any slight rotation of the analyzer in either direction of  $AB$  produces sharp differences in the illumination of the two halves.

*To determine the specific rotation of an optically active substance, e.g., a sugar solution, the procedure to be adopted is as follows:*

The Nicol  $N_1$  is illuminated with some monochromatic light. The analyzer  $N_2$  is set in the position for equal brightness (or darkness) in the field of view, first without any solution in the tube  $T$ . The readings in the circular scale  $SC$  is noted. Then the tube  $T$  is filled up with a solution of known concentration. On introducing the tube  $T$  filled with the solution, the field of view will no longer be equally bright (or dark). The analyzer is rotated in the clockwise direction until the whole field of view is again equally bright (or dark). The readings in the circular scale is noted again. The difference between the two readings *i.e.*, the angle through which



the analyzer has been rotated, gives the angle through which the plane of vibration of the incident beam has been rotated by the sugar solution. In an actual experiment, angles of rotations are determined corresponding to various concentrations of sugar solution. When  $\theta$  is plotted against concentration  $c$ , the graph is a straight line (Fig. 30.15). Specific rotation is then determined from the relation

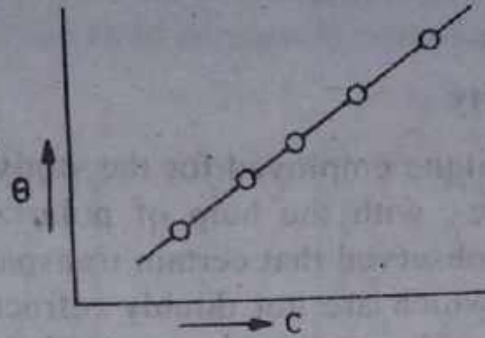


Fig. 30.15

$$s = \frac{100}{l \cdot c}$$

where  $\theta$  is the angle of rotation corresponding to a concentration as obtained from the graph.

The specific rotation of a substance depends upon the wavelength of light used and the temperature of the substance. It is usually denoted by  $s_{\lambda}^t$ . The specific rotations of some substances for the sodium  $D$  lines ( $\lambda_1 = 5890 \text{ \AA}$ ,  $\lambda_2 = 5896 \text{ \AA}$ ) at  $20^\circ\text{C}$  are given below.

Specific rotation for light of  $\lambda = 5893 \text{ \AA}$  at  $20^\circ\text{C}$ .

active substance	solvent	$s_{\lambda}^t$
cane sugar	water	+ 66.5°
glucose	water	+ 52.5°
fructose	water	- 91.5°
invert sugar	water	- 19.5°



tartaric	—	+ 15.0°
rochelle salt	—	+ 29.7°
turpentine	pure	— 37°
nicotine	pure	— 162°
comphor	alcohol	+ 41°

### 30.17 Photo-elasticity

The optical technique employed for the study of stress analysis in girders, beams, etc., with the help of polarized light is called photo-elasticity. It is observed that certain transparent materials like glass, bakelite, etc., which are not doubly refracting under normal conditions, exhibit the phenomenon temporarily when subjected to mechanical stresses. When such a specimen is placed between two crossed Nicol prisms, no change in the field of view is observed. But when the specimen is subjected to some mechanical stress, the field of view is covered by an interference pattern. The photograph of the pattern is taken and the stress at various points is studied. The stress is maximum at points where the interference pattern is close. The stress produced in a girder when loaded is estimated by this method.

In the cases of complex structures like bridges, boiler plates, gear teeth etc., a transparent model of the structure is constructed by materials like bakelite or zylonite and the structure is subjected to loads and stresses. The stresses produced at various parts of the model can be analyzed and estimated from the study of interference pattern. From this the actual effect of stresses for the materials can be calculated which helps in design of the structures. This method of stress analysis is commonly used in preference to other methods.

#### Photo Elastic Bench

As already mentioned above, when polarized light is passed through a *transparent elastic object* such as glass or bakelite, colours are observed when a stress is applied to the object. This fact has been developed into a full fledged and accurate technique for measurement of complicated stress distributions. Fig. 30.16 shows the experimental arrangement for this technique. *S* is a source of

monochromatic light (usually blue-green mercury arc lamp) and is placed at the focus of the lens  $L_1$ .  $L_1$  thus renders the beam of light coming from  $S$  parallel. This parallel beam of light is made to pass through two polaroids  $P_1$  and  $P_2$ . Light emerging from  $P_2$  is made to converge on a photographic camera  $C$  by the second lens  $L_2$ . For visual observation the lens  $L_2$  and the camera  $C$  are replaced by a milk glass screen which is placed behind  $P_2$ . The two polaroids are kept crossed, so that the field of view is completely dark.

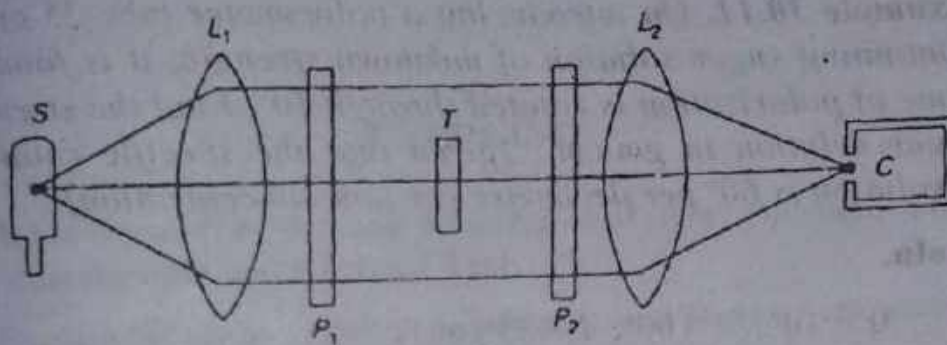


Fig. 30.16

The test piece  $T$  of the material is made of special yellowish sheet of *transparent bakelite*. The test piece is cut from a flat plate of thickness between 0.6 cm to 1.2 cm and given the proper form. The sheet is then placed between the two crossed polaroids. When no stress is applied to the plate, the field of view is completely dark. As the stress is applied on the test piece, some coloured patterns of fringes will appear. The information of the stress distribution at various points can be known from this pattern.

The stress analysis of various complicated structures and machine can be done by using the *photoelasticity technique*. The stress distribution in a railway wagon wheel and aircraft components can also be studied by this technique. Photoelasticity is also being used in solving thermal problems, high speed photography, computer technology and laser.

**Example 30.10.** Calculate the specific rotation of sugar if the plane of vibration is turned through  $26.4^\circ$ , traversing 20 cm length of 20% sugar solution.

**Soln.**



$$\begin{aligned}
 s &= \frac{100}{l \cdot c} & c &= 20\% = 0.2 \text{ gm/cm}^3 \\
 &= \frac{10 \times 26.4}{20 \times 0.2} & l &= 20 \text{ cm} \\
 &= 66^\circ & \theta &= 26.4^\circ
 \end{aligned}$$

**Example 30.11.** On introducing a polarimeter tube 25 cm long and containing sugar solution of unknown strength, it is found that the plane of polarization is rotated through  $10^\circ$ . Find the strength of the sugar solution in  $\text{gm/cm}^3$ . [given that the specific rotation of sugar solution is  $60^\circ$  per decimetre per unit concentration].

**Soln.**

$$\theta = 10^\circ, s = 60^\circ, l = 25 \text{ cm}, c = ?$$

$$\begin{aligned}
 s &= \frac{100}{l \cdot c}; \text{ or, } c = \frac{100}{s \cdot l} = \frac{10 \times 10}{25 \times 60} = \frac{1}{15} \\
 &= 0.067 \text{ gm/cc.}
 \end{aligned}$$

**Example 30.12.** A solution of an optically active solute produces a rotation of  $20^\circ$  of the plane of polarization in a path length of 10 cm when the concentration is 20 gm/litre. What is the concentration in a solution which produces a rotation of  $30^\circ$  in a path length of 5 cm?

**Soln.**

$$\begin{aligned}
 s &= \frac{100}{l \cdot c} & \theta &= 20^\circ, l = 10 \text{ cm} \\
 &= \frac{10 \times 20 \times 1000}{10 \times 20} & c &= 20 \text{ gm/litre} \\
 &= 1000^\circ & &= 20 \text{ gm/1000 cm}^3
 \end{aligned}$$

$$\begin{aligned}
 \text{Again } c &= \frac{100}{l \cdot s} & \theta &= 30^\circ, l = 5 \text{ cm}
 \end{aligned}$$



$$= \frac{10 \times 30}{5 \times 1000}$$

$$s = 1000^{\text{g}}$$

$$= 6 \text{ gm}/100 \text{ cm}^3 = 60 \text{ gm/litre.}$$

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### EXERCISES

- [1] What is meant by the term polarization of light? Distinguish between ordinary light and polarized light.
- [2] Explain the terms 'plane of polarization and plane of vibration'.
- [3] State and explain Brewster's law. Show that at the polarizing angle of incidence, the reflected and refracted rays are mutually perpendicular to each other.
- [4] Explain the phenomenon of double refraction. What are ordinary and extraordinary rays?
- [5] Explain the terms (i) bi-refringence (ii) optic axis and (iii) principal section.
- [6] Describe the construction and use of a Nicol prism and explain how it produces plane polarized light.
- [7] Describe the construction of a Nicol prism and show how it can be used as a polarizer or an analyzer. What are 'parallel' and 'crossed' positions of two Nicol prisms?
- [8] What is meant by 'optical activity' of a substance? Distinguish between a dextro-rotatory substance and a laevo-rotatory substance.
- [9] What is specific rotation? Give with details the construction and working of a Laurent's half-shade polarimeter. Explain how you would use it to determine the specific rotation of sugar.
- [10] Write short notes on :
  - (i) Malus' law
  - (ii) Retardation plates

- (iii) Brewster's law
  - (iv) Dichroism
  - (v) Polaroids and their uses
  - (vi) Double refraction
  - (vii) Quarter and half-wave plates
  - (viii) Uni-axial and bi-axial crystals
  - (ix) Photo-elasticity
- [11] The polarizing angle for a certain type of glass is  $54^\circ$ . Find its refractive index. (1.376).
- [12] Calculate the thickness of a quartz half-wave plate for the Fraunhofer C line ( $\lambda = 6563 \text{ A.U.}$ ) for which the extra-ordinary and ordinary refractive indices of quartz are 1.55085 and 1.54181 respectively. (0.003631 cm.)
- [13] Calculate the thickness of a quarter-wave plate of quartz for sodium light ( $\lambda = 5893 \text{ A.U.}$ ). The refractive indices of quartz for the *E*- and *O*-waves are 1.5533 and 1.5442 respectively.
- [14] Calculate the thickness of a doubly refracting crystal to introduce a path difference of  $\lambda/2$  between the ordinary and extra-ordinary rays when  $\lambda = 6000 \text{ A.U.}$ ,  $\mu_o = 1.55$  and  $\mu_e = 1.54$ .
- [15] How will you orient the polarized and the analyzer so that a beam of natural light is reduced to (i) 0.5 (ii) 0.25 (iii) 0.75 and (iv) 0.125 of its original intensity. [(i)  $45^\circ$ , (ii)  $60^\circ$ , (iii)  $30^\circ$ , (iv)  $69^\circ$ ]
- [16] A length of 25 cm of a solution, containing 50 gram of solute per litre causes a rotation of light by  $5^\circ$ . Find the rotation of the plane of polarization by a length of 75 cm of a solution containing 100 gm of solute per litre. ( $30^\circ$ ).
- [17] The specific rotation of tartaric acid is  $28^\circ$  for sodium light. Find the concentration of the salt, if 40 cm length of it rotates the plane of polarization of sodium light through  $8.4^\circ$ . (0.075 gram per  $\text{cm}^3$  of solution)
- [18] A solution of camphor in alcohol in a tube 20 cm long is found to rotate the plane of vibration of light by  $27^\circ$ . What is the mass of camphor in unit volume of the solution? The specific rotation of camphor is  $+54^\circ$ .

**Some Conversion Factors**

$$1 \text{ m} = 39.37 \text{ in} = 3.281 \text{ ft} = 1.094 \text{ yd}$$

$$1 \text{ m} = 10^{15} \text{ fm} = 10^{10} \text{ \AA} = 10^9 \text{ nm}$$

$$1 \text{ mi} = 5280 \text{ ft} = 1.609 \text{ km}$$

$$1 \text{ in} = 2.540 \text{ cm}$$

$$1 \text{ l (litre)} = 10^3 \text{ cm}^3 = 10^{-3} \text{ m}^3 = 1.057 \text{ qt}$$

$$1 \text{ y} = 365.24 \text{ d} = 3.156 \times 10^7 \text{ sec}$$

$$1 \text{ ft/sec} = 0.3048 \text{ m/sec} = 0.6818 \text{ mi/h}$$

$$1 \text{ rev} = 2\pi \text{ rad} = 360^\circ$$

$$1 \text{ rad} = 57.30^\circ$$

$$1 \text{ rev/min} = 0.1047 \text{ rad/sec}$$

$$1 \text{ slug} = 14.59 \text{ kg}$$

$$1 \text{ atm} = 1.013 \times 10^5 \text{ N/m}^2 = 1.013 \times 10^6 \text{ dynes/cm}^2 = 1.013 \text{ bars}$$

$$= 76.00 \text{ cm Hg} = 14.70 \text{ lb/in}^2 \text{ (psi)}$$

$$1 \text{ N} = 10^5 \text{ dynes} = 0.2248 \text{ lb}$$

$$1 \text{ lb} = 4.448 \text{ N}$$

$$1 \text{ J} = 10^7 \text{ ergs} = 0.7373 \text{ ft-lb} = 9.869 \times 10^{-3} \text{ l-atm}$$

$$1 \text{ cal} = 4.184 \text{ J} = 4.129 \times 10^{-2} \text{ l-atm}$$

$$1 \text{ l-atm} = 101.3 \text{ J} = 24.22 \text{ cal}$$

$$1 \text{ eV} = 1.602 \times 10^{-19} \text{ J} = 1.602 \times 10^{-12} \text{ ergs}$$

$$1 \text{ btu} = 778 \text{ ft-lb} = 252 \text{ cal} = 1054 \text{ J}$$

$$1 \text{ horse power} = 550 \text{ ft-lb/sec} = 746 \text{ W}$$

$$1 \text{ Tesla} = 10^4 \text{ Gauss}$$

$$1 \text{ kg} = 2.205 \text{ lb}$$



## Some Fundamental Constants

Symbol	Constant	Value	Units
$c$	velocity of light in vacuum	$2.997925 \times 10^8$	$\text{ms}^{-1}$
		$2.997925 \times 10^{10}$	$\text{cm/sec}$
		$1.86 \times 10^5$	$\text{miles/sec}$
$e$	elementary charge	$1.602\,192 \times 10^{-19}$	C
$F$	Faraday constant	$9.648\,67 \times 10^4$	$\text{C mol}^{-1}$
$g$	standard acceleration of free fall	$9.806\,65$	$\text{ms}^{-2}$
$h$	Planck's constant	$6.626\,196 \times 10^{-34}$	J s
		$4.136 \times 10^{-15}$	eV s
$\hbar = h/2\pi$	Plank's constant (reduced)	$1.054\,592 \times 10^{-34}$	J s
		$6.582\,182 \times 10^{-16}$	eV s
$k = R/N_A$	Boltzmann's constant	$1.380\,622 \times 10^{-23}$	$\text{JK}^{-1}$
		$8.617\,082 \times 10^{-5}$	$\text{eV K}^{-1}$
$m_e$	electron rest mass	$9.109\,558 \times 10^{-31}$	kg
$e/m_e$	electron charge/mass ratio	$1.7588028 \times 10^{11}$	$\text{Ckg}^{-1}$
$m_e c^2$	electron rest energy	$8.187\,265 \times 10^{-14}$	J
		0.51100	Mev
$m_p$	proton rest mass	$1.672\,614 \times 10^{-27}$	kg
$m_n$	neutron rest mass	$1.674920 \times 10^{-27}$	kg
$N_A$	Avogadro's number	$6.022\,169 \times 10^{23}$	$\text{mol}^{-1}$
$r_e$	classical electron radius	$2.817\,941 \times 10^{-15}$	m
$a_0$	Bohr radius	$5.29177 \times 10^{-11}$	m
$R$	gas constant	8.314 34	$\text{J mol}^{-1} \text{K}^{-1}$
		$8.20 \times 10^{-2}$	$\text{l-atm mol}^{-1} \text{K}^{-1}$
		1.987	$\text{cal mol}^{-1} \text{K}^{-1}$
$J$	Mechanical equivalent of heat	4.1860	$\text{J cal}^{-1}$
		$4.1860 \times 10^7$	$\text{ergs cal}^{-1}$
		778	$\text{ft. obs/B.Th.U}$
$u$	mass unit ( $^{12}\text{C}$ standard)	$1.660\,631 \times 10^{-27}$	kg

$u_c^2$	mass unit (energy units)	$9.314\ 81 \times 10^8$	eV
$\epsilon_0$	electrical permittivity of space	$8.854\ 18 \times 10^{-12}$	$C^2 N^{-1} m^{-2}$
$1/4\pi\epsilon_0$	Coulomb constant	$8.987\ 56 \times 10^9$	$N\ m^2\ C^{-2}$
$\sigma$	Stefan-Boltzmann constant	$5.669\ 61 \times 10^{-8}$	$W\ m^{-2} K^{-4}$
$\lambda_c$	Compton wavelength of electron	$2.426\ 31 \times 10^{-12}$	m
		0.0242631	A.U.
$\mu_B = eh/2m_e$	Bohr magneton	$9.274\ 096 \times 10^{-24}$	$J\ T^{-1}$
$\mu_N$	Nuclear Magnetron	$5.050951 \times 10^{-27}$	$J\ T^{-1}$
$\mu_0$	Magnetic permeability of space	$4\pi \times 10^{-7}$	$T\ m\ A^{-1}$
$k_m = \mu_0/4\pi$	Magnetic constant	$10^{-7}$ (exact)	$T\ m\ A^{-1}$

## Standard temperature and pressure (STP) :

$T$	273.15 K
$P$	1.00 atm
	$1.01325 \times 10^5\ N/m^2$
Molecular weight of air	28.97 gm/mole
Density of air (STP), $\rho_{air}$	$1.293\ kg/m^3$
Speed of sound in dry air	331.45 m/sec
at 0°C and 760 mm-Hg	1087 ft/sec
	741.5 miles/hr
Heat of fusion of $H_2O$ (0°C, 1 atm)	79.7 cal/gm
Heat of vapourization of $H_2O$	539.6 cal/gm
(100°C, 1 atm)	

## Mathematical Formulas

### Quadratic formula

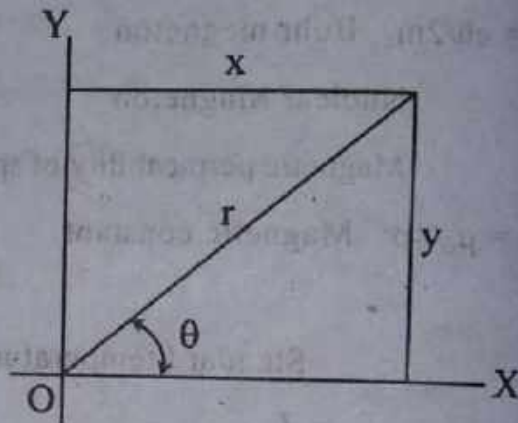
$$\text{If } ax^2 + bx + c = 0, \text{ then } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

### Trigonometric functions of angle $\theta$

$$\sin \theta = \frac{y}{r}, \quad \cos \theta = \frac{x}{r}$$

$$\tan \theta = \frac{y}{x}, \quad \cot \theta = \frac{x}{y}$$

$$\sec \theta = \frac{r}{x}, \quad \csc \theta = \frac{r}{y}$$



### Pythagorean theorem

$$x^2 + y^2 = r^2$$

### Logarithms

#### 1. Definition of e,

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$$

$$= 2.7182818 \dots$$

= natural base of logarithms

#### 2. Natural logarithm, base e,

$$y = \ln x \quad \text{if } x = e^y$$

#### 3. Common logarithm, base 10,

$$y = \log x \quad \text{if } x = 10^y$$

The natural and common logarithms are related by

$$\ln x = 2.303 \log x \quad \log x = 0.434 \ln x$$



*Numerical constants*

$$\pi = 3.14159 \dots$$

$$\sqrt{2} = 1.41421 \dots$$

$$e = 2.71828 \dots$$

$$\sqrt{3} = 1.73205 \dots$$

$$e^{-1} = 0.3679 \dots$$

$$\sqrt{5} = 2.23606 \dots$$

$$e^0 = 1$$

$$\sqrt[3]{2} = 1.25992 \dots$$

$$\sqrt[3]{3} = 1.44224 \dots$$

$$\log_{10} 3 = 0.47712 \dots$$

$$\sqrt[5]{2} = 1.14869 \dots$$

$$\log_{10} e = 0.43429 \dots$$

$$\sqrt[5]{3} = 1.24573 \dots$$

$$\log_{10} \pi = 0.49714 \dots$$

$$e^\pi = 23.14069 \dots$$

$$\log_e 10 = \ln 10 = 2.30258 \dots$$

$$\pi^e = 22.45915 \dots$$

$$\log_e 2 = \ln 2 = 0.69314 \dots$$

$$e_e = 15.15426 \dots$$

$$\log_e 3 = \ln 3 = 1.09861 \dots$$

$$\log_{10} 2 = 0.30102 \dots$$

$$\sqrt{e} = 1.64872 \dots$$

$$\sqrt{\pi} = 1.77245 \dots$$

$$1 \text{ radian} = 180^\circ/\pi = 57.29577 \dots$$

$$1^\circ = \pi/180 \text{ radians} = 0.01745 \dots \text{ radians}$$

$$\sin 30^\circ = \cos 60^\circ = \frac{1}{2} = 0.500$$

$$\sin 60^\circ \cos 30^\circ = \frac{\sqrt{3}}{2} = 0.866$$

$$\sin 45^\circ = \cos 45^\circ = \frac{\sqrt{2}}{2} = 0.707$$

For small  $\theta$  (in radians),

$$\sin \theta \simeq \theta, \quad \tan \theta \simeq \theta.$$

### FUNCTIONS OF NEGATIVE ANGLES

$$\sin(-\theta) = -\sin \theta$$

$$\cos(-\theta) = \cos \theta$$

$$\tan(-\theta) = -\tan \theta$$

$$\csc(-\theta) = -\csc \theta$$

$$\sec(-\theta) = -\sec \theta$$

$$\cot(-\theta) = -\cot \theta$$

### TRIGONOMETRIC FUNCTIONS

#### REDUCTION FORMULAS FOR TRIGONOMETRY

$$90^\circ \quad \sin(90^\circ + \theta) = +\cos \theta$$

$$\sin(90^\circ - \theta) = +\cos \theta$$

$$\cos(90^\circ + \theta) = -\sin \theta$$

$$\cos(90^\circ - \theta) = +\sin \theta$$

$$\tan(90^\circ + \theta) = -\cot \theta$$

$$\tan(90^\circ - \theta) = +\cot \theta$$

$$180^\circ \quad \sin(180^\circ + \theta) = -\sin \theta$$

$$\sin(180^\circ - \theta) = +\sin \theta$$

$$\cos(180^\circ + \theta) = -\cos \theta$$

$$\cos(180^\circ - \theta) = -\cos \theta$$

$$\tan(180^\circ + \theta) = +\tan \theta$$

$$\tan(180^\circ - \theta) = -\tan \theta$$

$$270^\circ \quad \sin(270^\circ + \theta) = -\sin \theta$$

$$\sin(270^\circ - \theta) = -\cos \theta$$

$$\cos(270^\circ + \theta) = +\sin \theta$$

$$\cos(270^\circ - \theta) = -\sin \theta$$

$$\tan(270^\circ + \theta) = -\cot \theta$$

$$\tan(270^\circ - \theta) = +\cot \theta$$

$$360^\circ \quad \sin(360^\circ + \theta) = +\sin \theta$$

$$\sin(360^\circ - \theta) = -\sin \theta$$

$$\cos(360^\circ + \theta) = +\cos \theta$$

$$\cos(360^\circ - \theta) = +\cos \theta$$

$$\tan(360^\circ + \theta) = +\tan \theta$$

$$\tan(360^\circ - \theta) = -\tan \theta$$

#### SIGNS AND VARIATIONS OF TRIGONOMETRIC FUNCTIONS

Quadrant	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\csc \theta$
I 0° to 90°	+	+	+	+	+	+
II 90° to 180°	+	-	-	-	1 to $\infty$	$\infty$ to 1
III 180° to 270°	-	-	+	+	-1 to $-\infty$	1 to $\infty$
IV 270° to 360°	-	+	-	-	$\infty$ to 1	-1 to $-\infty$

**Trigonometric identities**

$$\sin^2\theta + \cos^2\theta = 1 \quad \sec^2\theta - \tan^2\theta = 1 \quad \csc^2\theta - \cot^2\theta = 1$$

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$$

$$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta}$$

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$e^{\pm i\theta} = \cos \theta \pm i \sin \theta$$

**HALF ANGLE FORMULAS**

$$\sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}} \quad \left[ \begin{array}{l} + \text{ if } \theta/2 \text{ is in quadrant I or II} \\ - \text{ if } \theta/2 \text{ is in quadrant III or IV} \end{array} \right]$$

$$\cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}} \quad \left[ \begin{array}{l} + \text{ if } \theta/2 \text{ is in quadrant I or II} \\ - \text{ if } \theta/2 \text{ is in quadrant III or IV} \end{array} \right]$$

$$\tan \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}} \quad \left[ \begin{array}{l} + \text{ if } \theta/2 \text{ is in quadrant I or II} \\ - \text{ if } \theta/2 \text{ is in quadrant III or IV} \end{array} \right]$$

$$= \frac{\sin \theta}{1 + \cos \theta} = \frac{1 - \cos \theta}{\sin \theta} = \csc \theta - \cot \theta$$

**MULTIPLE ANGLE FORMULAS**

$$\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$$

$$\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$$

$$\tan 3\theta = \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}$$

$$\sin 4\theta = 4 \sin \theta \cos \theta - 8 \sin^3 \theta \cos \theta$$



$$\cos 4\theta = 8 \cos^4 \theta - 8 \cos^2 \theta + 1$$

$$\tan 3\theta = \frac{4 \tan \theta - 4 \tan^3 \theta}{1 - 6 \tan^2 \theta + \tan^4 \theta}$$

$$\cos 5\theta = 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta$$

$$\tan 5\theta = \frac{\tan^5 \theta - 10 \tan^3 \theta + 5 \tan \theta}{1 - 10 \tan^2 \theta + 5 \tan^4 \theta}$$

### POWERS OF TRIGONOMETRIC FUNCTIONS

$$\sin^2 \theta = \frac{1}{2} - \frac{1}{2} \cos 2\theta$$

$$\sin^4 \theta = \frac{3}{8} - \frac{1}{2} \cos 2\theta + \frac{1}{8} \cos 4\theta$$

$$\cos^2 \theta = \frac{1}{2} + \frac{1}{2} \cos 2\theta$$

$$\cos^4 \theta = \frac{3}{8} + \frac{1}{2} \cos 2\theta + \frac{1}{8} \cos 4\theta$$

$$\sin^3 \theta = \frac{3}{4} \sin \theta - \frac{1}{4} \cos 3\theta$$

$$\sin^5 \theta = \frac{5}{8} \sin \theta - \frac{5}{16} \sin 3\theta + \frac{1}{16} \sin 5\theta$$

$$\cos^3 \theta = \frac{3}{4} \sin \theta + \frac{1}{4} \cos 3\theta$$

$$\cos^5 \theta = \frac{5}{8} \cos \theta + \frac{5}{16} \cos 3\theta + \frac{1}{16} \cos 5\theta$$

### SUM, DIFFERENCE AND PRODUCT OF TRIGONOMETRIC FUNCTIONS

$$\sin \alpha + \sin \beta = 2 \sin \frac{1}{2} (\alpha + \beta) \cos \frac{1}{2} (\alpha - \beta)$$

$$\sin \alpha - \sin \beta = 2 \cos \frac{1}{2} (\alpha + \beta) \sin \frac{1}{2} (\alpha - \beta)$$

$$\cos \alpha + \cos \beta = 2 \cos \frac{1}{2} (\alpha + \beta) \cos \frac{1}{2} (\alpha - \beta)$$

$$\cos \alpha - \cos \beta = 2 \sin \frac{1}{2} (\alpha + \beta) \sin \frac{1}{2} (\beta - \alpha)$$

$$\sin \alpha + \sin \beta = \frac{1}{2} [\cos (\alpha - \beta) - \cos (\alpha + \beta)]$$

$$\cos \alpha + \cos \beta = \frac{1}{2} [\cos (\alpha - \beta) + \cos (\alpha + \beta)]$$

$$\sin \alpha \cos \beta = \frac{1}{2} [\sin (\alpha + \beta) + \sin (\alpha - \beta)]$$

## DERIVATIVES AND INDEFINITE INTEGRALS

The letters  $u$  and  $v$  stand for any functions of  $x$ , and  $a$  and  $m$  are constants. To each of the integrals should be added an arbitrary of integration.

$$1. \frac{dx}{dx} = 1$$

$$1. \int f dx = x$$

$$2. \frac{d}{dx} (au) = a \frac{du}{dx}$$

$$2. \int f au dx = a \int f u dx$$

$$3. \frac{d}{dx} (u + v) = \frac{du}{dx} + \frac{dv}{dx}$$

$$3. \int f(u + v) dx = \int f u dx + \int f v dx$$

$$4. \frac{d}{dx} x^m = mx^{m-1}$$

$$4. \int f x^m dx = \frac{x^{m+1}}{m+1} \quad (m \neq -1)$$

$$5. \frac{d}{dx} \ln x = \frac{1}{x}$$

$$5. \int \frac{f}{x} dx = \ln |x|$$

$$\frac{d}{dx} (uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$6. \int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

$$7. \frac{d}{dx} e^x = e^x$$

$$7. \int f e^x dx = e^x$$

$$8. \frac{d}{dx} \tan x = \frac{1}{1+x^2}$$

$$8. \int \frac{f}{1+x^2} = \arctan x$$

$$9. \frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$9. \int \frac{f dx}{\sqrt{1-x^2}} = \arcsin x$$

$$10. \frac{d}{dx} \operatorname{arc sec} x = \frac{1}{x \sqrt{1-x^2}}$$

$$10. \int \frac{f dx}{x \sqrt{1-x^2}} = \operatorname{arc sec} x$$

$$11. \frac{d}{dx} \cos x = -\sin x$$

$$11. \int f \sin x dx = -\cos x$$

$$12. \frac{d}{dx} \sin x = \cos x$$

$$12. \int f \cos x dx = \sin x$$

$$13. \frac{d}{dx} \tan x = \sec^2 x$$

$$13. \int \tan x \, dx = \ln |\sec x|$$

$$14. \frac{d}{dx} \cot x = -\csc^2 x$$

$$14. \int \cot x \, dx = \ln |\sin x|$$

$$15. \frac{d}{dx} \sec x = \tan x \sec x$$

$$15. \int \sec x \, dx = \ln |\sec x + \tan x|$$

$$16. \frac{d}{dx} \csc x = -\cot x \csc x$$

$$16. \int \csc x \, dx = \ln |\csc x + \cot x|$$



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